(1)

Key

BC Calculus – 10.5b Notes – Alternating Series Error Bound

*Must be a converging alternating series!

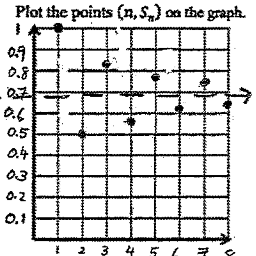
Use the alternating harmonic series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ to fill in the table below.

	n	1	2	3	4	5	6	7	8
terms a_n	fractions	1	$-\frac{1}{2}$	$\frac{1}{3}$	$-\frac{1}{4}$	$\frac{1}{5}$	$-\frac{1}{6}$	$\frac{1}{7}$	$-\frac{1}{8}$
	decimals	1	-0.5	$0.\overline{33}$	-0.25	0.2	$-0.16\overline{6}$	0.143	-0.125
partial Sum S_n	fractions	1	$\frac{1}{2}$	$5/6$	$\frac{7}{12}$	$\frac{47}{60}$	$\frac{37}{60}$	$\frac{319}{420}$	$\frac{533}{840}$
	decimals	1	0.5	0.833	0.583	0.783	0.616	0.759	0.634

partial Sums

*This series will keep alternating and bouncing above and below the line until eventually the partial sums approach $\ln 2$ (closer to the line)

Plot the points (n, S_n) on the graph.



*The actual Sum of this Series is $\ln 2 \approx 0.6931$

	R1	R2						
Error: $\lvert S-S_n\rvert$	$\ln 2 - 1$ 0.3068	$\ln 2 - \frac{1}{2}$ 0.193	$\ln 2 - 5/6$ 0.14	$\ln 2 - 7/12$ 0.1098	$\ln 2 - 47/60$ 0.0901	0.0764	0.0663	0.0586
$\le$	$\le a_2 (0.5)$	$\le a_3$ (0.33)	≤ 0.25	≤ 0.2	≤ 0.166	≤ 0.143	≤ 0.125	

Alternating Series Error Bound

If you have an alternating series that converges, we can approximate the sum of the series!

$$|S-S_n| = |R_n| \le |a_{n+1}|$$

Actual Remainder many times is not known to us.

Error bound is the next term (can be found)

- S: Sum of the series
- S_n: Partial sum
- R_n: Remainder (or error)
- $R_n = S - S_n$
- a_{n+1} = next term (Error Bound)

*Error Bound is the boundary of how far off your actual error is.

(2)

1. Determine the number of terms required to approximate the sum of the series $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ with an error less than 10^{-3}.

$|S-S_n| \le |a_{n+1}|$

*what makes $\frac{1}{n+1} < 10^{-3}$

or $\frac{1}{n+1} < 0.001$

$\frac{1}{n+1} < \frac{1}{1000}$

$1000 < n+1$

$999 < n$

If $n > 999$ terms, the partial sum's approximation is off by no more than $\frac{1}{1000}$ or 0.001

2. If the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{5n+2}$ is approximated by the partial sum with 10 terms, what is the alternating series $\underbrace{\text{error bound}}_{a_{n+1}}$?

$a_{11} = \left|\frac{1}{5(11)+2}\right| = \frac{1}{57}$

The partial sum S_{10} is off by no more than $\frac{1}{57}$ of the Actual sum

3. **Calculator active.** Approximate an interval of the sum of the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{4}{n^2}$ using the Alternating Series Error Bound for the first 5 terms.

*Math→0 (in calculator)

Approx: $S_5 = \sum_{n=1}^{5} a_n = 3.35\overline{4}$

$3.354 - 0.111 \le S \le 3.354 + 0.111$

$\boxed{3.24\overline{3} \le S \le 3.46\overline{5}}$

$|a_6| = \left|\frac{4}{6^2}\right| = \frac{1}{9} = 0.\overline{11}$ is the boundary for how far off our approximation is.

4. Let $f(x) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n+1)!} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots$. Show that $1 - \frac{1}{3!}$ approximates $f(1)$ with an error less than 0.01.

$1 - \frac{1}{3!}$ ↑ S_n $f(1)$ ↑ S

$|S - S_n| \le |a_{n+1}|$

$\left|S - \left(1 - \frac{1}{3!}\right)\right| \le \frac{1^4}{5!} \le \frac{1}{120} \approx 0.0083\overline{3} \le 0.01$ ✓

$\frac{1}{120} \le \frac{1}{100}$ ✓

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Alternating Series Error Bound

Calculus

Practice

A calculator may be used on all problems in this practice. For 1-2, approximate an interval of the sum of the alternating series using the Alternating Series Error Bound for the first 6 terms.

*select Math→0

1. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{3^n}$

$\sum_{x=1}^{6} a_n \approx 0.185185$

$|a_7| = \frac{7}{3^7} \approx 0.0032$

Sum is 0.185185 ± 0.0032

$\boxed{0.1819 \le S \le 0.188}$

2. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 4}{\ln(n+2)}$

$\sum_{x=1}^{6} a_n \approx 1.14046$

$|a_7| = \frac{4}{\ln(9)} \approx 1.8204$

$S = 1.14046 \pm 1.8204$

$\boxed{-0.68 \le S \le 2.9609}$

3. Determine the number of terms needed to approximate the sum of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ with an error less than 10^{-3}.

$|a_{n+1}| < 10^{-3}$

$\frac{1}{(n+1)^2} < \frac{1}{1000}$

$(n+1)^2 > 1000$

$n+1 > \sqrt{1000}$

$n+1 > 31.6227$

$n > 30.6227$

$\boxed{n = 31 \text{ or more terms}}$

4. The series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$ converges to S. Using the alternating series bound, what is the least number of terms that must be summed to guarantee a partial sum that is within 0.05 of S?

$|a_{n+1}| < 0.05$

$\frac{1}{\sqrt{n+1}} < 0.05$

$\sqrt{n+1} > \frac{1}{0.05}$

$\sqrt{n+1} > 20$

$n+1 > 400$

$n > 399$

(A) 20 (B) 55 (C) 399 $\boxed{\text{(D) } 400}$

5. If the infinite series $S = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{4}{n}$ is approximated by $P_k = \sum_{n=1}^{k} (-1)^{n+1} \frac{4}{n}$, what is the least value of k for which the alternating series error bound guarantees that $|S - P_k| < \frac{7}{100}$?

error less than $\frac{7}{100}$

$\frac{4}{k+1} < \frac{7}{100}$

$7(k+1) > 400$

$k+1 > \frac{400}{7}$

$k > \frac{400}{7} - 1$

$k > 56.1428$

(A) 55 (B) 56 $\boxed{\text{(C) } 57}$ (D) 60

6. If the series $S = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^3}$ is approximated by the partial sum $S_k = \sum_{n=1}^{k} (-1)^{n+1} \frac{1}{n^3}$, what is the least value of k for which the alternating series error bound guarantees that $|S - S_k| \le \frac{7}{10000}$?

error less than $\frac{7}{10000}$

$\frac{1}{(k+1)^3} < \frac{7}{10000}$

$7(k+1)^3 > 10000$

$(k+1)^3 > \frac{10000}{7}$

$k+1 > \sqrt[3]{\frac{10000}{7}}$

$k > 10.262$

(A) 10 $\boxed{\text{(B) } 11}$ (C) 12 (D) 13

7. The series $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ converges by the alternating series test. If $S_n = \sum_{k=1}^{n} (-1)^{k+1} a_k$ is the nth partial sum of the series, which of the following statements must be true?

(A) $\lim_{n\to\infty} S_n = 0$ (B) $\lim_{n\to\infty} a_n = S$ (C) $|S - S_{20}| \le a_{26}$ $\boxed{\text{(D) } |S - S_{25}| \le a_{26}}$

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8. If the series $\sum_{n=1}^{\infty}(-1)^n\frac{1}{5n+1}$ is approximated by the partials sum with 15 terms, what is the alternating series error bound?

$$|a_{n+1}| = \frac{1}{5(n+1)+1} \rightarrow \frac{1}{5(16)+1} = \boxed{\frac{1}{81}}$$

(A) $\frac{1}{15}$ (B) $\frac{1}{16}$ (C) $\frac{1}{76}$ $\boxed{\text{(D) } \frac{1}{81}}$

---

9. The function f is defined by the power series $f(x)=\sum_{n=0}^{\infty}\frac{(-1)^n x}{(2n+1)!}$ for all real numbers x. Show that $1-\frac{1}{3!}+\frac{1}{5!}$ approximates $f(1)$ with an error less than $\frac{1}{4000}$.

$$|S-S_n| < |a_{n+1}|$$

$$\left|f(1)-\left[1-\tfrac{1}{3!}+\tfrac{1}{5!}\right]\right| < \frac{1}{7!}$$

next term in the series
(error bound for partial Sum $(1-\frac{1}{3!}+\frac{1}{5!})$)

$\frac{1}{7!} = \frac{1}{5040}$ which is less than $\frac{1}{4000}$

[redacted] Alternating Series Error Bound

Test Prep

10. **Calculator active!** Let $f(x)=\sum_{n=1}^{\infty}\frac{x^n n^n}{n!}$ for all x for which the series converges.

a. Use the first three terms of the series to approximate $f\left(-\frac{1}{3}\right)$.

$$f(-1/3) \approx \sum_{n=1}^{3}\frac{(-1/3)^n n^n}{n!} = -0.2778$$

b. How far off is this estimate from the value of $f\left(-\frac{1}{3}\right)$? Justify your answer.

$$a_4 = \frac{(-1/3)^4 \cdot 4^4}{4!} = 0.13168$$

The estimate of -0.2778 is off by at most 0.13168 or less.

a_4 is the error bound for the partial Sum S_3

(the next unused term)

11. If the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2}$ is approximated with the series $\sum_{n=1}^{7}(-1)^{n+1}\frac{1}{n^2}$, what is the error bound?

Error Bound for partial Sum S_7 is $|a_8|$ ← 8^{th} term

$$|a_8| = \frac{1}{8^2} = \boxed{\frac{1}{64}}$$

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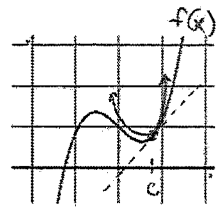
Key

BC Calculus – 10.10a Notes - Finding Taylor Polynomial Approximations of Functions

Taylor polynomials are created to help us approximate other functions. Why would we do this? Because polynomials are easy to work with in calculus (i.e., taking a derivative or integral).

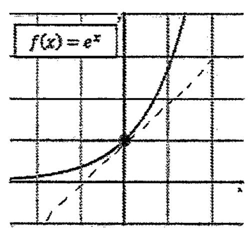
A **Maclaurin polynomial** is a special type of Taylor polynomial.



To start, we choose an x-value to center our polynomial approximation. Let's call that $x = c$. Our approximation will have the same y-value as the original function at $x = c$.

$f(c) = p(c)$

We expand the approximation about $x = c$. Another way of saying this: "**the functions are centered at $x = c$.**"



Explore with an example: $f(x) = e^x$. Let $c = 0$.

We know $f(0) = 1$. We need $p(0) = 1$

We want to make the graphs have a similar shape at the point $x = c$. They should have the same slope. That leads us to

$f'(c) = p'(c)$

In this example, that means $f'(0) = p'(0)$

The polynomial approximation will look like this:

$y - y_1 = m(x - x_1)$

$p(x) - f(c) = f'(c)(x - c)$ ← point-slope form

Or rewritten:

$p(x) = f(c) + f'(c)(x-c)$

$p(x) = f(0) + f'(0)(x-0)$

$p(x) = 1 + 1x$

$\boxed{p(x) = 1 + x}$

$* f'(x) = e^x$

$f'(0) = e^0 = 1$

This is called a $\boxed{\text{first order}}$ approximation. It works for a small interval around our point of center.

To improve the approximation, make the second derivatives agree at $x = c$.

We want $f(c) = p(c), f'(c) = p'(c), f''(c) = p''(c)$.
For our example this is $f(0) = p(0), f'(0) = p'(0), f''(0) = p''(0)$

If we worked through a similar process, we'd end up with the following:

Second-order approximation: $p(x) = 1 + x + \frac{1}{2}x^2$

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If we are centered at $x = 0$, then we have the following pattern:

$$p_n(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n$$

N[th] Taylor Polynomial
If $f(x)$ is a differentiable function, then an approximation of f centered about $x = c$ can be modeled by

$$p_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)(x-c)^2}{2!} + \frac{f'''(c)(x-c)^3}{3!} + \ldots + \ldots \boxed{\frac{f^n(c)(x-c)^n}{n!}}$$

where n is the order of the approximation.

Maclaurin Polynomial
A Maclaurin polynomial is a Taylor polynomial centered about $x = 0$. It can be modeled by

$$p_n(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \ldots + \boxed{\frac{f^n(0)x^n}{n!}}$$

where n is the order of the approximation.

centered at $x=0$

1. Find the third-degree Maclaurin polynomial for $f(x) = e^{2x}$

$f(x) = e^{2x}$	$f(0) = 1$
$f'(x) = 2e^{2x}$	$f'(0) = 2$
$f''(x) = 2e^{2x} \cdot 2$	$f''(0) = 4$
$f'''(x) = 8e^{2x}$	$f'''(0) = 8$

$P_3(x) = f(0) + f'(0) \cdot x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!}$

$P_3(x) = 1 + 2x + \frac{4x^2}{2!} + \frac{8x^3}{3!}$

$\boxed{P_3(x) = 1 + 2x + 2x^2 + \frac{4}{3}x^3}$

Evaluate at $f(0.2)$ and $p_3(0.2)$

$f(0.2) = e^{2(0.2)} \approx 1.4918$

$P_3(0.2) \approx 1.490\overline{6}$ ← this is a Maclaurin polynomial approximation of the actual y-value for $f(0.2)$

2. Find a fourth-degree Taylor Polynomial for $f(x) = \ln x$ centered at $x = 1$.

$f(x) = \ln x$	$f(1) = 0$
$f'(x) = \frac{1}{x} = x^{-1}$	$f'(1) = 1$
$f''(x) = -1x^{-2}$	$f''(1) = -1$
$f'''(x) = +2x^{-3}$	$f^3(1) = 2$
$f^4(x) = -6x^{-4}$	$f^4(1) = -6$

$P_4(x) = f(1) + f'(1)(x-1) + \frac{f''(1)(x-1)^2}{2!} + \frac{f'''(1)(x-1)^3}{3!} + \frac{f^4(1)(x-1)^4}{4!}$

$P_4(x) = 0 + 1(x-1) + \frac{-1(x-1)^2}{2!} + \frac{2(x-1)^3}{3!} + \frac{-6(x-1)^4}{4!}$

$P_4(x) = x - 1 - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4$

Evaluate at $f(1.1)$ and $p_4(1.1)$

$f(1.1) = 0.0953101798$

$P_4(1.1) = 0.0953083333$

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Coefficients of a Taylor Polynomial

The coefficient of the nth degree term in a Taylor polynomial for a function f cenetered at $x = c$ is

$$\frac{f^n(c)}{n!}$$

3. Let f be a function with third derivative $f'''(x) = (8x+2)^{\frac{3}{2}}$. What is the coefficient of $(x-2)^4$ in the fourth-degree Taylor Polynomial for f about $x = 2$.

$$\frac{f^4(2)}{4!}$$

$$f^4(x) = \tfrac{3}{2}(8x+2)^{1/2}(8)$$

$$f^4(2) = \tfrac{3}{2}(18)^{1/2}\cdot 8$$

$$f^4(2) = 12\cdot\sqrt{9\cdot 2}$$

$$f^4(2) = 36\sqrt{2}$$

$$\frac{36\sqrt{2}}{4!} \rightarrow \frac{36\sqrt{2}}{4\cdot 3\cdot 2\cdot 1}$$

$$\boxed{\frac{3\sqrt{2}}{2}}$$

Practice Problems

1. Find the fourth-degree Maclaurin Polynomial for e^{4x}.

$c = 0$

$$P_4(x) = f(0) + f'(0)(x) + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^4(0)}{4!}x^4$$

$f(x) = e^{4x}$ $\quad f(0) = e^0 = 1$

$f'(x) = 4e^{4x}$ $\quad f'(0) = 4$

$f''(x) = 4^2e^{4x}$ $\quad f''(0) = 4^2$

$f'''(x) = 4^3e^{4x}$ $\quad f'''(0) = 4^3$

$f^4(x) = 4^4e^{4x}$ $\quad f^4(0) = 4^4$

$$P_4(x) = 1 + 4x + \frac{4^2}{2!}x^2 + \frac{4^3}{3!}x^3 + \frac{4^4}{4!}x^4$$

$$\boxed{P_4(x) = 1 + 4x + 8x^2 + \frac{32}{3}x^3 + \frac{32}{3}x^4}$$

2. Find the fifth-degree Maclaurin Polynomial for the function $f(x) = \sin x$.

$c = 0$

$$P_5(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^4(0)}{4!}x^4 + \frac{f^5(0)}{5!}x^5$$

$f = \sin(x)$ $\quad f(0) = 0$

$f' = \cos x$ $\quad f'(0) = 1$

$f'' = -\sin x$ $\quad f''(0) = 0$

$f''' = -\cos x$ $\quad f'''(0) = -1$

$f^4 = \sin x$ $\quad f^4(0) = 0$

$f^5 = \cos x$ $\quad f^5(0) = 1$

$$0 + 1x + \frac{0}{2!}x^2 - \frac{1}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5$$

$$\boxed{P_5(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5}$$

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$$* \; P_n(x) = \frac{f^n(c)}{n!}(x-c)^n$$

3. Find the third-degree Taylor Polynomial for $f(x) = \ln(2x)$ about $x = 1$. $c=1$

$$P_3(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3$$

$f = \ln(2x)$ $\quad f(1) = \ln 2$

$f' = \frac{2}{2x} \to \frac{1}{x} = x^{-1}$ $\quad f'(1) = 1$

$f'' = -1x^{-2}$ $\quad f''(1) = -1$

$f''' = 2x^{-3}$ $\quad f'''(1) = 2$

$P_3(x) = \ln 2 + 1(x-1) + \frac{-1}{2}(x-1)^2 + \frac{2}{3!}(x-1)^3$

$$\boxed{P_3(x) = \ln 2 + (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3}$$

4. Find the third-degree Taylor Polynomial about $x = 0$ for $\ln(1-x)$. $c=0$

$$P_3(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3$$

$f = \ln(1-x)$ $\quad f(0) = \ln 1 = 0$

$f' = \frac{-1}{1-x} = -(1-x)^{-1}$ $\quad f'(0) = -1$

$f'' = 1(1-x)^{-2}(-1)$ $\quad f''(0) = -1$

$f''' = 2(1-x)^{-3}(-1)$ $\quad f'''(0) = -2$

$P_3(x) = 0 + -1(x-0) - \frac{1}{2!}(x-0)^2 - \frac{2}{3!}(x-0)^3$

$P_3(x) = -1(x-0) - \frac{1}{2}(x-0)^2 - \frac{1}{3}(x-0)^3$

$$\boxed{P_3(x) = -1x - \frac{1}{2}x^2 - \frac{1}{3}x^3}$$

5. Find the third-degree Taylor Polynomial about $x = 4$ of $\sqrt{x}$. $c=4$

$$P_3(x) = f(4) + f'(4)(x-4) + \frac{f''(4)(x-4)^2}{2!} + \frac{f'''(4)}{3!}(x-4)^3$$

$f = x^{1/2}$ $\quad f(4) = \sqrt{4} = 2$

$f' = \frac{1}{2}x^{-1/2}$ $\quad f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$

$f'' = -\frac{1}{4}x^{-3/2}$ $\quad f''(4) = \frac{-1}{4(4)^{3/2}} = \frac{-1}{4(8)} = -\frac{1}{32}$

$f''' = \frac{3}{8}x^{-5/2} \to f'''(4) = \frac{3}{8(4)^{5/2}} \to \frac{3}{8(2)^5} \to \frac{3}{256}$

$P_3(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{32}\cdot\frac{1}{2}(x-4)^2 + \frac{3}{256}\cdot\frac{1}{3!}(x-4)^3$

$$\boxed{P_3(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3}$$

6. The function f has derivatives of all orders for all real numbers with $f(1) = -1$, $f'(1) = 4$, $f''(1) = 6$, and $f'''(1) = 12$. Using the third-degree polynomial for f about $x = 1$, what is the approximation of $f(1.1)$? $c=1$

$$f(x) = f(1) + f'(1)(x-1) + \frac{f''(1)(x-1)^2}{2!} + \frac{f'''(1)(x-1)^3}{3!}$$

$f(x) = -1 + 4(x-1) + \frac{6}{2!}(x-1)^2 + \frac{12}{3!}(x-1)^3$

$f(x) = -1 + 4(x-1) + 3(x-1)^2 + 2(x-1)^3$

$f(1.1) = -1 + 4(0.1) + 3(0.1)^2 + 2(0.1)^3$

$$\boxed{f(1.1) = -0.568}$$

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7. A function f has a Maclaurin series given by $3 + 4x + 2x^2 + \frac{1}{3}x^3 + \cdots$, and the series converges to $f(x)$ for all real numbers x. If g is the function defined by $g(x) = e^{f(x)}$, what is the coefficient of x in the Maclaurin series for g? $(c=0)$

$f(0)=3$ $\quad f'(0)=4$

$P_n(x) = g(0) + \underline{g'(0)}(x-0) + \frac{g''(0)}{2}(x-0)^2$

$g'(x) = e^{f(x)} \cdot f'(x)$

$g'(0) = e^{f(0)} \cdot f'(0) \to e^3 \cdot 4 \to \boxed{4e^3}$

8. Let f be a function with third derivative $f'''(x) = (7x+2)^{\frac{3}{2}}$. What is the coefficient of $(x-2)^4$ in the fourth-degree Taylor Polynomial for f about $x = 2$?

$f^4(x) = \frac{3}{2}(7x+2)^{1/2}(7)$

$f^4(2) = \frac{3}{2}(16)^{1/2}(7)$

$= \frac{3}{2}(4)(7) = 42$

$\frac{f^4(2)}{4!}(x-2)^4$

$\frac{42}{4!} \to \frac{42}{24} = \boxed{\frac{7}{4}}$

9. Let $P(x) = 4x^2 - 6x^3 + 8x^4 + 4x^5$ be the fifth-degree Taylor Polynomial for the function f about $x = 0$. What is the value of $f'''(0)$? $c=0$

$\frac{f'''(0)}{3!}(x-0)^3$ $\quad \frac{f'''(0)}{3!} = -6$ $\quad f'''(0) = -6 \cdot 3!$

$$\boxed{f'''(0) = -36}$$

10. Let P be the second-degree Taylor Polynomial for $f(x) = e^{-3x}$ about $x = 3$. What is the slope of the line tangent to the graph of P at $x = 3$?

at $x=3$, $P'(3) = f'(3)$

$f'(x) = e^{-3x}(-3)$

$f'(3) = e^{-3(3)} \cdot -3$

$$\boxed{f'(3) = -3e^{-9} = \frac{-3}{e^9}}$$

11. Let f be a function with $f(4) = 2$, $f'(4) = -1$, $f''(4) = 6$, and $f'''(4) = 12$. What is the third-degree Taylor Polynomial for f about $x = 4$? $c=4$

$$P_3(x) = f(4) + f'(4)(x-4) + \frac{f''(4)}{2!}(x-4)^2 + \frac{f'''(4)}{3!}(x-4)^3$$

$P_3(x) = 2 - 1(x-4) + \frac{6}{2}(x-4)^2 + \frac{12}{3!}(x-4)^3$

$$\boxed{P_3(x) = 2 - 1(x-4) + 3(x-4) + 2(x-4)^3}$$

12

12. Let f be a function that has derivates of all orders for all real numbers. Assume $f(1) = 3, f'(1) = -2, f''(1) = 2$, and $f'''(1) = 4$. Use a second-degree Taylor Polynomial to approximate $f(0.7)$.

$P_2(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2$

$P_2(x) = 3 - 2(x-1) + \frac{2}{2!}(x-1)^2$

$f(0.7) \approx P_2(0.7) =$

$3 - 2(-0.3) + (-0.3)^2 = \boxed{3.69}$

13. The function f has derivatives of all orders for all real numbers with $f(0) = 4, f'(0) = -3, f''(0) = 3$, and $f'''(0) = 2$. Let g be the function given by $g(x) = \int_0^x f(t)\,dt$. Find the third-degree Taylor Polynomial for g about $x = 0$. $c = 0$

$P_3(x) = g(0) + g'(0)(x-0) + \frac{g''(0)}{2!}(x-0)^2 + \frac{g'''(0)}{3!}(x-0)^3$

$g(x) = \int_0^x f(t)dt \qquad g(0) = 0$

$g'(x) = f(x) \qquad g'(0) = f(0) = 4$

$g''(x) = f'(x) \qquad g''(0) = f'(0) = -3$

$g'''(x) = f''(x) \qquad g'''(0) = f''(0) = 3$

$P_3(x) = 0 + 4x - \frac{3}{2}x^2 + \frac{3}{3!}x^3$

$\boxed{P_3(x) = 4x - \frac{3}{2}x^2 + \frac{1}{2}x^3}$

14.

x	$f(x)$	$f'(x)$	$f''(x)$	$f'''(x)$
-4	1	-2	-4	2

Selected values for $f(x)$ and its first three derivatives are shown in the table above. What is the approximation for the value of $f(-3)$ about $x = -4$ obtained using the third-degree Taylor Polynomial for f.

$P_3(x) = f(-4) + f'(-4)(x+4) + \frac{f''(-4)}{2!}(x+4)^2 + \frac{f'''(-4)}{3!}(x+4)^3$

$P_3(x) = 1 - 2(x+4) - \frac{4}{2}(x+4)^2 + \frac{2}{3!}(x+4)^3$

$f(-3) \approx P_3(-3) = 1 - 2(1) - 2(1)^2 + \frac{2}{6}(1)^3 \boxed{= \frac{-8}{3} \approx -2.667}$

Taylor Polynomial Approximations

Test Prep

15. Which of the following polynomial approximations is the best for $\cos(3x)$ near $x = 0$?

$f(x) = \cos(3x) \qquad f(0) = \cos 0 = 1$

$f'(x) = -\sin(3x)\cdot 3 \qquad f'(0) = 0$

$f''(x) = -3\cos(3x)\cdot 3 \qquad f''(0) = -9$

$P(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2$

$P(x) = 1 + 0(x) - \frac{9}{2}x^2$

(A) $1 + \frac{3}{2}x$ (B) $\boxed{1 - \frac{9}{2}x^2}$ (C) $1 + x$ (D) $1 - \frac{9}{2}x + x^2$

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16. Consider the logistic differential equation $\frac{dy}{dt} = \frac{y}{6}(4 - y)$. Let $y = f(t)$ be the particular solution to the differential equation with $f(0) = 6$.

a. Write the second-degree Taylor Polynomial for f about $t = 0$.

$P_2(t) = f(0) + f'(0)(t-0) + \frac{f''(0)}{2!}(t-0)^2$

$f(0) = 6$

$f'(0) = \frac{6}{6}(4-6) = -2$

$f''(x) = \frac{1}{6}\left(\frac{dy}{dt}\right)(4-y) + \frac{y}{6}(-1)\left(\frac{dy}{dt}\right)$

$f''(0) = \frac{1}{6}(-2)(4-6) + \frac{6}{6}(-1)(-2)$

$= \frac{2}{3} + 2 = 8/3$

$P_2(t) = 6 - 2(t-0) + \frac{8}{3}\cdot\frac{1}{2}(t)^2$

$\boxed{P_2(t) = 6 - 2t + \frac{4}{3}t^2}$

b. Use the results from part a to approximate $f(1)$.

$f(1) \approx 6 - 2(1) + \frac{4}{3}(1) = \boxed{\frac{16}{3}}$

t (seconds)	0	4	10
$x'(t)$ meters per second	5.0	5.8	4.0

17. The position of a particle moving along a straight line is modeled by $x(t)$. Selected values of $x'(t)$ are shown in the table above and the position of the particle at time $t = 10$ is $x(10) = 8$.

a. Approximate $x''(8)$ using the average rate of change of $x'(t)$ over the interval $4 \le t \le 10$. Show computations that lead to your answer.

$\frac{x'(10) - x'(4)}{10-4} = \frac{4.0 - 5.8}{10-4} = \frac{-1.8}{6} = \boxed{-0.3}$

b. Using correct units, explain the meaning of $x''(8)$ in the context of the problem.

$x''(8)$ is the rate at which velocity is changing (which is acceleration) at $t = 8$ with units of meters/sec^2

c. Use a right Riemann sum with two subintervals to approximate $\int_0^{10} |x'(t)|\,dt$. Absolute value

$\int_0^{10} |x'(t)|dt \approx 4(5.8) + 6(4) = \boxed{47.2}$

d. Let s be a function such that the third derivative of s with respect to t is $(t-3)^7$. Write the fourth-degree term of the fourth-degree Taylor Polynomial for s about $t = 1$.

$f^4(x) = 7(t-3)^6$

$f^4(1) = 7(1-3)^6 = 7(-2)^6 = 448$

4th term is $\frac{f^4(1)}{4!}(t-1)^4 \rightarrow \frac{448}{4!}(t-1)^4$

$\boxed{\frac{56}{3}(t-1)^4}$

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Key

BC Calculus – 10.8a Notes – Radius and Interval of Convergence of Power Series

Power Series

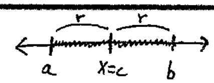
$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1x^1 + a_2x^2 + a_3x^3 + \cdots + a_nx^n$$ centered at $x=0$

$$\sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c)^1 + a_2(x-c)^2 + a_3(x-c)^3 + \cdots + a_n(x-c)^n$$

The domain of a power series is the set of all x-values for which the power series converges.

Note! The center is always part of the domain.

Three ways a power series may converge:



1. converges to an interval
   a. The radius is the distance from the center to the edge of interval
2. Converges to all real numbers
3. Converges to the center $(x=c)$ only

The Interval of Convergence is the set of values for convergence. We use the Ratio Test to find the interval of convergence.

Ratio Test for Interval of Convergence

If you have a power series $\sum_{n=1}^{\infty} a_n$, find $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$.

- $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$, then the series converges on an interval
- $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0$, then the series converges for all values of x
- $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series converges only to center $(x=c)$ (diverges everywhere else!)

Find the radius and interval of convergence.

1. $\sum_{n=1}^{\infty} \frac{n}{3^n}(x-5)^n$ $\lim_{n\to\infty}\left|\frac{n+1}{3^{n+1}}(x-5)^{n+1} \cdot \frac{3^n}{n(x-5)^n}\right|$

$\lim_{n\to\infty}\left|\frac{n+1}{n}\cdot\frac{(x-5)}{3}\right| < 1 \rightarrow \left|\frac{x-5}{3}\right| < 1$

$|x-c|<r$

$|x-5|<3$

$-1 < \frac{x-5}{3} < 1$

$-3 < x-5 < 3$

$2 < x < 8$

*check convergence at endpoints!!

$\sum_{n=1}^{\infty}\frac{n}{3^n}(-3)^n = (-1)^n n$ diverges (at $x=2$)

test $x=8$: $\sum_{n=1}^{\infty}\frac{n}{3^n}(3)^n$ diverges (n^{th} term test)

Interval of convergence

IOC: $2<x<8$

Radius: $r=3$

2. $\sum_{n=0}^{\infty} 3(x-2)^n$ $\lim_{n\to\infty}\left|3(x-2)^{n+1}\cdot\frac{1}{3(x-2)^n}\right|$

$\lim_{n\to\infty}|x-2|<1$

$|x-c|<r$

$-1<x-2<1$

$1<x<3$

*check endpts:

at $x=1$: $\sum 3(1-2)^n$, $\sum_{n=0}^{\infty} 3(-1)^n$ diverges

at $x=3$: $\sum 3(3-2)^n$, $\sum 3(1)^n$ diverges

IOC: $1<x<3$

Radius: $r=1$

3. $\sum_{n=1}^{\infty}\frac{(x+2)^{n+1}}{n^3}$ $\lim_{n\to\infty}\left|\frac{(x+2)^{n+2}}{(n+1)^3}\cdot\frac{n^3}{(x+2)^{n+1}}\right|$

$\lim_{n\to\infty}\left|(x+2)\cdot\frac{n^3}{(n+1)^3}\right| < 1$

$|x+2|<1$

$-1<x+2<1$

$-3<x<-1$

test $x=-3$: $\sum\frac{(-1)^{n+1}}{n^3}$ converges

test $x=-1$: $\sum\frac{(1)^{n+1}}{n^3}$ converges

$\boxed{\text{IOC: } -3\le x\le -1 \quad \text{Radius: } r=1}$

4. $\sum_{n=0}^{\infty}\frac{(2n)!x^{2n}}{n!}$ $\lim_{n\to\infty}\left|\frac{(2n+2)!x^{2n+2}}{(n+1)!}\cdot\frac{n!}{(2n)!x^{2n}}\right|$

$\lim_{n\to\infty}\left|\frac{(2n+2)(2n+1)\cancel{(2n)!}\cancel{x^{2n}}\cdot x^2\cancel{n!}}{(n+1)\cancel{n!}\cancel{(2n)!}\cancel{x^{2n}}}\right| = \infty$

$\boxed{\text{Converges only at } x=0 \quad \text{Radius} = 0}$

since there are no range of values

5. $\sum_{n=0}^{\infty}\frac{x^{3n}}{n!}$

$\lim_{n\to\infty}\left|\frac{x^{3n+3}}{(n+1)!}\cdot\frac{n!}{x^{3n}}\right|$

$\lim_{n\to\infty}\left|\frac{x^3\cdot n!}{(n+1)\cdot n!}\right|$

$\lim_{n\to\infty}\left|\frac{1}{n!}\cdot x^3\right| = 0$

$\boxed{\text{Converges for all values of } x \quad \text{IOC: } (-\infty,\infty) \quad \text{radius} = \infty}$

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Radius and Interval of Convergence of Power Series

Calculus

Practice

Find the interval of convergence for each power series.

1. $\sum_{n=0}^{\infty} \frac{(x-1)^n}{4^n}$ $\lim_{n\to\infty}\left|\frac{(x-1)^{n+1}}{4^{n+1}}\cdot\frac{4^n}{(x-1)^n}\right|$

$\lim_{n\to\infty}\left|\frac{x-1}{4}\right|<1 \to -1<\frac{x-1}{4}<1$

$-4<x-1<4$

$-3<x<5$ *check endpoints to see if they converge or diverge

test $x=-3$

$\sum\frac{(-4)^n}{4^n} \to \sum(-1)^n$ diverges

$\boxed{-3<x<5}$

test $x=5$

$\sum\frac{4^n}{4^n}\to(1)^n$ diverges

2. $\sum_{n=0}^{\infty} \frac{(x+2)^n}{3^n}$ $\lim_{n\to\infty}\left|\frac{(x+2)^{n+1}}{3^{n+1}}\cdot\frac{3^n}{(x+2)^n}\right|$

$\lim_{n\to\infty}\left|\frac{(x+2)}{3}\right|<1 \to -1<\frac{x+2}{3}<1$

$-3<x+2<3$

$-5<x<1$

test $x=-5$

$\sum\frac{(-3)^n}{3^n}\to(-1)^n$ diverges

test $x=1$

$\sum\frac{(3)^n}{3^n}\to(1)^n$ diverges

$\boxed{-5<x<1}$

3. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-2)^n}{n2^n}$ $\lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{(n+1)2^{n+1}}\cdot\frac{n\cdot 2^n}{(x-2)^n}\right|$

$\lim_{n\to\infty}\left|\frac{(x-2)}{2}\cdot\frac{n}{n+1}\right|<1$ $\quad -1<\frac{x-2}{2}<1$

$-2<x-2<2$

$0<x<4$

test $x=0$

$\sum\frac{(-1)^{n+1}(-2)^n}{n(2^n)}$

$\sum\frac{(-1)^{n+1}(-1)^n}{n}$ diverges

test $x=4$

$\sum\frac{(-1)^{n+1}\cancel{(2)^n}}{n\cdot\cancel{2^n}}$ converges by AST

$\boxed{0<x\le 4}$

4. $\sum_{n=0}^{\infty}(2n)!\left(\frac{x}{3}\right)^n$ $\lim_{n\to\infty}\left|\frac{[2(n+1)]!\left(\frac{x}{3}\right)^{n+1}}{1}\cdot\frac{}{(2n)!\left(\frac{x}{3}\right)^n}\right|$

$\lim_{n\to\infty}\left|\frac{(2n+2)!\left(\frac{x}{3}\right)}{(2n)!}\right| \to \lim_{n\to\infty}\left|\frac{(2n+2)(2n+1)\cancel{(2n)!}}{\cancel{(2n)!}}\left(\frac{x}{3}\right)\right|$

$=\boxed{\infty}$

Converges only at center $x=0$

Find the radius of convergence for each series.

5. $\sum_{n=1}^{\infty}\frac{(4x)^n}{n^2}$ $\lim_{n\to\infty}\left|\frac{(4x)^{n+1}}{(n+1)^2}\cdot\frac{n^2}{(4x)^n}\right|$

$\lim_{n\to\infty}\left|\frac{(4x)^n(4x)}{(4x)^n}\cdot\frac{n^2}{(n+1)^2}\right| = \lim_{n\to\infty}|4x|<1$

$|4x|<1 \to -1<4x<1$

$-\frac{1}{4}<x<\frac{1}{4} \to |x|<1/4$

$\boxed{\text{Radius} = 1/4}$

6. $\sum_{n=0}^{\infty}\frac{(x-4)^{n+1}}{2\cdot 3^{n+1}}$ $\lim_{n\to\infty}\left|\frac{(x-4)^{n+2}}{2\cdot 3^{n+2}}\cdot\frac{2\cdot 3^{n+1}}{(x-4)^{n+1}}\right|$

$\lim_{n\to\infty}\left|\frac{(x-4)}{3}\right|<1$

$|x-4|<3$

* $|x-c|<r$

$\boxed{\text{radius} = 3}$

7. $\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}$ $\lim_{n\to\infty}\left|\frac{x^{2(n+1)}}{[2(n+1)]!}\cdot\frac{(2n)!}{x^{2n}}\right|$

$\lim_{n\to\infty}\left|\frac{x^2\cdot(2n)!}{(2n+2)!}\right| \to \lim_{n\to\infty}\left|\frac{x^2\cdot(2n)!}{(2n+2)(2n+1)(2n)!}\right|$

$\lim_{n\to\infty}\left|\frac{x^2}{(2n+2)(2n+1)}\right| = 0$ (always less than 1)

$\boxed{\text{Radius} = \infty}$

8. $\sum_{n=0}^{\infty}\frac{(2n)!\,x^{2n}}{n!}$ $\lim_{n\to\infty}\left|\frac{(2(n+1))!\,x^{2(n+1)}}{(n+1)!}\cdot\frac{n!}{(2n)!\,x^{2n}}\right|$

$\lim_{n\to\infty}\left|\frac{(2n+2)!\,x^{2n+2}\cdot n!}{(n+1)!\,(2n)!\,x^{2n}}\right|$

$\lim_{n\to\infty}\left|\frac{(2n+2)(2n+1)\cancel{(2n)!}\,\cancel{x^{2n}}\,x^2\,\cancel{n!}}{(n+1)\cancel{n!}\,\cancel{(2n)!}\,\cancel{x^{2n}}}\right| = \infty$

$\boxed{\text{Radius} = 0}$ *converges only at center ($c=0$)

What are all values of x for which each series converges?

9. $\sum_{n=1}^{\infty}\left(\frac{4}{x^2+1}\right)^n$

$\lim_{n\to\infty}\left|\left(\frac{4}{x^2+1}\right)^{n+1}\cdot\left(\frac{x^2+1}{4}\right)^n\right|$

$\lim_{n\to\infty}\left|\frac{4}{x^2+1}\right|<1$

$4<(x^2+1)$

$x^2+1>4$

$x^2>3$

$x>\sqrt{3}$ or $x<-\sqrt{3}$

*test endpoints:

at $x=\sqrt{3}$:

$\sum\left(\frac{4}{\sqrt{3}^2+1}\right)^n$ diverges

at $x=-\sqrt{3}$

$\sum\left(\frac{4}{\sqrt{3}^2+1}\right)^n$ diverges

$\boxed{\text{converges when } x>\sqrt{3} \text{ or } x<-\sqrt{3}}$

10. $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left(x+\frac{3}{2}\right)^n$ $\lim_{n\to\infty}\left|\frac{(x+3/2)^{n+1}}{n+1}\cdot\frac{n}{(x+3/2)^n}\right|$

$\lim_{n\to\infty}\left|\frac{(x+3/2)}{1}\right|<1$ $\quad |x+3/2|<1$

$-1<x+3/2<1$ $\quad$ * $|x-c|<r$ $\quad r=1$

$-5/2<x<-1/2$

test $x=-5/2$:

$\sum\frac{(-1)^n}{n}\left(-\frac{5}{2}+\frac{3}{2}\right)^n$

$\sum\frac{(-1)^n}{n}(-1)^n$ diverges (p-series)

test $x=-1/2$:

$\sum\frac{(-1)^n}{n}\cdot(1)^n$ converges by AST

$\boxed{-\frac{5}{2}<x\le-1/2}$

11. $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n \cdot 3^n}$ $\lim_{n\to\infty} \left| \frac{(x-2)^{n+1}}{(n+1)\cdot 3^{n+1}} \cdot \frac{n \cdot 3^n}{(x-2)^n} \right|$

$\lim_{n\to\infty} \left| \frac{(x-2)}{3} \cdot \frac{n}{n+1} \right| < 1$ $\left|\frac{x-2}{3}\right| < 1$

$|x-2| < 3$

$-3 < x-2 < 3$

$-1 < x < 5$

test $x=-1$:

$\sum \frac{(-1-2)^n}{n \cdot 3^n}$

$\sum \frac{(-3)^n}{n \cdot 3^n} \to \frac{(-1)^n}{n}$ converges by AST

test $x=5$:

$\sum \frac{(5-2)^n}{n \cdot 3^n} \to \sum \frac{(3)^n}{n \cdot 3^n}$ diverges (p-series)

$\boxed{-1 \le x < 5}$

12. $\sum_{n=0}^{\infty} \frac{x^{5n}}{n!}$ $\lim_{n\to\infty} \left| \frac{x^{5(n+1)}}{(n+1)!} \cdot \frac{n!}{x^{5n}} \right|$

$\lim_{n\to\infty} \left| \frac{x^5 \cdot n!}{(n+1)\cdot n!} \right| \to \lim_{n\to\infty} \left| \frac{x^5}{n+1} \right| = 0$ (always less than 1)

converges for all values of x:

$\boxed{(-\infty, \infty)}$

~~10.13~~ Radius and Interval of Convergence of Power Series **Test Prep**

13. The radius of convergence for the power series $\sum_{n=1}^{\infty} \frac{(x-4)^{2n}}{n}$ is equal to 1. What is the interval of convergence?

$c = 4$, $r = 1$

$|x-c| < r$

$|x-4| < 1$

$-1 < x-4 < 1$

$3 < x < 5$

*check endpoints:

test $x=3$:

$\sum \frac{(3-4)^{2n}}{n} \to \frac{(-1)^{2n}}{n}$

diverges by p-series

test $x=5$:

$\sum \frac{(5-4)^{2n}}{n}$

$\sum \frac{(1)^{2n}}{n}$

diverges by p-series

$\boxed{3 < x < 5}$

14. If the power series $\sum_{n=0}^{\infty} a_n(x-5)^n$ converges at $x = 8$ and diverges at $x = 10$, which of the following must be true?

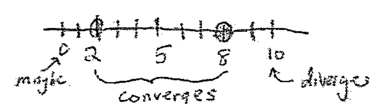
*center at $x=5$

*Radius is between 3 and 5

maybe I. The series converges at $x = 2$.

yes II. The series converges at $x = 3$.

maybe III. The series diverges at $x = 0$.



(A) I only

$\boxed{\text{(B) II only}}$

(C) I and II only

(D) II and III only

15. The coefficients of the power series $\sum_{n=0}^{\infty} a_n(x-3)^n$ satisfy $a_0 = 6$ and $a_n = \left(\frac{2n+1}{3n+1}\right) a_{n-1}$ for all $n \ge 1$. What is the radius of convergence?

$\lim_{n\to\infty} \left| \frac{b_{n+1}}{b_n} \right| = ?$

$b_n = a_n(x-3)^n$

$b_{n+1} = a_{n+1}(x-3)^{n+1}$

$a_{n+1} = \left(\frac{2(n+1)+1}{3(n+1)+1} \right) \cdot a_n$

$a_{n+1} = \frac{2n+3}{3n+4} a_n$

$\lim_{n\to\infty} \left| \frac{a_{n+1}(x-3)^{n+1}}{a_n (x-3)^n} \right|$

$\frac{a_{n+1}}{a_n} = \frac{2n+3}{3n+4}$

$\lim_{n\to\infty} \left| \frac{2n+3}{3n+4} \cdot \frac{(x-3)^{n+1}}{(x-3)^n} \right|$

$\lim_{n\to\infty} \left| \frac{2}{3} \cdot (x-3) \right| < 1$

$-1 < \frac{2}{3}(x-3) < 1$

$-\frac{3}{2} < x-3 < \frac{3}{2}$

$\frac{3}{2} < x < \frac{9}{2}$

$\left|\frac{2}{3}(x-3)\right| < 1$

$|x-3| < 3/2$

$|x-c| < r$ *

$\boxed{\text{Radius} = \frac{3}{2}}$

16. The radius of convergence for the power series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-5)^n}{n5^n}$ is 5, what is the interval of convergence?

Center is $x=5$

radius $= 5$

$|x-5| < 5$

$-5 < x-5 < 5$

$0 < x < 10$

test endpoints:

$x=0$:

$\sum \frac{(-1)^{n+1}(-5)^n}{n \cdot 5^n} \to \frac{(-1)^n(5)^n}{5^n} \cdot \frac{(-1)}{n}$ diverges

test $x=10$

$\sum \frac{(-1)^{n+1}(5)^n}{n \cdot 5^n}$ converges by AST

$\boxed{0 < x \le 10}$

(A) $-5 < x < 5$

(B) $-5 < x \le 5$

(C) $0 < x < 10$

$\boxed{\text{(D) } 0 < x \le 10}$

17. Let $a_n = \frac{1}{n \ln n}$ for $n \ge 3$ and let f be the function given by $f(x) = \frac{1}{x \ln x}$.

a. The function f is continuous, decreasing, and positive. Use the Integral Test to determine the convergence or divergence of the series $\sum_{n=3}^{\infty} a_n$.

$\int_3^{\infty} \frac{1}{x \ln x} dx$

$u = \ln x$

$\frac{du}{dx} = \frac{1}{x}$

$dx = x\,du$

$\lim_{b\to\infty} \int_3^b \frac{1}{\cancel{x} u} \cdot \cancel{x} du$

$\ln|u| \to \ln|\ln x|$

$\ln|\ln x| \Big]_3^b \to \lim_{b\to\infty} \ln|\ln b| - \ln|\ln 3|$

$\infty - \ln|\ln 3| = \infty$

$\boxed{\text{Series Diverges by Integral Test}}$

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b. Find the interval of convergence of the power series $\sum_{n=3}^{\infty} \frac{(x-2)^{n+1}}{n \ln n}$.

$$\lim_{n\to\infty} \left| \frac{(x-2)^{n+2}}{(n+1)\cdot \ln(n+1)} \cdot \frac{n\ln(n)}{(x-2)^{n+1}} \right|$$

$$\lim_{n\to\infty} \left| \frac{(x-2)}{1} \cdot \frac{n\ln(n)}{(n+1)\ln(n+1)} \right| < 1$$

$$\lim_{n\to\infty} |x-2| < 1$$

$$-1 < x-2 < 1$$

$$1 < x < 3$$

*test endpoints:

x = 1:

$\sum \frac{(-1)^{n+1}}{n\cdot \ln(n)}$ converges by AST

x = 3

$\sum \frac{(1)^{n+1}}{(n)\ln(n)}$ diverges by Integral Test

$$\boxed{1 \le x < 3}$$

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Key

BC Calculus – 10.9 Notes – Finding Taylor & Maclaurin Series for a Function

Taylor Series
If $f(x)$ has derivatives of all orders at $x = c$, then a Taylor Series may be formed that is equal to the function for many common functions.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(c)}{n!}(x-c)^n = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \ldots$$

If $c = 0$ it is a Maclaurin Series.

You need to know the following series: e^x, $\cos x$, $\sin x$, $\frac{1}{1+x}$

The Taylor series of these functions are exact when we go to ∞. They must be memorized!

Maclaurin Series for e^x	Maclaurin Series for $\sin x$
$f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \ldots$ e^x $e^0 = 1$ $e^x = 1 + 1x + \frac{1x^2}{2!} + \frac{1x^3}{3!} + \ldots + \frac{x^n}{n!}$ $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ | IOC: $(-\infty, \infty)$	$f(x) = \sin x$ $f(0) = 0$ $f'(x) = \cos x$ $f'(0) = 1$ $f''(x) = -\sin x$ $f''(0) = 0$ $f'''(x) = -\cos x$ $f'''(0) = -1$ $f^4(x) = \sin x$ $f^4(0) = 0$ $\sin x = 0 + 1x + \frac{0x^2}{2!} - \frac{x^3}{3!} + \frac{0x^4}{4!} - \frac{x^5}{5!} + \ldots \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

Memorize the following!

Function	Series (expanded)	Series Notation	Int of Conv.
$e^x =$	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$-\infty < x < \infty$
$\sin x =$ (odd)	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$	$(-\infty, \infty)$
$\cos x =$ (even)	$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$	$-\infty < x < \infty$
$\frac{1}{1+x} =$	$1 - x + x^2 - x^3 + \ldots$	$\sum_{n=0}^{\infty} (-1)^n x^n$	$-1 < x < 1$

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The function $f(x) = \frac{1}{1-x}$ is actually a geometric series.

Recall: $\sum_{n=k}^{\infty} ar^n = \frac{a_1}{1-r}$, $|r| < 1$

$$f(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} 1(x)^n \text{ or } \boxed{\sum_{n=0}^{\infty} x^n}$$

$a_1 = 1$
$r = x$

Find the Maclaurin Series for each of the following functions.

3. $\sin x^2$

*look for parent function:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

$$\sin(x^2) = x^2 - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} - \frac{(x^2)^7}{7!} + \ldots$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\sin(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n+1}}{(2n+1)!}$$

$$\boxed{\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!}}$$

4. $x^2 e^x$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!}$$

$$x^2 e^x = x^2\left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!}\right]$$

$$= x^2 + x^3 + \frac{x^4}{2!} + \frac{x^5}{3!} + \ldots + \frac{x^{n+2}}{n!}$$

$$\boxed{\sum_{n=0}^{\infty} \frac{x^{n+2}}{n!}}$$

Practice Problems:

1. What is the coefficient of x^6 in the Taylor Series about $x = 0$ for the function $f(x) = \frac{e^{2x^2}}{2}$? $\frac{1}{2}e^{2x^2}$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \frac{x^6}{6!}$$

$$e^{2x^2} = 1 + (2x^2) + \frac{(2x^2)^2}{2} + \frac{(2x^2)^3}{3!}$$

$$\frac{1}{2}e^{2x^2} = \frac{1}{2}\left[1 + 2x^2 + \frac{4x^4}{2} + \frac{8x^6}{6}\right]$$

$$= \frac{1}{2} + x^2 + x^4 + \boxed{\frac{4}{6}x^6}$$

$\boxed{\text{coefficient is } \frac{2}{3}}$

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2. If $f(x) = x \sin 3x$, what is the Taylor Series for f about $x = 0$? Write the first four non-zero terms.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

$$\sin(3x) = (3x) - \frac{(3x)^3}{3!} + \frac{(3x)^5}{5!} - \frac{(3x)^7}{7!}$$

$$x\sin(3x) = x(3x) - \frac{x \cdot 27x^3}{6} + \frac{x \cdot 3^5 x^5}{5!} - \frac{x \cdot 3^7 x^7}{7!}$$

$$\boxed{3x^2 - \frac{27x^4}{3!} + \frac{243x^6}{5!} - \frac{2187x^8}{7!}}$$

3. What is the Maclaurin Series for $\frac{1}{(1-x)^2}$? Write the first four non-zero terms.

$$\frac{1}{1-x} \cdot \frac{1}{1-x}$$

$$[1+x+x^2+x^3+x^4 \dots] \cdot [1+x+x^2+x^3]$$

$$\begin{array}{l} 1 + x + x^2 + x^3 \\ \quad + x + x^2 + x^3 + x^4 \\ \quad\quad + x^2 + x^3 + x^4 + x^5 + \dots \\ \quad\quad\quad + x^3 + x^4 + x^5 + \dots \\ + \quad\quad\quad\quad + x^4 + x^5 + \dots \end{array}$$

$$\boxed{1 + 2x + 3x^2 + 4x^3}$$

4. What is the Maclaurin Series for the function $f(x) = \frac{1}{2}(e^x + e^{-x})$? Write the first four non-zero terms.

$$\frac{1}{2}e^x = \frac{1}{2}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!}\right)$$

$$\frac{1}{2}e^{-x} = \frac{1}{2}\left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \frac{x^6}{6!}\right)$$

$+$

$$\frac{1}{2}\left[2 + 2\left(\frac{x^2}{2}\right) + 2\left(\frac{x^4}{4!}\right) + 2\left(\frac{x^6}{6!}\right)\right]$$

$$\boxed{1 + \frac{x^2}{2} + \frac{x^4}{4!} + \frac{x^6}{6!}}$$

5. Find the Maclaurin Series for the function $f(x) = \cos\sqrt{x}$. Write the first four non-zero terms.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$$

$$\cos(\sqrt{x}) = 1 - \frac{\sqrt{x}^2}{2!} + \frac{(\sqrt{x})^4}{4!} - \frac{(\sqrt{x})^6}{6!}$$

$$\boxed{= 1 - \frac{x}{2} + \frac{x^2}{4!} - \frac{x^3}{6!}}$$

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6. Find the Maclaurin Series for the function $f(x) = \sin 5x$. Write the first four non-zero terms.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

$$\sin(5x) = 5x - \frac{(5x)^3}{3!} + \frac{(5x)^5}{5!} - \frac{(5x)^7}{7!}$$

$$\boxed{\sin(5x) = 5x - \frac{125}{3!}x^3 + \frac{5^5x^5}{5!} - \frac{5^7x^7}{7!}}$$

7. What is the Taylor series expansion about $x = 0$ for the function $f(x) = \frac{\sin x}{x}$? Write the first four non-zero terms.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

$$\frac{1}{x} \cdot \sin x = \frac{1}{x}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}\right) = \boxed{1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!}}$$

8. The sum of the series $1 + \frac{3}{1!} + \frac{3^2}{2!} + \frac{3^3}{3!} + \dots + \frac{3^n}{n!}$ is

$$e^x = \sum \frac{x^n}{n!} \qquad e^3 = \sum_{n=0}^{\infty} \frac{3^n}{n!}$$

(A) $\ln 3$ (B) e^3 (C) $\cos 3$ (D) $\sin 3$

9. What is the sum of the series $1 + \ln 3 + \frac{(\ln 3)^2}{2!} + \dots + \frac{(\ln 3)^n}{n!}$?

$$e^x = \sum \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \dots \frac{x^n}{n!}$$

$$e^{(\ln 3)} = 1 + \ln 3 + \frac{(\ln 3)^2}{2!} + \dots + \frac{(\ln 3)^n}{n!}$$

$$e^{\ln 3} = \boxed{3}$$

10. What is the Taylor Series about $x = 0$ for the function $f(x) = 1 + x^2 + \cos x$? Write the first four non-zero terms.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$$

$$f(x) = 1 + x^2 + \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}\right)$$

$$f(x) = 1 + x^2 + 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!}$$

$$\boxed{f(x) = 2 + \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}}$$

11. What is the sum of the infinite series $1-\left(\frac{\pi}{2}\right)^2\left(\frac{1}{3!}\right)+\left(\frac{\pi}{2}\right)^4\left(\frac{1}{5!}\right)-\left(\frac{\pi}{2}\right)^6\left(\frac{1}{7!}\right)+\cdots+\frac{\left(\frac{\pi}{2}\right)^{2n}(-1)^n}{(2n+1)!}$?

$$\sin x=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\frac{\sin x}{x}=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}\cdot x^1}{x\cdot(2n+1)!}\to\frac{(-1)^n x^{2n}}{(2n+1)!}$$

$$\frac{\sin(\pi/2)}{\pi/2}=\frac{1}{\pi/2}=\boxed{\frac{2}{\pi}}$$

12. Find the Maclaurin Series for the function $f(x)=e^{-3x}$. Write the first four non-zero terms.

$$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}$$

$$e^{-3x}=1+(-3x)+\frac{(-3x)^2}{2!}+\frac{(-3x)^3}{3!}$$

$$e^{-3x}=1-3x+\frac{9x^2}{2}-\frac{27x^3}{6}$$

OR

$$\boxed{1-3x+\frac{9x^2}{2}-\frac{9}{2}x^3}$$

13. Find the Maclaurin Series for the function $f(x)=\frac{\sin x^2}{x}+\cos x$. Write the first four non-zero terms.

$$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}$$

$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}$$

$$\sin(x^2)=x^2-\frac{(x^2)^3}{3!}+\frac{(x^2)^5}{5!}-\frac{(x^2)^7}{7!}$$

$$\frac{1}{x}\left(\sin(x^2)\right)=\frac{1}{x}\left[x^2-\frac{x^6}{3!}+\frac{x^{10}}{5!}-\frac{x^{14}}{7!}\right]$$

$$=x-\frac{x^5}{3!}+\frac{x^9}{5!}-\frac{x^{13}}{7!}$$

$$\boxed{\frac{\sin x^2}{x}+\cos x\approx 1+x-\frac{x^2}{2!}+\frac{x^4}{4!}}$$

14. Which of the following is the Maclaurin Series for the function $f(x)=x\cos 2x$?

$$\cos x=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$$

$$x\cos(2x)=\sum_{n=0}^{\infty}\frac{(-1)^n\cdot(2x)^{2n}\cdot x}{(2n)!}$$

$$\sum_{n=0}^{\infty}\frac{(-1)^n\cdot 2^{2n}\cdot x^{2n}\cdot x^1}{(2n)!}$$

(A) $\sum_{n=0}^{\infty}\frac{(-1)^n 2x^{2n}}{(2n)!}$ (B) $\sum_{n=0}^{\infty}\frac{(-1)^n 2^{2n}x^{2n}}{(2n)!}$ $\boxed{\text{(C) } \sum_{n=0}^{\infty}\frac{(-1)^n 2^{2n}x^{2n+1}}{(2n)!}}$ (D) $\sum_{n=0}^{\infty}\frac{(-1)^n 2x^{2n+1}}{(2n)!}$

Finding Taylor or Maclaurin Series

15. The Maclaurin series $x+\frac{x^3}{3!}+\frac{x^5}{5!}+\frac{x^7}{7!}+\cdots+\frac{x^{2n+1}}{(2n+1)!}$ represents which function $f(x)$

A) $\sin x=x-\frac{x^3}{3!}+\ldots$ (incorrect)

B) $-\sin x=-\left(x-\frac{x^3}{3!}+\ldots\right)$ (incorrect)

C) $\frac{1}{2}e^x-\frac{1}{2}e^{-x}\to\frac{1}{2}\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}\right)$

$$-\frac{1}{2}\left(1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\frac{x^4}{4!}-\frac{x^5}{5!}\right)$$

$$x+\frac{1}{2}\left(\frac{2x^3}{3!}\right)+\frac{1}{2}\left(\frac{2x^5}{5!}\right)\to x+\frac{x^3}{3!}+\frac{x^5}{5!}+\ldots$$

(A) $\sin x$ (B) $-\sin x$ $\boxed{\text{(C) } \frac{1}{2}(e^x-e^{-x})}$ (D) e^x-e^{-x}

16. The function f satisfies the equation $f'(x)=f(x)+x+1$ and $f(0)=2$. The Taylor Series for f about $x=0$ converges to $f(x)$ for all x.

a. Write an equation for the line tangent to the curve of $y=f(x)$ at $x=0$.

$f'(x)=f(x)+x+1$

$f'(0)=f(0)+0+1$

$f'(0)=2+1=3$

point: $(0,2)$

slope: $m=3$

$$\boxed{y-2=3(x-0) \quad \text{OR} \quad y=3x+2}$$

b. Find $f''(0)$ and find the second-degree Taylor Polynomial for f about $x=0$.

$f''(x)=f'(x)+1$

$f''(0)=f'(0)+1$

$=3+1=4$

$$P_2(x)=f(0)+f'(0)(x-0)+\frac{f''(0)}{2!}(x-0)^2$$

$$P_2(x)=2+3(x-0)+\frac{4}{2!}(x-0)^2$$

$$\boxed{P_2(x)=2+3x+2x^2}$$

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c. Find the fourth-degree Taylor Polynomial for f about $x = 0$.

$$f'''(x) = f''(x) \rightarrow f'''(0) = f''(0) = 4$$

$$f^4(x) = f'''(x) \rightarrow f^4(0) = f'''(0) = 4$$

$$P_4(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 + \frac{f^4(0)}{4!}(x-0)^4$$

$$P_4(x) = 2 + 3(x-0) + \frac{4}{2!}(x-0)^2 + \frac{4}{3!}(x-0)^3 + \frac{4}{4!}(x-0)^4$$

$$\boxed{P_4(x) = 2 + 3x + 2x^2 + \frac{2}{3}x^3 + \frac{1}{6}x^4}$$

d. Find $f^{(n)}(0)$, the n^{th} derivative of f about $x = 0$, for $n \geq 2$. Use the Taylor Series for f about $x = 0$ and the Taylor Series for e^x about $x = 0$ to find $f(x) - 4e^x$.

$f^n(0) = 4$ for $n \geq 2$

$$f(x) = 2 + 3x + \frac{4x^2}{2!} + \frac{4}{3!}x^3 + \frac{4}{4!}x^4 + \ldots$$

$$4e^x = 4\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}\right)$$

$$= 4 + 4x + \frac{4x^2}{2!} + \frac{4x^3}{3!} + \frac{4x^4}{4!}$$

$$f(x) - 4e^x = 2 - 4 + 3x - 4x + \cancel{\frac{4x^2}{2!}} - \cancel{\frac{4x^2}{2!}} + \cancel{\frac{4x^3}{3!}} - \cancel{\frac{4x^3}{3!}} + \ldots$$

$$\boxed{f(x) - 4e^x = -2 - x}$$

Key

BC Calculus – 10.8b Notes – Representing Functions as a Power Series

Recall:

Function	Series (expanded)	Series Notation	Int. of Conv.
$e^x =$	$1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$	$\sum_{n=0}^{\infty}\frac{x^n}{n!}$	$-\infty<x<\infty$
$\sin x =$	$x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots$	$\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$	$-\infty<x<\infty$
$\cos x =$	$1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots$	$\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$	$-\infty<x<\infty$
$\frac{1}{1+x}=$	$1-x+x^2-x^3+\cdots$	$\sum_{n=0}^{\infty}(-1)^n x^n$	$-1<x<1$

1. If $f(x)=\sum_{n=0}^{\infty}\frac{x^{5n}}{n!}$ then $f'(x)=$

$$f(x)=1+\frac{x^5}{1}+\frac{x^{10}}{2!}+\frac{x^{15}}{3!}+\dots$$

$$f'(x)=0+5x^4+\frac{10x^9}{2}+\frac{15x^{14}}{3!}+\dots$$

simplified

$$\boxed{f'(x)=5x^4+5x^9+\frac{5x^{14}}{2}+\dots}$$

OR take the derivative of the Rule

$$\sum_{n=1}^{\infty}\frac{5n\cdot x^{5n-1}}{n!} \rightarrow \frac{5\cancel{n}\cdot x^{5n-1}}{\cancel{n}(n-1)!}$$

$$\frac{5x^4}{1}+\frac{10x^9}{2!}+\dots$$

simplified $\rightarrow \boxed{\sum_{n=1}^{\infty}\frac{5x^{5n-1}}{(n-1)!}}$

2. Write the first 4 nonzero terms for the Maclaurin series that represents $\int_0^x \sin(t^7)\,dt$.

$$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\dots+\frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\sin(t^7)=t^7-\frac{t^{21}}{3!}+\frac{t^{35}}{5!}-\frac{t^{49}}{7!}+\dots+\frac{(-1)^n(t^7)^{2n+1}}{(2n+1)!}$$

$$\int_0^x \sin(t^7)dt=\frac{t^8}{8}-\frac{t^{22}}{22\cdot3!}+\frac{t^{36}}{36\cdot5!}-\frac{t^{50}}{50\cdot7!}\Bigg]_0^x=\boxed{\frac{x^8}{8}-\frac{x^{22}}{22\cdot3!}+\frac{x^{36}}{36\cdot5!}-\frac{x^{50}}{50\cdot7!}}$$

Practice Problems:

1. What is the coefficient of x^2 in the Taylor Series for the function $f(x)=\sin^2 x$ about $x=0$?

$$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}$$

$$\sin^2 x=(\sin x)(\sin x)=\left[x-\frac{x^3}{3!}\dots\right]\left[x-\frac{x^3}{3!}+\dots\right]\rightarrow 1x^2-\frac{x^4}{3!}-\frac{x^4}{3!}+\dots$$

coefficient is $\boxed{1}$

2. If the function f is defined as $f(x)=\sum_{n=1}^{\infty}\frac{x^{2n}}{n!}$, then what is $f'(x)$? Write the first four nonzero terms and the general term.

$$f'(x)=\sum_{n=1}^{\infty}\frac{1}{n!}(2n)x^{2n-1}=2x+\frac{4}{2!}x^3+\frac{6}{3!}x^5+\frac{8}{4!}x^7$$

$$=\boxed{2x+2x^3+x^5+\frac{1}{3}x^7}$$

3. Use the power series expansion for $\cos x^6$ to evaluate the integral $\int_0^x \cos t^6\,dt$. Write the first four nonzero terms and the general term.

$$\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}\rightarrow\frac{(-1)^n x^{2n}}{(2n)!}$$

$$\cos(t^6)=1-\frac{(t^6)^2}{2!}+\frac{(t^6)^4}{4!}-\frac{(t^6)^6}{6!}\rightarrow\frac{(-1)^n(t^6)^{2n}}{(2n)!}$$

$$\int_0^x \cos(t^6)=x-\frac{t^{13}}{26}+\frac{t^{25}}{25\cdot4!}-\frac{t^{37}}{37\cdot6!}\rightarrow\boxed{\frac{(-1)^n t^{12n+1}}{(12n+1)\cdot(2n)!}}$$

4. For $x>0$, the power series defined by $1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\dots+\frac{(-1)^n x^{2n}}{(2n+1)!}$ converges to which of the following?

(A) $\cos x$ (B) $\sin x$ $\boxed{\text{(C) } \frac{\sin x}{x}}$ (D) $e^x - e^{x^2}$

$$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\dots$$

$$\boxed{\frac{1}{x}\cdot\sin x}=\frac{1}{x}\left[x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\dots\right]$$

$$=\boxed{1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\dots}$$

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5. It is known that the Maclaurin series for $\frac{1}{1-x}$ is $\sum_{n=0}^{\infty} x^n$. Use this fact to assist in finding the first four nonzero terms and the general term for the power series expansion for the function $\frac{x^2}{1-x^2}$.

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$

$$\frac{1}{1-x^2} = 1 + x^2 + x^4 + x^6 + \dots$$

$$\frac{x^2}{1-x^2} = x^2\left[1 + x^2 + x^4 + x^6 \dots\right] \rightarrow \boxed{x^2 + x^4 + x^6 + x^8 + \dots \qquad \sum_{n=0}^{\infty} x^{2n+2}}$$

6. Let f be the function with initial condition $f(0) = 0$ and derivative $f'(x) = \frac{1}{1+x^7}$. Write the first four nonzero terms of the Maclaurin series for the function f.

$$\frac{1}{1+x} \rightarrow \frac{1}{1-(-x)} = 1 - x + x^2 - x^3 + x^4 - \dots$$

$$\frac{1}{1+x^7} \rightarrow 1 - (x^7) + (x^7)^2 - (x^7)^3 + (x^7)^4 \dots$$

$$1 - x^7 + x^{14} - x^{21} + x^{28} + \dots$$

$$\int \frac{1}{1+x^7}dx = \boxed{x - \frac{x^8}{8} + \frac{x^{15}}{15} - \frac{x^{22}}{22}} + C$$

$C = 0$ since $f(0) = 0$

7. Find the Maclaurin series for the function $f(x) = e^{3x}$. Write the first four nonzero terms and the general term.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \dots \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^{3x} = 1 + (3x) + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} \dots \boxed{\sum_{n=0}^{\infty} \frac{(3x)^n}{n!}}$$

$$\boxed{e^{3x} \approx 1 + 3x + \frac{9x^2}{2!} + \frac{27x^3}{3!}}$$

8. If a function has the derivative $f'(x) = \sin(x^2)$ and initial conditions $f(0) = 0$, write the first four nonzero terms of the Maclaurin series for f.

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \quad \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\sin(x^2) = x^2 - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} - \frac{(x^2)^7}{7!}$$

$$\boxed{\int \sin(x^2)dx \approx \frac{x^3}{3!} - \frac{x^7}{7 \cdot 3!} + \frac{x^{11}}{11 \cdot 5!} - \frac{x^{15}}{15 \cdot 7!}}$$

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9. The function f has derivatives of all orders and the Maclaurin series for the function f is given by $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+3}$. Find the Maclaurin series for the derivative $f'(x)$. Write the first four nonzero terms and the general term.

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+3} = \frac{x}{3} - \frac{x^3}{5} + \frac{x^5}{7} - \frac{x^7}{9} + \dots$$

$$\boxed{f'(x) = \frac{1}{3} - \frac{3}{5}x^2 + \frac{5}{7}x^4 - \frac{7}{9}x^6 \dots\dots + \frac{(-1)^n (2n+1) x^{2n}}{2n+3}}$$

10. Let the function f be defined by $f(x) = \frac{1}{1-x}$. Find the Maclaurin series for the derivative f'. Write the first four nonzero terms and the general term.

$$f = \frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \dots$$

$$f'(x) = 1 + 2x + 3x^2 + 4x^3 + \dots nx^{n-1}$$

11. Find the second-degree Taylor Polynomial for the function $f(x) = \frac{\cos x}{1-x}$ about $x = 0$.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$$

$$\frac{1}{1-x} = 1 + x + x^2 + \dots$$

$$(\cos x)\left(\frac{1}{1-x}\right) = \left[1 - \frac{x^2}{2!} \dots\right]\left[1 + x + x^2 \dots\right]$$

$$= 1 + x + x^2 - \frac{x^2}{2!} - \frac{x^3}{2!} - \frac{x^4}{2!} + \dots$$

$$f(x) = 1 + x + x^2 - \frac{x^2}{2}$$

$$\boxed{f(x) = 1 + x + \frac{x^2}{2}}$$

12. What is the coefficient of x^2 in the Maclaurin series for the function $f(x) = \left(\frac{1}{1+x}\right)^2$?

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = 1 - x + x^2 - x^3 \dots$$

$$\left(\frac{1}{1+x}\right)^2 = \left(\frac{1}{1+x}\right)\left(\frac{1}{1+x}\right) \rightarrow \left[1 - x + x^2 \dots\right]\left[1 - x + x^2 + \dots\right]$$

$$= 1 - x + \underline{x^2} - x + \underline{x^2} - x^3 + \underline{x^2} - x^3 + x^4$$

$$= 1 - 2x + \underline{\underline{3x^2}} - 2x^3 + x^4$$

Coefficient for x^2 term is $\boxed{3}$

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13. Find the Maclaurin series for the function $f(x) = x\cos x^2$. Write the first four nonzero terms and the general term.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\cos(x^2) = 1 - \frac{(x^2)^2}{2!} + \frac{(x^2)^4}{4!} - \frac{(x^2)^6}{6!} + \dots \frac{(-1)^n (x^2)^{2n}}{(2n)!}$$

$$x\cdot\cos(x^2) = x\left[1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!}\dots\right] + \dots \frac{(-1)^n x^{4n}\cdot x}{(2n)!} \rightarrow \boxed{\frac{(-1)^n x^{4n+1}}{(2n)!}}$$

$$\boxed{x - \frac{x^5}{2!} + \frac{x^9}{4!} - \frac{x^{13}}{6!}}$$

14. Given that f is a function that has derivatives of all orders and $f(1) = 3, f'(1) = -2, f''(1) = 2$, and $f'''(1) = 4$. Write the second-degree Taylor Polynomial for the derivative f' about $x = 1$ and use it to find the approximate value of $f'(1.2)$.

$$P_3(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3$$

$$P_3(x) = 3 + -2(x-1) + \frac{2}{2!}(x-1)^2 + \frac{4}{3!}(x-1)^3$$

$$P_2'(x) = -2 + 2(x-1) + \frac{4}{3!}(3)(x-1)^2 = -2 + 2(x-1) + 2(x-1)^2$$

$$P_2'(1.2) \approx -2 + 2(0.2) + 2(0.2)^2 = \boxed{-1.52}$$

15. Let the fourth-degree Taylor Polynomial be defined by $T = 7 - 3(x-4) + 5(x-4)^2 - 2(x-4)^3 + 6(x-4)^4$ for the function f about $x = 4$. Find the third-degree Taylor Polynomial for f' about $x = 4$ and then use it to approximate $f'(4.2)$.

$$T_3'(x) = -3 + 10(x-4) - 6(x-4)^2 + 24(x-4)^3$$

$$T_3'(4.2) = -3 + 10(0.2) - 6(0.2)^2 + 24(0.2)^3 \approx \boxed{-1.048}$$

~~16.5~~ Representing Functions as Power Series

Test Prep

16. Given a function defined by $f(x) = \frac{\cos(2x)-1}{x^2}$ for $x \neq 0$ and is continuous for all real numbers x.

a. What is the limit of the function $f(x)$ as x approaches 0?

$$\lim_{x\to 0} \frac{\cos(2x)-1}{x^2} \rightarrow \frac{1-1}{0} \rightarrow \frac{0}{0}$$

L'Hopital's $$\lim_{x\to 0} \frac{-\sin(2x)\cdot 2}{2x} \rightarrow \frac{0}{0}$$

L'Hopital's $$\lim_{x\to 0} \frac{-2\cos(2x)\cdot 2}{2} \rightarrow \frac{-4(1)}{2} = \boxed{-2}$$

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b. Write the first four nonzero terms and the general term of the power series that represents the function $h(x) = \cos 2x$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\cos(2x) = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \dots \frac{(-1)^n (2x)^{2n}}{(2n)!}$$

$$\boxed{= 1 - \frac{4x^2}{2!} + \frac{2^4x^4}{4!} - \frac{2^6x^6}{6!} + \dots \frac{(-1)^n 2^{2n} x^{2n}}{(2n)!}}$$

c. Use the results from part (b) to write the first three nonzero terms for $f(x) = \frac{\cos(2x)-1}{x^2}$.

$$\frac{1}{x^2}\left[\cos(2x) - 1\right]$$

$$\frac{1}{x^2}\left[1 - 2x^2 + \frac{16}{4!}x^4 - \frac{4}{45}x^6 - 1\right]$$

$$\boxed{-2 + \frac{2}{3}x^2 - \frac{4}{45}x^4}$$

d. Use the results from part (c) to determine if the function $f(x) = \frac{\cos(2x)-1}{x^2}$ has a relative maximum, a relative minimum or neither at $x = 0$. Justify your answer.

$$f(x) \approx -2 + \frac{2}{3}x^2 - \frac{4}{45}x^4$$

$$f'(x) = \frac{2}{3}\cdot 2x - \frac{16}{45}x^3$$

$$0 = x\left(\frac{4}{3} - \frac{16}{45}x^2\right)$$

$x=0$ is a critical point where slope $= 0$

$$f''(x) = \frac{4}{3} - \frac{48}{45}x^2$$

$$f''(0) = \frac{4}{3} - \frac{48}{45}(0)^2 = \frac{4}{3} > 0$$

so concave up at $x=0$

$\boxed{x=0 \text{ is a relative minimum because } f'(0)=0 \text{ and } f''(0)>0}$

BC Calculus – 10.10b Notes – Lagrange Error Bound

Error bound means how far off is the approximation from the actual value or answer

Key

Exact value = Approximate value + Remainder

$f(x) = \underset{\text{(Taylor polynomial)}}{P(x)} + \underset{\text{(Remainder)}}{R(x)}$

$R(x) = f(x) - P(x)$

Error: $|R_n(x)| = |f(x) - P_n(x)|$

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)(x-c)^2}{2!} + \frac{f'''(c)(x-c)^3}{3!} + \dots + \frac{f^{(n)}(c)(x-c)^n}{n!} + \boxed{R(x)}$$

$$\frac{f^{(n+1)}(c)(x-c)^{n+1}}{(n+1)!} \rightarrow \frac{\boxed{f^{(n+1)}(z)}(x-c)^{n+1}}{(n+1)!}$$

*we want to know what is the greatest possible value of this (n+1) derivative on an interval

Lagrange Error Bound

Let $f(x)$ be differentiable through the order $n+1$. The error between the Taylor Polynomial and $f(x)$ is bounded by:

$$|R_n(x)| \le \left| \frac{\text{Max}\left[f^{(n+1)}(z)\right](x-c)^{n+1}}{(n+1)!} \right|$$

where z is some number between c and x.

1. The fourth degree Maclaurin polynomial (centered at $x=0$) for $\cos x$ is given by $p_4(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$. If this polynomial is used to approximate $\cos(0.2)$, what is the Lagrange error bound?

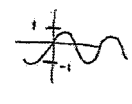
$\cos(0.2) \approx P_4(0.2) = 0.980067$

$x = 0.2$
$c = 0$
$n = 4$

$$R_4(x) \le \left| \frac{\text{Max}(-\sin(z))(0.2-0)^5}{(4+1)^5} \right|$$

* $c \le z \le x$
$0 \le z \le 0.2$

sinx curve



$f = \cos x$
$f'(x) = -\sin x$
$f''(x) = -\cos x$
$f'''(x) = \sin x$
$f^4(x) = \cos x$
$f^5(x) = -\sin x$

$R_4(x) \le \left| \frac{1 \cdot (0.2)^5}{5!} \right|$

$R_4(x) \le 2.667 \times 10^{-6}$
$R_4(x) \le 0.000002667$

2. Use a third degree Taylor polynomial on the interval [0, 1] for e^x centered about $x = 0$ to approximate e^1. What is the error bound of this approximation?

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$

$e^1 \approx P_3(1) = 1 + 1 + \frac{1^2}{2!} + \frac{1^3}{3!} = 2.66667$

$$R_3(x) \le \left| \frac{\max\left[e^z\right] \cdot (1-0)^4}{4!} \right|$$

$0 \le z \le 1$

$R_3(x) \le \left| \frac{e^1 (1)^4}{4!} \right|$

$\boxed{R_3 \le 0.1132617}$

3. What is the smallest order Taylor Polynomial centered at $x = 1$ which will approximate e^{x-1} on the interval [0, 3] with a Lagrange error bound less than 1?

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$

$e^{x-1} = 1 + (x-1) + \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} + \dots$

* $0 < z < 3$

$$\left| \frac{\text{Max}\left(f^{(n+1)}(z)\right) \cdot (x-1)^{n+1}}{(n+1)!} \right| < 1$$

$\underline{\underline{x=3}}$ $\left| \frac{e^2 \cdot (3-1)^{n+1}}{(n+1)!} \right| < 1$

$\frac{e^2 \cdot 2^{n+1}}{(n+1)!} < 1$

n	R
1	14.778
2	9.8521
3	4.926
4	1.9704
5	0.6568

5th order Taylor Polynomial

Practice Problems:

1. The third Maclaurin polynomial for $\sin x$ is given by $p(x) = x - \frac{x^3}{3!}$. If this polynomial is used to approximate $\sin(0.1)$, what is the Lagrange error bound?

Interval $[\overset{c}{0}, \overset{x}{0.1}]$

$$\frac{\max\left[f^{n+1}(z)\right]\left[x-c\right]^{n+1}}{(n+1)!}$$

$$\frac{\text{Max}\left[f^4(z)\right]\left[0.1-0\right]^4}{4!}$$

$f = \sin x$
$f'(x) = \cos x$
$f''(x) = -\sin x$
$f^3(x) = -\cos x$
$f^4(x) = \sin x$

max of $\sin x$ is $\boxed{1}$

$$\frac{[1][0.1]^4}{4!} \rightarrow \boxed{4.1667 \times 10^{-6}}$$

2. If the Taylor Polynomial for approximating $\cos x$ is given by $1-\frac{x^2}{2!}+\frac{x^4}{4!}$, what is the upper bound for the error in the approximation of cos(0.3)?

Interval $[\overset{c}{0}, \overset{x}{0.3}]$

$n=4$

$f=\cos x$, $f'=-\sin x$, $f''=-\cos x$, $f'''=\sin x$, $f^4=\cos x$, $f^5=-\sin x$

Max of $-\sin x = 1$

$$\frac{Max[f^5(z)][0.3-0]^5}{5!} \quad \frac{(1)(0.3)^5}{5!} = \boxed{2.025\times10^{-5}}$$

3. If the Taylor Polynomial about $x = 0$ for the approximation of e^x is given by $1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}$, what is the upper bound for the error in the approximation of e? → e means e^1 → interval $[\overset{c}{0}, \overset{x}{1}]$

$f^6(x)=e^x$ Max on interval $[0,1]$ is $\boxed{e}$

$$\frac{Max[f^6(z)][1-0]^6}{6!} \to \frac{(e)(1)^6}{6!} = \boxed{0.00377}$$

4. Let f be a function that has derivatives of all orders for all real numbers and let $P_3(x)$ be the third-degree Taylor Polynomial for f about $x = 0$. $|f^{(n)}(x)| \le \frac{n}{n+1}$, for $1 \le n \le 5$ and all values of x. Of the following, which is the smallest value of k for which the Lagrange error bound guarantees that $|f(1) - P_3(1)| \le k$?

$R_3(x) \le K$ $\quad Max[f^4(z)] \le \frac{4}{4+1} = \frac{4}{5}$ on interval $[\overset{c}{0}, \overset{x}{1}]$

$$\frac{Max[f^4(z)][x-c]^4}{4!} \to \frac{(\frac{4}{5})(1-0)^4}{4!} \to \boxed{\frac{4}{5}\cdot\frac{1}{4!}}$$

(A) $\frac{5}{6}$ (B) $\frac{5}{6}*\frac{1}{5!}$ (C) $\frac{5}{6}*\frac{1}{4!}$ $\boxed{\text{(D) } \frac{4}{5}*\frac{1}{4!}}$

5. The function f has derivatives of all orders for all real numbers, $f^{(4)}(x) = e^{\cos x}$. If the third-degree Taylor Polynomial for f about $x = 0$ is used to approximate f on the interval [0,1], what is the Lagrange error bound?

$$\frac{Max[f^4(z)][1-0]^4}{4!} \quad \frac{(e^1)(1)^4}{4!} \approx \boxed{0.11326}$$

Max of $e^{\cos x}$ on $[0,1]$ is $x=0 \to e^{\cos 0} \to e^1$

6. The Taylor series for a function f about $x = 3$ is given by $\sum_{n=0}^{\infty}(-1)^n\frac{3n+1}{2^n}(x-3)^n$ and converges to f for $0 \le x \le 5$. If the third-degree Taylor Polynomial for f about $x = 3$ is used to approximate $f\left(\frac{13}{4}\right)$, what is the alternating series error bound?

$$a_4 = \frac{3(4)+1}{2^4}(x-3)^4 \to \frac{13}{16}\left[\frac{13}{4}-3\right]^4$$

$$\boxed{|R_3| \le \frac{13}{16}\left(\frac{1}{4}\right)^4}$$

7. Let f be a polynomial function with nonzero coefficients such that $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5$. $T_4(x)$ is the fourth-degree Taylor Polynomial for f about $x = c$ such that $T_4 = b_0 + b_1(x-c) + b_2(x-c)^2 + b_3(x-c)^3 + b_4(x-c)^4$. Based on the Lagrange error bound, $f(x) - T_4(x)$ must equal which of the following?

$|R_4(x)| = |f(x) - T_4(x)|$ is the next term

(A) x (B) $(x-c)^5$ $\boxed{\text{(C) } a_5(x-c)^5}$ (D) $\frac{a_5(x-c)^5}{5!}$

8. Let $P(x)$ be the sixth-degree Taylor Polynomial for a function f about $x = 0$. Information about the maximum of the absolute value of selected derivatives of f over the interval $0 \le x \le 1.5$ is given below.

$\max_{0\le x\le1.5}|f^{(5)}(x)| = 9.3$ $\quad \max_{0\le x\le1.5}|f^{(6)}(x)| = 62.1$ $\quad \max_{0\le x\le1.5}|f^{(7)}(x)| = 481.3$

What is the smallest value of k for which the Lagrange error bound guarantees that $|f(1.5) - P(1.5)| \le k$?

$$\frac{max[f^7(z)][x-c]^7}{7!} \to \frac{(481.3)(1.5-0)^7}{7!} \approx \boxed{1.6316}$$

on interval $[\overset{c}{0}, \overset{x}{1.5}]$

9. The function f has derivatives of all orders for all real numbers. Values of f and its first four derivatives at $x = 2$ are given in the table.

x	$f(x)$	$f'(x)$	$f''(x)$	$f'''(x)$	$f^{(4)}(x)$
2	6	−12	18	−24	34

a. Write the third-degree Taylor Polynomial for f about $x = 2$, and use it to approximate $f(1.5)$.

$$P_3(x) = f(x) + f'(x)[x-2] + \frac{f''(x)}{2!}(x-2)^2 + \frac{f'''(x)}{3!}(x-2)^3$$

$$P_3(x) = 6 + -12(x-2) + \frac{18}{2}(x-2)^2 - \frac{24}{3!}(x-2)^3$$

$$P_3(x) = 6 - 12(x-2) + 9(x-2)^2 - 4(x-2)^3$$

$$f(1.5) \approx P_3(1.5) = 6 - 12(-0.5) + 9(-0.5)^2 - 4(-0.5)^3 = \boxed{14.75}$$

b. The fourth derivative of f satisfies the inequality $|f^{(4)}(x)| \le 48$ for all $x > 1$. Use the Lagrange error bound to show that the approximation found in part (a) differs from $f(1.5)$ by no more than $\frac{1}{8}$.

$$R_1(1.5) \le \frac{\max\left[f^4(z)\right](1.5-2)^4}{4!} = \frac{48(0.5)^4}{4!} = 0.125$$

$$\boxed{R_3(1.5) \le 0.125 = 1/8}$$

10. Let h be a function having derivatives of all orders for $x > 0$. Selected values for the first four derivatives of h are given for $x = 3$. Use the Lagrange error bound to show that the third-degree Taylor polynomial for h about $x = 3$ approximates $h(2.9)$ with an error less than 3×10^{-4}.

x	$h(x)$	$h'(x)$	$h''(x)$	$h'''(x)$	$h^{(4)}(x)$
3	317	$\frac{753}{4}$	$\frac{1383}{4}$	$\frac{3483}{8}$	$\frac{1125}{16}$

interval $[2.9, 3]$ (x, c)

$$R_3(x) \le \frac{\max\left[h^4(z)\right](x-c)^4}{4!}$$

$$\le \frac{\left(\frac{1125}{16}\right)(2.9-3)^4}{4!}$$

$$\le \boxed{2.9297 \times 10^{-4}}$$

10.12 Lagrange Error Bound

Test Prep

11. **Calculator allowed.**

x	$f(x)$	$f'(x)$	$f''(x)$	$f'''(x)$	$f^{(4)}(x)$
3	4	-8	14	-22	30

The function f has derivatives of all orders for all real numbers. Values of f and its first four derivatives at $x = 3$ are given in the table.

a. Write an equation for the line tangent to the graph of f at $x = 3$ and use it to approximate $f(2.5)$.

point: $(3, 4)$

slope: $f'(3) = -8$

$y - 4 = -8(x-3)$

$y = -8(x-3) + 4$

$y(2.5) = -8(2.5) + 28 = 8$

$\boxed{f(2.5) \approx 8}$

b. Write the third-degree Taylor polynomial for f about $x = 3$, and use it to approximate $f(2.5)$.

$c = 3$

$$P_3(x) = f(x) + f'(x)(x-3) + \frac{f''(x)}{2!}(x-3)^2 + \frac{f'''(x)}{3!}(x-3)^3$$

$$P_3(x) = f(3) + f'(3)(x-3) + \frac{f''(3)}{2!}(x-3)^2 + \frac{f'''(3)}{3!}(x-3)^3$$

$$P_3(x) = 4 - 8(x-3) + \frac{14}{2}(x-3)^2 - \frac{22}{3!}(x-3)^3$$

$$f(2.5) \approx P_3(2.5) = 4 - 8(-0.5) + 7(-0.5)^2 - \frac{11}{3}(-0.5)^3 = \boxed{10.208}$$

c. Is there enough information to determine whether f has a critical point at $x = 2.5$? If not, explain why not. If so, determine whether $f(2.5)$ is a relative maximum, relative minimum, or neither, and give a reason for your answer.

There is not enough information. We don't know if $f'(2.5) = 0$ or if $f'(2.5)$ does not exist.

The Taylor Polynomial only gives us an approximation of $f(x)$.

d. The fourth derivative of f satisfies the inequality $|f^{(4)}(x)| \le 48$ for all $x > 2$. Use the Lagrange error bound to show that the approximation found in part (b) differs from $f(2.5)$ by no more than $\frac{1}{8}$.

$$R_3(x) \le \frac{\max\left[f^4(z)\right]\left[x-c\right]^4}{4!}$$

interval $[2.5, 3]$ (x, c)

$$\le \frac{(48)(2.5-3)^4}{4!}$$

$$\le \boxed{0.125}$$

e. What is the coefficient of the $(x-3)^3$ term in the Taylor series for f', the derivative of f, about $x = 3$?

$$f(x) = \ldots\ldots \frac{f^4(x)}{4!}(x-3)^4$$

$$= \frac{30}{4!}(x-3)^4$$

$$f'(x) = \ldots\ldots \frac{4 \cdot 30}{4!}(x-3)^3$$

$$\frac{30}{3!}(x-3)^3 \quad \text{or} \quad \frac{30}{6}(x-3)^3 \rightarrow \boxed{5}(x-3)^3$$

$\boxed{\text{coefficient is } 5}$

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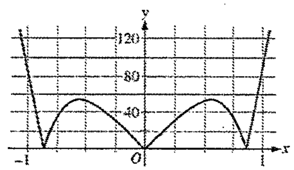
Question 6

Let $f(x) = \sin(x^2) + \cos x$. The graph of $y = \left|f^{(5)}(x)\right|$ is shown above.



Graph of $y = \left|f^{(5)}(x)\right|$

(a) Write the first four nonzero terms of the Taylor series for $\sin x$ about $x = 0$, and write the first four nonzero terms of the Taylor series for $\sin(x^2)$ about $x = 0$.

(b) Write the first four nonzero terms of the Taylor series for $\cos x$ about $x = 0$. Use this series and the series for $\sin(x^2)$, found in part (a), to write the first four nonzero terms of the Taylor series for f about $x = 0$.

(c) Find the value of $f^{(6)}(0)$.

(d) Let $P_4(x)$ be the fourth-degree Taylor polynomial for f about $x = 0$. Using information from the graph of $y = \left|f^{(5)}(x)\right|$ shown above, show that $\left|P_4\left(\frac{1}{4}\right) - f\left(\frac{1}{4}\right)\right| < \frac{1}{3000}$. $\ast\ R_n(x) = \left|\frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}\right|$

a) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$

$\sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \dots$

b) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$

$f(x) = 1 + x^2 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{3!} - \frac{x^6}{6!}$

$f(x) = 1 + \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6} - \frac{x^6}{720}$

$1 + \frac{x^2}{2} + \frac{x^4}{4!} - \frac{120x^6}{720} - \frac{1x^6}{720}$

$f(x) = 1 + \frac{x^2}{2} + \frac{x^4}{4!} - \frac{121}{6!}x^6$

c) Since $\frac{f^{(6)}(0)}{6!}x^6 = \frac{-121}{6!}x^6$,

$\boxed{f^{(6)}(0) = -121}$

d) $R_4(x) = \left|P_4(x) - f(x)\right| = \left|P_4\left(\tfrac{1}{4}\right) - f\left(\tfrac{1}{4}\right)\right| = \left|\frac{f^{(5)}(z)}{5!}(x-c)^5\right| \le \left|\frac{40}{5!}\left(\tfrac{1}{4}-0\right)^5\right|$

$x = 1/4$
$c = 0$

max y-value in interval $0 < x < 1/4$

$= \left|\frac{40}{5!}\left(\frac{1}{4}\right)^5\right| = \frac{40}{120 \cdot 4^5} = \boxed{\frac{1}{3072} < \frac{1}{3000}}$

$= \frac{1}{3 \cdot 4^5} = \frac{1}{3 \cdot 1024}$

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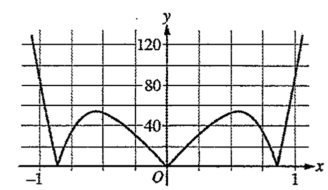
Question 6

Let $f(x) = \sin(x^2) + \cos x$. The graph of $y = \left|f^{(5)}(x)\right|$ is shown above.



Graph of $y = \left|f^{(5)}(x)\right|$

(a) Write the first four nonzero terms of the Taylor series for $\sin x$ about $x = 0$, and write the first four nonzero terms of the Taylor series for $\sin(x^2)$ about $x = 0$.

(b) Write the first four nonzero terms of the Taylor series for $\cos x$ about $x = 0$. Use this series and the series for $\sin(x^2)$, found in part (a), to write the first four nonzero terms of the Taylor series for f about $x = 0$.

(c) Find the value of $f^{(6)}(0)$.

(d) Let $P_4(x)$ be the fourth-degree Taylor polynomial for f about $x = 0$. Using information from the graph of $y = \left|f^{(5)}(x)\right|$ shown above, show that $\left|P_4\left(\frac{1}{4}\right) - f\left(\frac{1}{4}\right)\right| < \frac{1}{3000}$.

(a) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$ $\sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots$	$3: \begin{cases} 1: \text{series for } \sin x \\ 2: \text{series for } \sin(x^2) \end{cases}$
(b) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$ $f(x) = 1 + \frac{x^2}{2} + \frac{x^4}{4!} - \frac{121x^6}{6!} + \cdots$	$3: \begin{cases} 1: \text{series for } \cos x \\ 2: \text{series for } f(x) \end{cases}$
(c) $\frac{f^{(6)}(0)}{6!}$ is the coefficient of x^6 in the Taylor series for f about $x = 0$. Therefore $f^{(6)}(0) = -121$.	1 : answer
(d) The graph of $y = \left\|f^{(5)}(x)\right\|$ indicates that $\max\limits_{0 \le x \le \frac{1}{4}} \left\|f^{(5)}(x)\right\| < 40$. Therefore $\left\|P_4\left(\frac{1}{4}\right) - f\left(\frac{1}{4}\right)\right\| \le \frac{\max\limits_{0 \le x \le \frac{1}{4}} \left\|f^{(5)}(x)\right\|}{5!} \cdot \left(\frac{1}{4}\right)^5 < \frac{40}{120 \cdot 4^5} = \frac{1}{3072} < \frac{1}{3000}$.	$2: \begin{cases} 1: \text{form of the error bound} \\ 1: \text{analysis} \end{cases}$