Name: ______________________________ Period: __________

BC Calculus

Miscellaneous BC Topics (Ch. 6 – 8)

(Integration by Parts, Partial Fractions, Improper Integrals, Revisiting L'Hopital's Rule, Euler's Method, Logistic Models, Arc Length of Curve & Distance Traveled)

BC Calculus - 6.6 Notes – Integration by Parts

Integration by Parts (I.B.P.) -a method of integration useful for problems involving the product of two different types of functions . (example: logs and polynomial)

Integration by parts is typically used for the integration of the product of two functions.

$$\int f(x)g'(x) =$$

Integration by parts is based on the product rule:

$$[fg]' = f'g + fg'$$

Basic rule for choosing f and g':

1. For f: choose something that becomes simpler when you differentiate.
2. For g': choose something that can easily be integrated.

Steps:

1. Determine the u-value by using the acronym L.I.P.E.T.
   a. LIPET shows the priority order for determining u-value
   b. Logs Inverse Trig Polynomial Exponential function Trigonometric function
2. Let dv be other function
3. Find u, dv, v, and dv
4. Plug into formula and integrate

Tabular Integration: Differentiate to 0 for the chosen $f(x)$. Integrate your chosen $g'(x)$ the same number of times. Follow the sign convention, which is plus/minus repeating.

2. $\int x^4 \sin x \, dx$

$f(x)$	$g'(x)$

(2)

Practice Problems:

Integrate the following.

1. $\int x\cos(x)\,dx$

2. $\int 2x\cos(3x+1)\,dx$

3. $\int x^2\sin(x)\,dx$

4. $\int 4xe^{3x+1}\,dx$

5. $\int_1^{e^2} x^4 \ln x\,dx$

6. $\int \ln x\,dx$

7. $\int_1^2 (3x^2 - 2x + 1) \ln x \, dx$

8. $\int x^3 e^x dx$

9. The table gives values of f, f', g, and g' for selected values of x. If $\int_0^3 f'(x)g(x)\,dx = 6$, then $\int_0^3 f(x)g'(x)\,dx = ?$

x	0	3
$f(x)$	1	5
$f'(x)$	5	−3
$g(x)$	−4	3
$g'(x)$	3	2

10. Let f be a twice-differentiable function with selected values of f and its derivatives shown in the table. What is the value of $\int_0^3 x f''(x)\,dx$?

x	$f(x)$	$f'(x)$	$f''(x)$
0	2	−2	5
3	5	7	−2

11. $\int x \cos 2x \, dx$

(A) $\frac{1}{2}x^2 \sin(2x) + C$

(B) $\frac{1}{2}x^2 \cos(2x) + \frac{1}{2}\sin(2x) + C$

(C) $\frac{1}{2}x \sin(2x) - \frac{1}{4}\cos(2x) + C$

(D) $\frac{1}{2}x \sin(2x) + \frac{1}{4}\cos(2x) + C$

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12. $\int_1^e x^4 \ln x \, dx$

A) $\frac{6e^5-1}{25}$ (B) $\frac{4e^5+1}{25}$ (C) $\frac{1-e^3}{3}$ (D) e^4

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13. Let f be a differentiable function such that $\int f(x) \cos x \, dx = f(x) \sin x - \int \frac{1}{2}x^3 \sin x \, dx$. Which of the following could be $f(x)$.

A) $\frac{1}{2}\sin x$ (B) $\frac{1}{2}\cos x$ (C) $\frac{1}{8}x^4$ (D) $\frac{1}{2}x^3$

BC Calculus – 6.10 Notes - Linear Partial Fractions

Let $F(x) = \frac{P(x)}{Q(x)}$ be a rational function where $P(x)$ and $Q(x)$ are polynomials. If the degree of the numerator is smaller than the degree of the denominator (**proper rational function**), then you might be able to use partial fraction decomposition to change the integrand to something easier to integrate. Let's "decompose" this fraction to make it easier to integrate.

Partial Fraction Decomposition

Every proper rational function can be written as a sum

$$\frac{P(x)}{Q(x)} = F_1(x) + F_2(x) + F_3(x) + \cdots + F_n(x).$$

Where $F_1(x), F_2(x), F_3(x), \ldots, F_n(x)$ are also rational functions, in which the denominators are factors of $Q(x)$. We are only concerned with nonrepeating linear factors.

With $Q(x)$ having n nonrepeating linear factors, we can write

$$Q(x) = (a_1x + b_1)(a_2x + b_2)(a_3x + b_3) \ldots (a_nx + b_n)$$

Therefore, using the process of partial fractions and with $Q(x)$ having n nonrepeating linear factors, we can write $\frac{P(x)}{Q(X)} =$

1. Evaluate $\int \frac{1}{(x-1)(x+2)} dx$

2. Evaluate $\int \frac{2x^2-7}{x^3-3x^2-4x} dx$

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Practice Problems

Evaluate using partial fractions.

1. $\int \frac{x-12}{x^2-4x}\,dx$

2. $\int \frac{2x}{x^2-4}\,dx$

3. $\int \frac{1}{(x+2)(x-3)(x+1)}\,dx$

4. $\int \frac{x+2}{x^2+5x}\,dx$

5. $\int \frac{2}{x(x-2)}\,dx$

6. $\int \frac{x^3-11x-15}{x^2-2x-8}\,dx$

7. For $0 < P < 50$, what is the antiderivative of $\frac{1}{P(50-P)}$?

8. $\int_0^1 \frac{1}{(x+5)(x+1)}\,dx$

8

9. $\int_0^2 \frac{3}{(4x+1)(x+1)}\,dx$

10. $\int_2^3 \frac{3}{(x-1)(x+2)}\,dx$

BC Calculus – 6.12 Notes – Improper Integrals

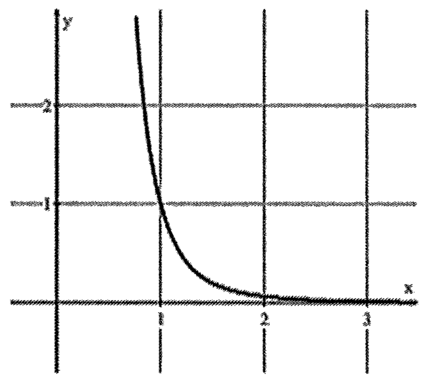
Improper integrals are integrals with infinite limits of integration or have an infinite discontinuity on the interval.

If $f(x)$ is continuous on $[a, \infty)$, then $\int_a^{\infty} f(x)\,dx =$

If $f(x)$ is continuous on $(-\infty, b]$, then $\int_{-\infty}^{b} f(x)\,dx =$

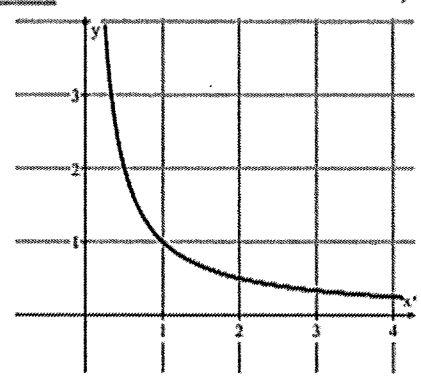
Provided the limits exist!

1. $\int_1^{\infty} \frac{1}{x^4}\,dx$



If the limit exists, the improper integral is said to __________. If the limit does not exist, the integral is said to ____________.

2. $\int_1^{\infty} \frac{1}{x}\,dx$



Improper *p-integral*: $\int_1^{\infty} \frac{1}{x^p}\,dx$ converges if   and diverges if

10

Remember the definite integral $\int_a^b f(x)\,dx$, requires the interval to be finite and the FTC requires that $f(x)$ be continuous on $[a, b]$. If the integral does not meet these requirements, we may need to manipulate the problem.

Another form of the Improper Integral is $\int_{-\infty}^{\infty} f(x)\,dx$, with $f(x)$ continuous on $(-\infty, \infty)$. Let $x = c$ be any real number in the interval $(-\infty, \infty)$, then

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx.$$

(It's easiest to use 0 here for c). If either of these integrals diverge, then the whole diverges.

3. $\int_{-\infty}^{\infty} e^x\,dx$

If $f(x)$ is continuous on $[a, b)$ and has an infinite discontinuity at b, then

$$\int_a^b f(x)\,dx =$$

If $f(x)$ is continuous on $(a, b]$ and has an infinite discontinuity at a, then

$$\int_a^b f(x)\,dx =$$

4. $\int_0^2 \frac{x+2}{\sqrt{x^2+4x}}\,dx$

If $f(x)$ is continuous on the interval $[a, b]$, except for some c in (a, b) at which f has an infinite discontinuity, then $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$.

5. $\int_{-1}^{1} \frac{1}{x}\,dx$

Evaluate each integral.

1. $\int_1^\infty \frac{1}{x^2}\,dx$

2. $\int_0^\infty \frac{2}{x^2+4x+3}\,dx$

3. $\int_0^1 \frac{x+1}{\sqrt{x^2+2x}}\,dx$

4. $\int_1^\infty xe^{-x}\,dx$

(12)

5. $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$

6. $\int_{-1}^{0} \frac{1}{x^5}\,dx$

7. $\int_{0}^{\infty} e^{-x}\,dx$

8. Determine all the values of p for which $\int_{0}^{1} \frac{1}{x^p}\,dx$ converges.

9. If g is a twice-differentiable function, where $g(2) = 1$ and $\lim_{x\to\infty} g(x) = 8$, then $\int_2^\infty g'(x)\,dx$ is

A) -7 (B) 7 (C) 9 (D) nonexistent

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10. If R is the unbounded region between the graph of $y = \frac{x}{(1+x^2)^2}$ and the x-axis for $x \geq 0$, then the area of R is

A) -1 (B) 0 (C) $\frac{1}{2}$ (D) infinite

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11. For what values of p will $\int_1^\infty \frac{1}{x^{7p-3}}\,dx$ converge?

A) $p < 0$ (B) $p > 0$ (C) $p > \frac{4}{7}$ (D) $p < \frac{4}{7}$

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BC Calculus **4.4 Notes** **L'Hopital's Rule**

L'Hôpital's Rule (or Bernoulli's Rule)

If $\lim_{x \to a} \frac{f(x)}{g(x)}$ yields either of the indeterminate forms $\frac{0}{0}$ or $\pm\frac{\infty}{\infty}$, then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$.

The rule works great, but it only works with the two forms $\frac{0}{0}$ or $\pm\frac{\infty}{\infty}$. There are other indeterminate forms including 0^0, 1^∞, $\infty - \infty$, $0 \cdot \infty$, and ∞^0. We can still use the rule, but we have to first convert them to $\frac{0}{0}$ or $\pm\frac{\infty}{\infty}$.

1.If Indeterminate form is 0^0, 1^∞, or ∞^0 → rewrite as equation and use Log Differentiation

2. If Indeterminate form is $\infty - \infty$ → find common denominator, which will get the expression into a single quotient, ready to evaluate.

3. If Indeterminate form is $0 \cdot \infty$ → rewrite as a quotient, bring ∞ or 0 down to denominator to create $\frac{0}{0}$ *or* $\pm\frac{\infty}{\infty}$

Example 1:

a) $\lim_{x \to \infty} e^{-x}\sqrt{x} =$

b) $\lim_{x \to 1^+} \left(\frac{1}{\ln x} - \frac{1}{x-1} \right) =$

Example 2:

a) $\lim_{x \to \infty} \left(1 + \frac{2}{x}\right)^x =$

b) $\lim_{x \to 0^+} x^x =$

c) $\lim_{x \to \infty} x^{1/x} =$

(15) Key (15)

BC Calculus **Ch. 8.7 Notes** **L'Hopital's Rule**

L'Hôpital's Rule (or Bernoulli's Rule)

If $\lim_{x\to a}\frac{f(x)}{g(x)}$ yields either of the indeterminate forms $\frac{0}{0}$ or $\pm\frac{\infty}{\infty}$, then $\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$.

The rule works great, but it only works with the two forms $\frac{0}{0}$ or $\pm\frac{\infty}{\infty}$. There are other indeterminate forms including 0^0, 1^∞, $\infty-\infty$, $0\cdot\infty$, and ∞^0. We can still use the rule, but we have to first convert them to $\frac{0}{0}$ or $\pm\frac{\infty}{\infty}$.

try direct substitution first.

Example 1:

a) $\lim_{x\to\infty} e^{-x}\sqrt{x} =$

$\lim_{x\to\infty}\frac{\sqrt{x}}{e^x}=\frac{\infty}{\infty}\xrightarrow{L'H}$

$\lim_{x\to\infty}\frac{\frac{1}{2}x^{-1/2}}{e^x}=\frac{1}{2\sqrt{x}e^x}=\boxed{0}$

b) $\lim_{x\to1^+}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)=$ $\infty-\infty$ $\quad\frac{x-1-\ln x}{\ln x(x-1)}$

$L'H\to\lim_{x\to1^+}\frac{1-\frac{1}{x}}{\frac{1}{x}(x-1)+\ln x(1)}\cdot\left(\frac{x}{x}\right)=\frac{x-1}{x-1+x\ln x}=$

$L'H\to\lim_{x\to1^+}\frac{1}{1+1\ln x+x(\frac{1}{x})}=\frac{1}{1+1}=\boxed{\frac{1}{2}}$ ✓

*use logs to bring exponent down.

Example 2: 1^∞

a) $\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x=$ $\quad y=\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x$

$\ln y=\lim_{x\to\infty}\ln\left(1+\frac{2}{x}\right)^x$

$\ln y=\lim_{x\to\infty}x\ln\left(1+\frac{2}{x}\right)$

$\ln y=\lim_{x\to\infty}\frac{\ln(1+2x^{-1})}{x^{-1}}=\frac{0}{0}$

$=\lim_{x\to\infty}\frac{\frac{-2x^{-2}}{1+2x^{-1}}}{-1x^{-2}}=\frac{2}{1+\frac{2}{x}}=\frac{2}{1}$

$\ln y=2$

$e^{\ln y}=e^2$

$\boxed{y=e^2}$

b) $\lim_{x\to0^+}x^x=$ $\quad y=\lim_{x\to0^+}x^x$

$\ln y=\lim_{x\to0^+}\ln x^x$

$=\lim_{x\to0^+}x\cdot\ln x=\frac{\ln x}{x^{-1}}=\frac{-\infty}{\infty}$

$=\lim_{x\to0^+}\frac{\frac{1}{x}}{-1x^{-2}}=\frac{\frac{1}{x}}{-\frac{1}{x^2}}=-x=0$

$\ln y=0$

$e^{\ln y}=e^0$

$\boxed{y=1}$

c) $\lim_{x\to\infty}x^{1/x}=$ $\quad y=\lim_{x\to\infty}x^{1/x}$

$\ln y=\lim_{x\to\infty}\ln x^{1/x}$

$\ln y=\lim_{x\to\infty}\frac{1}{x}\ln x$

$\ln y=\lim_{x\to\infty}\frac{\ln x}{x}$

$\ln y=\lim_{x\to\infty}\frac{\frac{1}{x}}{1}=0$

$\ln y=0$

$e^{\ln y}=e^0$

$\boxed{y=1}$

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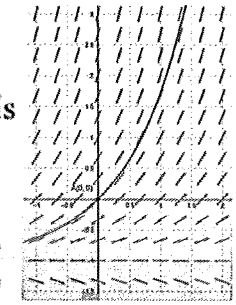
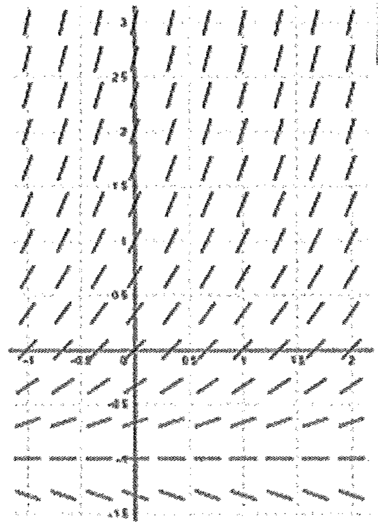
BC Calculus – 7.4 Notes – Euler's Method



In lesson 7.6, we will show how to find a solution $y = f(x)$ to a differential equation. In this lesson, we are going to APPROXIMATE a solution to a differential equation. This approximation method is called **Euler's method**.

1. $\frac{dy}{dx} = 1 + y$ and $y(0) = 0$. $\Delta x = 0.5$. Using Euler's Method, show an approximation to the solution curve $y = f(x)$.



Step 1: Construct a tangent line at $(0, 0)$ for $0 \leq x \leq 0.5$.

Starting point was $(0, 0)$. New point to work with is

Step 2: Construct a tangent line at for $0.5 \leq x \leq 1$.

Starting point was New point to work with is

Step 3: Construct a tangent line at for $1 \leq x \leq 1.5$.

Starting point was New point to work with is

Here is a way Euler Method questions often appear on the AP Exam.

2. $\frac{dy}{dx} = 2x$ and let $f(x) = y$ be a solution to this differential equation. If $f(1) = 3$, what is the approximation to $f(2)$ obtained by using Euler's method with 5 steps of equal size?

First, find the step size. $\Delta x =$

$$y - y_1 = m(x - x_1)$$

$$y = y_1 + m(x - x_1)$$

x	y	y'	New y

Practice problems:

1. The table below gives the values of f', the derivative of f. If $f(1) = 2$, what is the approximation to $f(2.5)$ obtained by using Euler's method with 3 steps of equal size?

x	1	1.5	2.0	2.5
$f'(x)$	0.3	0.7	1.2	1.8

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2. The table below gives the values of f', the derivative of f. If $f(2) = 3$, what is the approximation to $f(2.6)$ obtained by using Euler's method with 2 steps of equal size?

x	2	2.3	2.6
$f'(x)$	−0.5	−0.3	−0.1

3. The table below gives the values of f', the derivative of f. If $f(3) = 5$, what is the approximation to $f(4.0)$ obtained by using Euler's method with 2 steps of equal size?

x	3	3.25	3.5	3.75	4.0	4.25
$f'(x)$	0.1	0.3	0.5	0.7	0.9	1.1

4. The table below gives the values of f', the derivative of f. If $f(1.5) = 4$, what is the approximation to $f(1)$ obtained by using Euler's method with 2 steps of equal size?

x	1	1.25	1.5	1.75	2.0
$f'(x)$	0.3	0.4	0.6	0.9	1.3

5. Let $h(x) = \int_1^x \sqrt{1 + t^2}\, dt$. Use Euler's method, starting at $x = 1$ with 2 steps of equal size, to approximate $h(3)$.

6. Let $h(x) = \int_0^x \sqrt{1+3t^2}\, dt$. Use Euler's method, starting at $x = 0$ with 3 steps of equal size, to approximate $h(3)$.

7. Let $y = f(x)$ be the solution to the differential equation $\frac{dy}{dx} = 2x - y$ with initial condition $f(1) = 0$. What is the approximation for $f(1.3)$ obtained using Euler's method with 3 steps of equal length, starting at $x = 1$?

8. Let $y = f(x)$ be the solution to the differential equation $\frac{dy}{dx} = -\frac{x}{y}$ with initial condition $f(0) = 1$. What is the approximation for $f(.3)$ obtained using Euler's method with 3 steps of equal length, starting at $x = 0$?

9. Let $y = f(x)$ be the solution to the differential equation $\frac{dy}{dx} = y$ with initial condition $f(0) = 1$. What is the approximation for $f(.5)$ obtained using Euler's method with a step size of $\Delta x = 0.1$, starting at $x = 0$?

10. Let $y = f(x)$ be the solution to the differential equation $\frac{dy}{dx} = x + y$ with initial condition $f(0) = 1$. What is the approximation for $f(.8)$ obtained using Euler's method with 4 steps of equal length, starting at $x = 0$?

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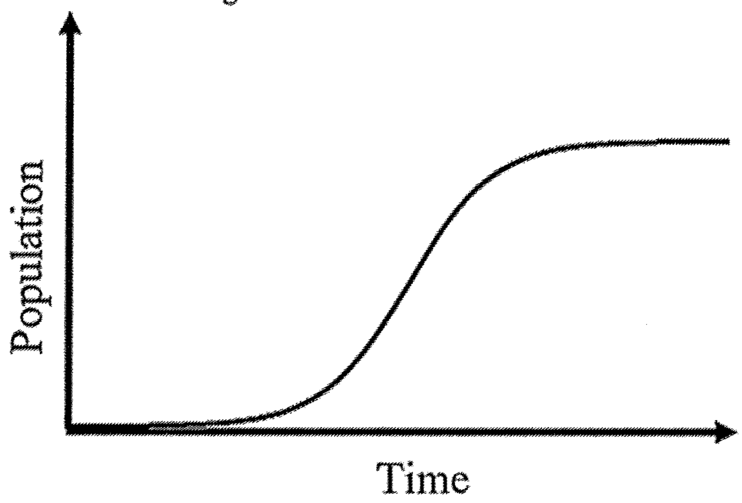
BC Calculus – 7.5 Notes – Logistic Models

Logistic population growth is when the growth rate increases quickly at first, but then slows as the population reaches carrying capacity.

Graphs would look something like this:



A real-world application could be with animals and how many the land can support.

Logistic Differential Equation

The derivative of a logistic function is typically written in one of the following forms:

$$\frac{dy}{dt} =$$

or if you manipulate this algebraically you could see it as

$$\frac{dy}{dt} =$$

In either form, k and L are positive constants and L is the limiting value.

Identify the limiting value and the y-value of the point of inflection for each solution of the given differential equation.

1. $\frac{dy}{dt} = 6y\left(1 - \frac{y}{3}\right)$

   Limiting value:

   y-value of the pt of inflection:

2. $\frac{dy}{dt} = 2y(16 - y)$

   Limiting value:

   y-value of the pt of inflection:

These two things will answer most questions you will see on the AP Exam for logistics.

1. The maximum value of the logistic function is the limiting value.
2. The maximum rate happens when y'' changes from positive to negative.

Identify the carrying capacity and where the maximum rate of change occurs.

3. $\frac{dy}{dt} = 20y\left(1 - \frac{y}{100}\right)$

Carrying capacity:

Maximum rate occurs at

4. $\frac{dy}{dt} = 4y(7 - y)$

Carrying capacity:

Maximum rate occurs at

You can derive a general solution, using separation of variables, to solve $\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right)$.
You end up with something that looks like this: $y = \frac{L}{1+be^{-kt}}$

5. The rate of change $\frac{dP}{dt}$ of the number of people entering a state park is modeled by a logistic differential equation. The capacity of the state park is 2500 people. At a certain time, the number of people in the state park is 1200 and is increasing at a rate of 100 people per hour. Create a differential equation that could represent this situation.

6. Find the carrying capacity $\frac{dy}{dt} = \frac{4}{5}y - \frac{1}{150}y^2$

Practice Problems:

1. A population y changes at a rate modeled by the logistic differential equation $\frac{dy}{dt} = 0.3y(4000 - y)$, where t is measured in years. What are all the values of y for which the population is increasing at a decreasing rate?

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2. A rumor spreads through a community at the rate $\frac{dy}{dt} = 2y(0.7 - y)$, where y is the proportion of the population that has heard the rumor at time t.

   a. What proportion of the population has heard the rumor when it is spreading the fastest?

   b. If at $t = 0$, 20% of the people have heard the rumor, find y as a function of t.

   c. At what time t is the rumor spreading the fastest? [no calculator, give an exact answer.]

3. The population P of a city at time t is increasing according to a logistic differential equation. Which of the following could be the differential equation?

A. $\frac{dP}{dt} = 0.375t$

B. $\frac{dP}{dt} = 0.375t(15000 - t)$

C. $\frac{dP}{dt} = 0.375P$

D. $\frac{dP}{dt} = 0.375(15000 - P)$

E. $\frac{dP}{dt} = 0.375P(15000 - P)$

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4. The total number of positive COVID cases in a city t days after the start of an outbreak is modeled by the function $y = C(t)$ that is the solution to the logistic differential equation $\frac{dy}{dt} = \frac{1}{7000}y(1600 - y)$. If there are 10 reported positive COVID cases initially, what is the limiting value for the total number of positive cases of the COVID virus as t increases?

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5. The size of a rabbit population is modeled by the function R that is a solution to the logistic differential equation $\frac{dR}{dt} = \frac{R}{3} - \frac{R^2}{2400}$, where t is measured in years for $t \geq 0$ and the initial population satisfies $R(0) > 0$. Which of the following statements could be true?

I. $\lim_{t\to\infty} R(t) > 1000$

II. The graph of R has a point of inflection for $t > 0$.

III. The maximum rate of change of R occurs at $t = 0$.

A. None

B. II only

C. I & II only

D. II & III only

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6. The rate of change $\frac{dP}{dt}$ of the number of people in a mall is modeled by a logistic differential equation. The maximum number of people allowed in the mall is 2000. At 10 A.M., the number of people in the mall is 200 and is increasing at a rate of 400 people per hour. Which of the following differential equations describe this situation?

   A. $\frac{dP}{dt} = \frac{1}{400}(2000 - P) + 200$

   B. $\frac{dP}{dt} = \frac{2}{5}(2000 - P)$

   C. $\frac{dP}{dt} = \frac{1}{900}P(2000 - P)$

   D. $\frac{dP}{dt} = 900P(2000 - P)$

   E. $\frac{dP}{dt} = \frac{1}{400}P(2000 - P)$

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7. The population P of deer in a preserve grows at a rate that is jointly proportional to the size of the deer population and the difference between the deer population and the carrying capacity of the population. If the carrying capacity of the preserve is 3000 deer, which of the following differential equations best models the growth rate of the deer population with respect to time t, where k is a constant?

   A. $\frac{dP}{dt} = 3000k(1 - P)$

   B. $\frac{dP}{dt} = 3000 - kP$

   C. $\frac{dP}{dt} = k(3000 - P)$

   D. $\frac{dP}{dt} = kP\left(1 - \frac{P}{3000}\right)$

   E. $\frac{dP}{dt} = \frac{k}{P}(2000 - P)$

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8. The rate of change, $\frac{dP}{dt}$, of the number of people entering an arena is modeled by a logistic differential equation. The capacity of the arena is 5000 people. At a certain time, the number of people in the arena is 1000 and is increasing at the rate of 500 people per minute. Which of the following differential equations could describe this situation?

   A. $\frac{dP}{dt} = \frac{1}{800}(5000 - P)$

   B. $\frac{dP}{dt} = \frac{1}{500}P(5000 - P)$

   C. $\frac{dP}{dt} = \frac{1}{8000}P(5000 - P)$

   D. $\frac{dP}{dt} = \frac{1}{5000}P(500 - P)$

9. If a certain population is modeled by the function P that satisfies the logistic differential equation $\frac{dP}{dt} = 0.5P\left(1 - \frac{P}{200}\right)$, where t is the time in years and $P(0) = 100$. What is $\lim_{t\to\infty} P(t)$?

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10. The function P satisfies the logistic differential equation $\frac{dP}{dt} = \frac{P}{20}\left(1 - \frac{P}{1700}\right)$, where $P(0) = 210$. Which of the following statements is false?

A. $\lim_{t\to\infty} P(t) = 1700$

B. $\frac{dP}{dt}$ has a maximum value when $P = 210$.

C. $\frac{d^2P}{dt^2} = 0$ when $P = 850$

D. When $P > 850, \frac{dP}{dt} > 0, \frac{d^2P}{dt^2} < 0$

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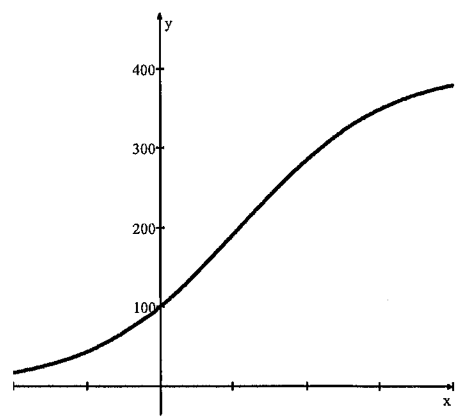
11. Which of the following differential equations for a population P could model the logistic growth shown in the figure?



A. $\frac{dP}{dt} = 0.1P - 0.00025P^2$

B. $\frac{dP}{dt} = 0.1P - 0.025P^2$

C. $\frac{dP}{dt} = 0.1P^2 - 0.00025P$

D. $\frac{dP}{dt} = 0.1P + 0.00025P^2$

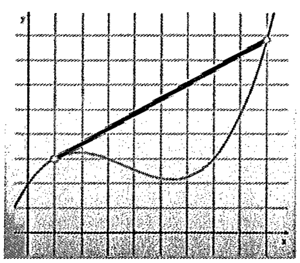
E. $\frac{dP}{dt} = 0.1P + 0.025P^2$

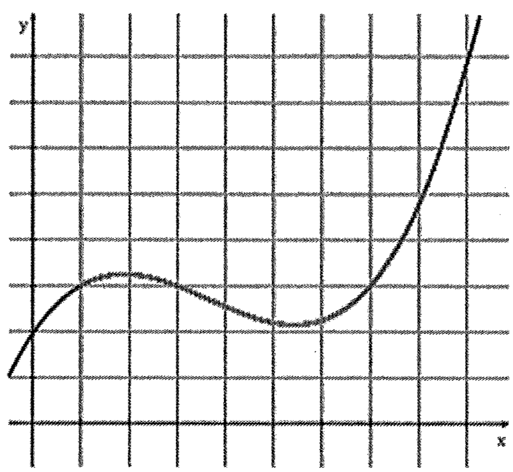
BC Calculus – 8.5 Notes – Arc Length of Curve and Distance Traveled

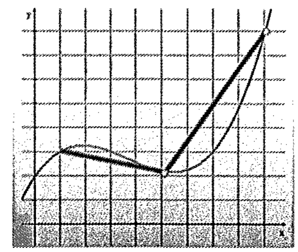
The idea behind finding arc length is very similar to the way we find area using calculus. We are going to divide the curve into a large quantity of small segments, find their lengths and then add them up.

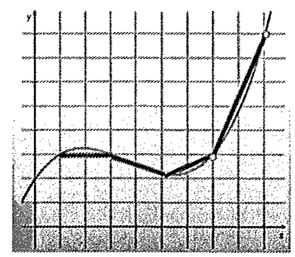
Recall the distance formula: $d =$

We can approximate the length of the graph of a function f by using n line segments whose endpoints are partitioned between $[a, b]$. Think of it this way: $a = x_0$ and $b = x_n$ where $x_0 < x_1 < x_2 < x_3 < \cdots < x_n$.









Arc Length

If a function $y = f(x)$ represents a smooth continuous curve on the closed interval $[a, b]$, the arc length of f between a and b is given by

1. Find the arc length of the graph $y = \frac{1}{12}(x^2 + 8)^{\frac{3}{2}}$ from $x = 1$ to $x = 4$.

Often the problem will only ask that you set up the integral, but we could take this further and use the calculator

2. Set up an integral that represents the length of the curve $y = \sin x$, for $0 \leq x \leq \pi$ and use a calculator to find the value.

Proof:

1. $$L \approx \sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}$$

2. $$L \approx \sum_{i=1}^{n} \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$$

3. $$L \approx \sum_{i=1}^{n} \sqrt{(\Delta x_i)^2 + \left(\frac{\Delta x_i}{\Delta x_i}\right)^2 (\Delta y_i)^2}$$

4. $$L \approx \sum_{i=1}^{n} \sqrt{(\Delta x_i)^2 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2 (\Delta x_i)^2}$$

5. $$L \approx \sum_{i=1}^{n} \sqrt{\left[1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2\right] \cdot (\Delta x_i)^2}$$

6. $$L \approx \sum_{i=1}^{n} \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2} \cdot \Delta x_i$$

7. The approximation improves as we take the number of segments to infinity.
$$L = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2} \cdot \Delta x_i$$

8. Because $f'(x)$ exists for every x in the interval (x_{i-1}, x_i) the MVT guarantees that there is a value c_i in the interval such that
$$\frac{f(x_i) - f(x_{i-1})}{(x_i - x_{i-1})} = f'(c_i)$$

9. $$f'(c_i) = \frac{\Delta y_i}{\Delta x_i}$$

10. $$L = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + (f'(c_i))^2} \cdot \Delta x_i$$

11. Using the definition of integration, we get
$$\int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

<u>Practice Problems:</u>

1. Find an expression for the length of the curve $y = \sin x$ from $x = 0$ to $x = \frac{5\pi}{6}$. Do Not Evaluate.

(28)

2. The length of a curve from $x = 1$ to $x = 3$ is given by $\int_1^3 \sqrt{1 + 4x^2}\, dx$. If the point $(1, 6)$ is on the curve, which of the following could be an equation for this curve?

A. $y = \frac{4}{3}x^3 + x + 1$

B. $y = 4x^2 + 1$

C. $y = x^2 + 5$

D. $y = x^2 - 6$

E. $1 + \frac{4}{3}x^3$

3. **Calculator active.** Suppose $G(x) = \int_0^x \sqrt{\sin(t)}\, dt$, for $0 \le x \le \pi$. What is the length of the arc along the curve $y = G(x)$ for $x = 0$ to $x = \pi/7$.

4. **No Calculator.** Let $g(x) = \sqrt{3x}$ and f be an antiderivative of g.

a. Find $f'(x)$

b. Find an expression for the length of the graph of f from $x = a$ to $x = b$.

c. If $a = 0$ and $b = 8$, find the length of the graph of f from a to b.

5. **Calculator active.** Consider the region bounded by the graphs of $f(x) = x^2 - 4$ and $g(x) = 5$.

a. Write an expression using one or more integrals that could be used to find the perimeter of this region.

b. Find the perimeter.

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6. Find an integral that gives the length of the graph $y = \cos\sqrt{x}$ between $x = a$ and $x = b$, where $0 < a < b$.

7. **Calculator active.** Let f be a function with derivative $f'(x) = \sqrt{x^5 + 1}$. What is the length of the graph of $y = f(x)$ from $x = 0$ to $x = 2.5$?

8. Find an integral that is equal to the length of the curve $f(x) = \frac{5x^3-2x-1}{7}$ from the point $(0, -0.143)$ to the point $(2, 5)$.

9. Find an expression for the length of the graph of $y = e^{3x}$ between $x = 1$ and $x = 3$.

10. **Calculator active.** The trajectory of a ball thrown from a height of 160 meters is given by the equation $y = 160 - \frac{x^2}{40}$ until it hits the water where y is the height of the ball above the water and x is the horizontal distance traveled in meters. Find the distance traveled by the ball from the time it is thrown until it hits the water.

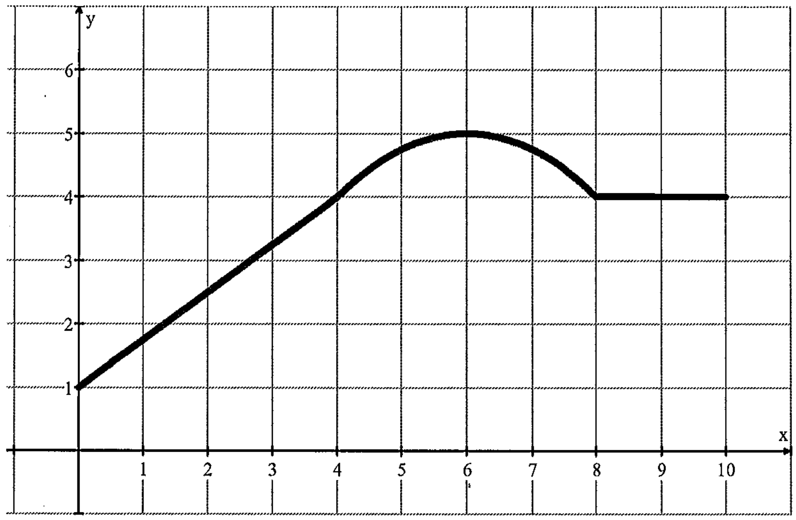
---

11. Which of the following integrals gives the length of the curve $y = \frac{1}{2}x^3$ from $x = 1$ to $x = 3$?

A. $\int_1^3 \sqrt{1 + \frac{1}{4}x^6}\, dx$ B. $\int_1^3 \sqrt{1 + \frac{1}{2}x^6}\, dx$ C. $\int_1^3 \frac{1}{2}\sqrt{4 + 9x^4}\, dx$ D. $\int_1^3 \sqrt{1 + \frac{3}{2}x^4}\, dx$

12. **Calculator active**. What is the length of the curve $y = 1 - \sin x$ from $x = 0$ to $x = 4\pi$?

13.



$$f(x) = \begin{cases} 1 + \dfrac{3}{4}x & \text{for } 0 \le x < 4 \\ 5 - \dfrac{1}{4}(x-6)^2 & \text{for } 4 \le x < 8 \\ 4 & \text{for } 8 \le x \le 10 \end{cases}$$

A mountain hike consists of a steady incline followed by a curved hill and then a flat valley. The mountain hike is modeled by the piecewise-defined function f above, and the graph of f is shown in the figure above. Which of the following expressions gives the total length of the hike from $x = 0$ to $x = 10$.

A. $2 + \int_0^8 \sqrt{1 + \left(\frac{3}{4} - \frac{1}{2}(x-6)\right)^2}\, dx$

B. $2 + \int_0^8 \sqrt{1 + \left(\frac{3}{4}\right)^2} + \sqrt{1 - \frac{1}{4}(x-6)^2}\, dx$

C. $7 + \int_4^8 \sqrt{1 + \left(1 - \frac{1}{2}(x-6)^2\right)^2}\, dx$

D. $7 + \int_4^8 \sqrt{1 + \frac{1}{4}(x-6)^2}\, dx$

EXAMPLE 10 **Finding the Limit of an Indeterminate Form of the Type 1^∞**

Find $\lim_{x\to 0^+}(1+x)^{1/x}$.

Solution

The expression $(1+x)^{1/x}$ is an indeterminate form at 0^+ of the type 1^∞.

Step 1 Let $y=(1+x)^{1/x}$. Then $\ln y = \frac{1}{x}\ln(1+x)$.

Step 2 $$\lim_{x\to 0^+}\ln y = \lim_{x\to 0^+}\frac{\ln(1+x)}{x} \underset{\uparrow}{=} \lim_{x\to 0^+}\frac{\frac{d}{dx}\ln(1+x)}{\frac{d}{dx}x} = \lim_{x\to 0^+}\frac{\frac{1}{1+x}}{1}=1$$

Type $\frac{0}{0}$; use L'Hôpital's Rule

Step 3 Since $\lim_{x\to 0^+}\ln y = 1$, $\lim_{x\to 0^+} y = e^1 = e$. ■

NOW WORK Problem 85.

4.4 Assess Your Understanding

Concepts and Vocabulary

1. *True or False* $\frac{f(x)}{g(x)}$ is an indeterminate form at c of the type $\frac{0}{0}$ if $\lim_{x\to c}\frac{f(x)}{g(x)}$ does not exist.
2. *True or False* If $\frac{f(x)}{g(x)}$ is an indeterminate form at c of the type $\frac{0}{0}$, then L'Hôpital's Rule states that $\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\left[\frac{d}{dx}\frac{f(x)}{g(x)}\right]$.
3. *True or False* $\frac{1}{x}$ is an indeterminate form at 0.
4. *True or False* $x\ln x$ is not an indeterminate form at 0^+ because $\lim_{x\to 0^+} x = 0$ and $\lim_{x\to 0^+}\ln x = -\infty$, and $0\cdot -\infty = 0$.
5. In your own words, explain why $\infty - \infty$ is an indeterminate form, but $\infty + \infty$ is not an indeterminate form.
6. In your own words, explain why $0\cdot\infty \neq 0$.

Skill Building

In Problems 7–26:
(a) Determine whether each expression is an indeterminate form at c.
(b) If it is, identify the type. If it is not an indeterminate form, state why.

[PAGE 292] 7. $\frac{1-e^x}{x}$, $c=0$

8. $\frac{1-e^x}{x-1}$, $c=0$

9. $\frac{e^x}{x}$, $c=0$

10. $\frac{e^x}{x}$, $c=\infty$

11. $\frac{\ln x}{x^2}$, $c=\infty$

12. $\frac{\ln(x+1)}{e^x-1}$, $c=0$

13. $\frac{\sec x}{x}$, $c=0$

14. $\frac{x}{\sec x - 1}$, $c=0$

15. $\frac{\sin x(1-\cos x)}{x^2}$, $c=0$

16. $\frac{\sin x - 1}{\cos x}$, $c=\frac{\pi}{2}$

17. $\frac{\tan x - 1}{\sin(4x-\pi)}$, $c=\frac{\pi}{4}$

18. $\frac{e^x-e^{-x}}{1-\cos x}$, $c=0$

19. x^2e^{-x}, $c=\infty$

20. $x\cot x$, $c=0$

21. $\csc\frac{x}{2}-\cot\frac{x}{2}$, $c=0$

22. $\frac{x}{x-1}+\frac{1}{\ln x}$, $c=1$

23. $\left(\frac{1}{x^2}\right)^{\sin x}$, $c=0$

24. $(e^x+x)^{1/x}$, $c=0$

25. $(x^2-1)^x$, $c=0$

26. $(\sin x)^x$, $c=0$

In Problems 27–42, identify each quotient as an indeterminate form of the type $\frac{0}{0}$ or $\frac{\infty}{\infty}$. Then find the limit.

27. $\lim_{x\to 2}\frac{x^2+x-6}{x^2-3x+2}$

28. $\lim_{x\to 1}\frac{2x^3+5x^2-4x-3}{x^3+x^2-10x+8}$

[PAGE 294] 29. $\lim_{x\to 1}\frac{\ln x}{x^2-1}$

30. $\lim_{x\to 0}\frac{\ln(1-x)}{e^x-1}$

31. $\lim_{x\to 0}\frac{e^x-e^{-x}}{\sin x}$

32. $\lim_{x\to 0}\frac{\tan(2x)}{\ln(1+x)}$

33. $\lim_{x\to 1}\frac{\sin(\pi x)}{x-1}$

34. $\lim_{x\to\pi}\frac{1+\cos x}{\sin(2x)}$

[PAGE 295] 35. $\lim_{x\to\infty}\frac{x^2}{e^x}$

36. $\lim_{x\to\infty}\frac{e^x}{x^4}$

37. $\lim_{x\to\infty}\frac{\ln x}{e^x}$

38. $\lim_{x\to\infty}\frac{x+\ln x}{x\ln x}$

[PAGE 294] 39. $\lim_{x\to 0}\frac{e^x-1-\sin x}{1-\cos x}$

40. $\lim_{x\to 0}\frac{e^x-e^{-x}-2\sin x}{3x^3}$

41. $\lim_{x\to 0}\frac{\sin x - x}{x^3}$

42. $\lim_{x\to 0}\frac{x^3}{\cos x - 1}$

32

In Problems 43–58, identify each expression as an indeterminate form of the type $0\cdot\infty$, $\infty-\infty$, 0^0, 1^∞, or ∞^0. Then find the limit.

43. $\lim_{x\to 0^+}(x^2\ln x)$

44. $\lim_{x\to\infty}(xe^{-x})$

PAGE 297 45. $\lim_{x\to\infty}[x(e^{1/x}-1)]$

46. $\lim_{x\to\pi/2}[(1-\sin x)\tan x]$

PAGE 298 47. $\lim_{x\to\pi/2}(\sec x-\tan x)$

48. $\lim_{x\to 0}\left(\cot x-\frac{1}{x}\right)$

49. $\lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{x}{\ln x}\right)$

50. $\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{e^x-1}\right)$

PAGE 298 51. $\lim_{x\to 0^+}(2x)^{3x}$

52. $\lim_{x\to 0^+}x^{x^2}$

53. $\lim_{x\to\infty}(x+1)^{e^{-x}}$

54. $\lim_{x\to\infty}(1+x^2)^{1/x}$

55. $\lim_{x\to 0^+}(\csc x)^{\sin x}$

56. $\lim_{x\to\infty}x^{1/x}$

57. $\lim_{x\to\pi/2^-}(\sin x)^{\tan x}$

58. $\lim_{x\to 0}(\cos x)^{1/x}$

In Problems 59–88, find each limit.

59. $\lim_{x\to 0^+}\frac{\cot x}{\cot(2x)}$

60. $\lim_{x\to\infty}\frac{\ln(\ln x)}{\ln x}$

PAGE 296 61. $\lim_{x\to 1/2^-}\frac{\ln(1-2x)}{\tan(\pi x)}$

62. $\lim_{x\to 1^-}\frac{\ln(1-x)}{\cot(\pi x)}$

63. $\lim_{x\to\infty}\frac{x^4+x^3}{e^x+1}$

64. $\lim_{x\to\infty}\frac{x^2+x-1}{e^x+e^{-x}}$

65. $\lim_{x\to 0}\frac{xe^{4x}-x}{1-\cos(2x)}$

66. $\lim_{x\to 0}\frac{x\tan x}{1-\cos x}$

67. $\lim_{x\to 0}\frac{\tan^{-1}x}{x}$

68. $\lim_{x\to 0}\frac{\tan^{-1}x}{\sin^{-1}x}$

69. $\lim_{x\to 0}\frac{\cos x-1}{\cos(2x)-1}$

70. $\lim_{x\to 0}\frac{\tan x-\sin x}{x^3}$

71. $\lim_{x\to 0^+}(x^{1/2}\ln x)$

72. $\lim_{x\to\infty}[(x-1)e^{-x^2}]$

73. $\lim_{x\to\pi/2}[\tan x\ln(\sin x)]$

74. $\lim_{x\to 0^+}[\sin x\ln(\sin x)]$

75. $\lim_{x\to 0}[\csc x\ln(x+1)]$

76. $\lim_{x\to\pi/4}[(1-\tan x)\sec(2x)]$

77. $\lim_{x\to a}\left[(a^2-x^2)\tan\left(\frac{\pi x}{2a}\right)\right]$

78. $\lim_{x\to 1^+}\left[(1-x)\tan\left(\frac{1}{2}\pi x\right)\right]$

79. $\lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)$

80. $\lim_{x\to 1}\left(\frac{x}{x-1}-\frac{1}{\ln x}\right)$

81. $\lim_{x\to\pi/2}\left(x\tan x-\frac{\pi}{2}\sec x\right)$

82. $\lim_{x\to\pi}(\cot x-x\csc x)$

83. $\lim_{x\to 1^-}(1-x)^{\tan(\pi x)}$

84. $\lim_{x\to 0^+}x^{\sqrt{x}}$

PAGE 299 85. $\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{1/x}$

86. $\lim_{x\to\infty}\left(1+\frac{5}{x}+\frac{3}{x^2}\right)^x$

87. $\lim_{x\to(\pi/2)^-}(\tan x)^{\cos x}$

88. $\lim_{x\to 0^+}(x^2+x)^{-\ln x}$

Applications and Extensions

89. **Wolf Population** In 2014 there were 229 wolves in Wyoming outside of Yellowstone National Park. Suppose the population w of wolves in the region at time t follows the logistic growth curve

$$w=w(t)=\frac{Ke^{rt}}{\frac{K}{40}+e^{rt}-1}$$

where $K=252$, $r=0.283$, and $t=0$ represents the population in the year 2000.

Source: Federal Wildlife Service.

(a) Find $\lim_{t\to\infty}w(t)$.

(b) Interpret the answer found in (a) in the context of the problem.

(c) Use technology to graph $w=w(t)$.

90. **Skydiving** The downward velocity v of a skydiver with nonlinear air resistance can be modeled by

$$v=v(t)=-A+RA\frac{e^{Bt+C}-1}{e^{Bt+C}+1}$$

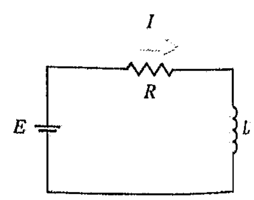
where t is the time in seconds, and A, B, C, and R are positive constants with $R>1$.

(a) Find $\lim_{t\to\infty}v(t)$.

(b) Interpret the limit found in (a).

(c) If the velocity v is measured in feet per second, reasonable values of the constants are $A=108.6$, $B=0.554$, $C=0.804$, and $R=2.62$. Graph the velocity of the skydiver with respect to time.

91. **Electricity** The equation governing the amount of current I (in amperes) in a simple RL circuit consisting of a resistance R (in ohms), an inductance L (in henrys), and an electromotive force E (in volts) is $I=\frac{E}{R}(1-e^{-Rt/L})$.



(a) Find $\lim_{t\to\infty}I(t)$ and $\lim_{R\to 0^+}I(t)$.

(b) Interpret these limits.

92. Find $\lim_{x\to 0}\frac{a^x-b^x}{x}$, where $a\neq 1$ and $b\neq 1$ are positive real numbers.

93. Show that $\lim_{x\to\infty}\frac{\ln x}{x^n}=0$, for $n\geq 1$ an integer.

94. Show that $\lim_{x\to\infty}\frac{x^n}{e^x}=0$ for $n\geq 1$ an integer.

95. Show that $\lim_{x\to 0^+}(\cos x+2\sin x)^{\cot x}=e^2$.

96. Find $\lim_{x\to\infty}\frac{P(x)}{e^x}$, where P is a polynomial function.

97. Find $\lim_{x\to\infty}[\ln(x+1)-\ln(x-1)]$.

98. Show that $\lim_{x\to 0^+}\frac{e^{-1/x^2}}{x}=0$. *Hint:* Write $\frac{e^{-1/x^2}}{x}=\frac{\frac{1}{x}}{e^{1/x^2}}$.

99. If n is an integer, show that $\lim_{x\to 0^+}\frac{e^{-1/x^2}}{x^n}=0$.

100. Show that $\lim_{x\to\infty}\sqrt[x]{x}=1$.

- When n is odd, repeated applications lead eventually to

$$\int \sec x\,dx = \ln|\sec x + \tan x| + C$$

For example, if $n = 3$,

$$\int \sec^3 x\,dx = \frac{\sec x\,\tan x}{2} + \frac{1}{2}\int \sec x\,dx = \frac{\sec x\,\tan x}{2} + \frac{1}{2}\ln|\sec x + \tan x| + C$$

NOW WORK Problem 67.

6.6 Assess Your Understanding

Concepts and Vocabulary

1. *True or False* Integration by parts is based on the Product Rule for derivatives.
2. The integration by parts formula states that $\int u\,dv =$ ______.

Skill Building

In Problems 3–34, use integration by parts to find each integral.

PAGE 464 3. $\int xe^{2x}\,dx$

4. $\int xe^{-3x}\,dx$

PAGE 465 5. $\int x\cos x\,dx$

6. $\int x\sin(3x)\,dx$

7. $\int \sqrt{x}\ln x\,dx$

8. $\int x^{-2}\ln x\,dx$

PAGE 466 9. $\int \cot^{-1} x\,dx$

10. $\int \sin^{-1} x\,dx$

11. $\int (\ln x)^2\,dx$

12. $\int x(\ln x)^2\,dx$

PAGE 467 13. $\int x^2\sin x\,dx$

14. $\int x^2\cos x\,dx$

15. $\int x\cos^2 x\,dx$

16. $\int x\sin^2 x\,dx$

PAGE 465 17. $\int x^2\ln x\,dx$

18. $\int \frac{(\ln x)^2}{x^2}\,dx$

19. $\int \frac{xe^x}{(x+1)^2}\,dx$

20. $\int \frac{xe^{3x}}{(3x+1)^2}\,dx$

21. $\int \sin(\ln x)\,dx$

22. $\int \cos(\ln x)\,dx$

23. $\int (\ln x)^3\,dx$

24. $\int (\ln x)^4\,dx$

25. $\int x^2(\ln x)^2\,dx$

26. $\int x^3(\ln x)^2\,dx$

27. $\int x^2\tan^{-1} x\,dx$

28. $\int x\tan^{-1} x\,dx$

29. $\int 7^x x\,dx$

30. $\int 2^{-x} x\,dx$

PAGE 467 31. $\int e^{-x}\cos(2x)\,dx$

32. $\int e^{-2x}\sin(3x)\,dx$

33. $\int e^{2x}\sin x\,dx$

34. $\int e^{3x}\cos(5x)\,dx$

In Problems 35–44, use integration by parts to find each definite integral.

35. $\int_0^{\pi} e^x\cos x\,dx$

36. $\int_0^{\pi/2} e^{-x}\sin x\,dx$

37. $\int_0^2 x^2 e^{-3x}\,dx$

38. $\int_0^1 x^2 e^{-x}\,dx$

39. $\int_0^{\pi/4} x\sec x\tan x\,dx$

40. $\int_0^{\pi/4} x\tan^2 x\,dx$

41. $\int_1^9 \ln\sqrt{x}\,dx$

42. $\int_{\pi/4}^{3\pi/4} x\csc^2 x\,dx$

PAGE 468 43. $\int_1^e (\ln x)^2\,dx$

44. $\int_0^{\pi/4} x\sec^2 x\,dx$

Applications and Extensions

PAGE 469 45. **Area under a Graph** Find the area under the graph of $y = e^x\sin x$ from 0 to π.

46. **Area under a Graph** Find the area under the graph of $y = x\cos x$ from $x = 0$ to $x = \frac{\pi}{2}$.

47. **Area under a Graph** Find the area under the graph of $y = xe^{-x}$ from $x = 0$ to $x = 1$.

48. **Area under a Graph** Find the area under the graph of $y = xe^{3x}$ from $x = 0$ to $x = 2$.

49. **Rectilinear Motion** The acceleration $a = a(t)$ of an object in rectilinear motion is given by

$$a(t) = e^{-2t}\sin t \text{ m/s}^2$$

(a) Find the velocity $v = v(t)$ of the object if the initial velocity is $v(0) = 8$ m/s.

(b) Find the position $s = s(t)$ of the object if the initial position is $s(0) = 0$ m.

50. **Rectilinear Motion** The acceleration a of an object in rectilinear motion is given by

$$a(t) = t^2 e^{-t} \text{ ft/s}^2$$

(a) Find the velocity $v = v(t)$ of the object if the initial velocity is $v(0) = 5$ ft/s.

(b) Find the position $s = s(t)$ of the object if the initial position is $s(0) = 0$ ft.

34

In the third integral on the right in (3), use the trigonometric substitution $x = 2\tan\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then $dx = 2\sec^2\theta\, d\theta$, and

$$\int \frac{dx}{(x^2+4)^2} = \int \frac{2\sec^2\theta\, d\theta}{(4\tan^2\theta + 4)^2} = \frac{2}{16}\int \frac{\sec^2\theta\, d\theta}{(\sec^2\theta)^2}$$

$$= \frac{1}{8}\int \cos^2\theta\, d\theta = \frac{1}{8}\int \frac{1+\cos(2\theta)}{2}\, d\theta$$

$$= \frac{1}{16}\left[\theta + \frac{1}{2}\sin(2\theta)\right] = \frac{1}{16}(\theta + \sin\theta\cos\theta)$$

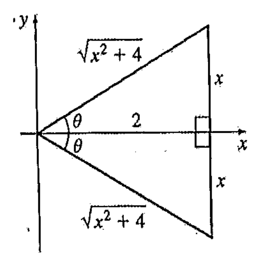


Figure 10 $\tan\theta = \frac{x}{2}$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

To express the solution in terms of x, either use the triangles in Figure 10 or use trigonometric identities as follows:

$$\sin\theta\cos\theta = \frac{\sin\theta}{\cos\theta}\cdot\cos^2\theta = \frac{\tan\theta}{\sec^2\theta} = \frac{\tan\theta}{\tan^2\theta + 1} = \frac{\frac{x}{2}}{\frac{x^2}{4}+1} = \frac{2x}{x^2+4}$$

Then

$$\int \frac{dx}{(x^2+4)^2} = \frac{1}{16}(\theta + \sin\theta\cos\theta) \underset{\substack{\uparrow \\ x = 2\tan\theta;\ \theta = \tan^{-1}\frac{x}{2}}}{=} \frac{1}{16}\left(\tan^{-1}\frac{x}{2} + \frac{2x}{x^2+4}\right) \qquad (6)$$

Now combine the results of (3), (4), (5), and (6):

$$\int \frac{x^3+1}{(x^2+4)^2}dx = \frac{1}{2}\ln(x^2+4) + \frac{2}{x^2+4} + \frac{1}{16}\tan^{-1}\frac{x}{2} + \frac{x}{8(x^2+4)} + K_1 \quad ■$$

NOTE Case 4 type integrands lead to sums of natural logarithms, inverse tangents, and/or rational functions.

NOW WORK Problem 17.

6.10 Assess Your Understanding

Concepts and Vocabulary

1. *True or False* The integration of a proper rational function always leads to a logarithm.
2. *True or False* The decomposition of $\frac{7x+1}{(x+1)^4}$ into partial fractions has three terms: $\frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}$, where A, B, and C are real numbers.

Skill Building

In Problems 3–8, find each integral.
Hint: Each of the denominators contains only distinct linear factors.

3. $\int \frac{dx}{(x-2)(x+1)}$
4. $\int \frac{dx}{(x+4)(x-1)}$
5. (PAGE 497) $\int \frac{x\,dx}{(x-1)(x-2)}$
6. $\int \frac{3x\,dx}{(x+2)(x-4)}$
7. $\int \frac{x\,dx}{(3x-2)(2x+1)}$
8. $\int \frac{dx}{(2x+3)(4x-1)}$

In Problems 9–12, find each integral.
Hint: Each of the denominators contains a repeated linear factor.

9. (PAGE 499) $\int \frac{x-3}{(x+2)(x+1)^2}dx$
10. $\int \frac{x+1}{x^2(x-2)}dx$
11. $\int \frac{x^2\,dx}{(x-1)^2(x+1)}$
12. $\int \frac{x^2+x}{(x+2)(x-1)^2}dx$

In Problems 13–16, find each integral.
Hint: Each of the denominators contains an irreducible quadratic factor.

13. $\int \frac{dx}{x(x^2+1)}$
14. $\int \frac{dx}{(x+1)(x^2+4)}$
15. (PAGE 501) $\int \frac{x^2+2x+3}{(x+1)(x^2+2x+4)}dx$
16. $\int \frac{x^2-11x-18}{x(x^2+3x+3)}dx$

In Problems 17–20, find each integral.
Hint: Each of the denominators contains a repeated irreducible quadratic factor.

17. $\int \frac{2x+1}{(x^2+16)^2}\,dx$

18. $\int \frac{x^2+2x+3}{(x^2+4)^2}\,dx$

19. $\int \frac{x^3\,dx}{(x^2+16)^3}$

20. $\int \frac{x^2\,dx}{(x^2+4)^3}$

In Problems 21–30, find each integral.

21. $\int \frac{x\,dx}{x^2+2x-3}$

22. $\int \frac{x^2-x-8}{(x+1)(x^2+5x+6)}\,dx$

23. $\int \frac{10x^2+2x}{(x-1)^2(x^2+2)}\,dx$

24. $\int \frac{x+4}{x^2(x^2+4)}\,dx$

25. $\int \frac{7x+3}{x^3-2x^2-3x}\,dx$

26. $\int \frac{x^5+1}{x^6-x^4}\,dx$

27. $\int \frac{x^2}{(x-2)(x-1)^2}\,dx$

28. $\int \frac{x^2+1}{(x+3)(x-1)^2}\,dx$

29. $\int \frac{2x+1}{x^3-1}\,dx$

30. $\int \frac{dx}{x^3-8}$

In Problems 31–34, find each definite integral.

31. $\int_0^1 \frac{dx}{x^2-9}$

32. $\int_2^4 \frac{dx}{x^2-25}$

33. $\int_{-2}^3 \frac{dx}{16-x^2}$

34. $\int_1^2 \frac{dx}{9-x^2}$

Applications and Extensions

In Problems 35–48, find each integral.
Hint: Make a substitution before using partial fraction decomposition.

35. $\int \frac{\cos\theta}{\sin^2\theta+\sin\theta-6}\,d\theta$

36. $\int \frac{\sin x}{\cos^2 x-2\cos x-8}\,dx$

37. $\int \frac{\sin\theta}{\cos^3\theta+\cos\theta}\,d\theta$

38. $\int \frac{4\cos\theta}{\sin^3\theta+2\sin\theta}\,d\theta$

39. $\int \frac{e^t}{e^{2t}+e^t-2}\,dt$

40. $\int \frac{e^x}{e^{2x}+e^x-6}\,dx$

41. $\int \frac{e^x}{e^{2x}-1}\,dx$

42. $\int \frac{dx}{e^x-e^{-x}}$

43. $\int \frac{dt}{e^{2t}+1}$

44. $\int \frac{dt}{e^{3t}+e^t}$

45. $\int \frac{\sin x\cos x}{(\sin x-1)^2}\,dx$

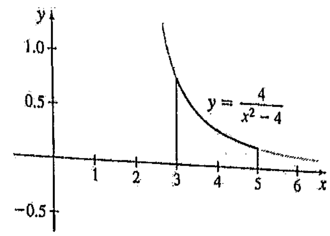
46. $\int \frac{\cos x\sin x}{(\cos x-2)^2}\,dx$

47. $\int \frac{\cos x}{(\sin^2 x+9)^2}\,dx$

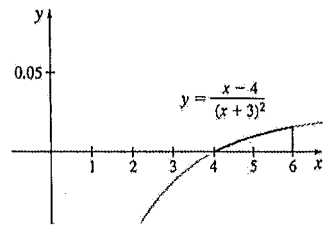
48. $\int \frac{\sin x}{(\cos^2 x+4)^2}\,dx$

35

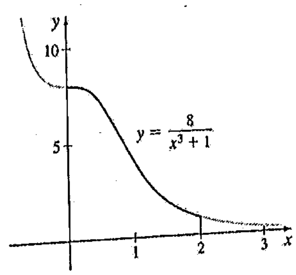
49. **Area** Find the area under the graph of $y=\frac{4}{x^2-4}$ from $x=3$ to $x=5$, as shown in the figure below.



50. **Area** Find the area under the graph of $y=\frac{x-4}{(x+3)^2}$ from $x=4$ to $x=6$, as shown in the figure below.



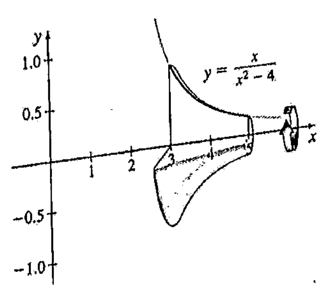
51. **Area** Find the area under the graph of $y=\frac{8}{x^3+1}$ from $x=0$ to $x=2$, as shown in the figure below.



52. **Volume of a Solid of Revolution** See the figure below. The volume V of the solid of revolution generated by revolving the region bounded by the graph of $y=\frac{x}{x^2-4}$ and the x-axis from $x=3$ to $x=5$ about the x-axis is given by

$$V=\pi\int_3^5\left(\frac{x}{x^2-4}\right)^2 dx$$

Find V.



You are asked to prove the theorem in Problem 96. The proof follows from a property of definite integrals: If the functions f and g are continuous on a closed interval $[a, b]$ and if $f(x) \geq g(x)$ on $[a, b]$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$. (See Section 6.4, Problem 128, p. 436.)

EXAMPLE 7 **Using the Comparison Test for Improper Integrals**

Determine whether $\int_1^\infty e^{-x^2}\,dx$ converges or diverges.

Solution

By definition, $\int_1^\infty e^{-x^2}\,dx = \lim_{b\to\infty} \int_1^b e^{-x^2}\,dx$ converges if the limit exists and equals a real number. Since e^{-x^2} has no antiderivative, we use the Comparison Test for Improper Integrals. We proceed as follows: For $x \geq 1$,

$$x^2 \geq x$$
$$-x^2 \leq -x$$
$$0 < e^{-x^2} \leq e^{-x} \qquad \text{Since } e > 1, \text{ if } a \leq b, \text{ then } e^a \leq e^b.$$

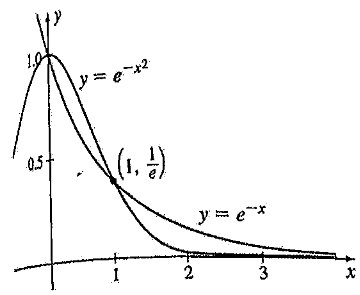


Figure 23

Figure 23 illustrates this.

Based on the Comparison Test for Improper Integrals, if $\int_1^\infty e^{-x}\,dx$ converges, so does $\int_1^\infty e^{-x^2}\,dx$. We investigate $\int_1^\infty e^{-x}\,dx$.

$$\int_1^\infty e^{-x}\,dx: \lim_{b\to\infty} \int_1^b e^{-x}\,dx = \lim_{b\to\infty} \left[-e^{-x}\right]_1^b = \lim_{b\to\infty} [-e^{-b} + e^{-1}] = \lim_{b\to\infty} \left[\frac{1}{e} - \frac{1}{e^b}\right]$$
$$= \lim_{b\to\infty} \frac{1}{e} - \lim_{b\to\infty} \frac{1}{e^b} = \frac{1}{e}$$

Since $\int_1^\infty e^{-x}\,dx$ converges, we conclude that $\int_1^\infty e^{-x^2}\,dx$ converges. ■

Notice that the Comparison Test for Improper Integrals does not give the value of $\int_1^\infty e^{-x^2}\,dx$. To find $\int_1^\infty e^{-x^2}\,dx$ requires numerical techniques. The Comparison Test does, however, tell us that $0 \leq \int_1^\infty e^{-x^2}\,dx \leq \frac{1}{e}$.

NOW WORK Problem 67.

6.12 Assess Your Understanding

Concepts and Vocabulary

1. *Multiple Choice* If a function f is continuous on the interval $[a, \infty)$, then $\int_a^\infty f(x)\,dx$ is called

   (a) a definite **(b)** an infinite **(c)** an improper **(d)** a proper integral.

2. *Multiple Choice* If the $\lim_{b\to\infty} \int_a^b f(x)\,dx$ does not exist, the improper integral $\int_a^\infty f(x)\,dx$

   (a) converges **(b)** diverges **(c)** equals ab.

3. *True or False* If a function f is continuous for all x, then the improper integral $\int_{-\infty}^\infty f(x)\,dx$ always converges.

4. *True or False* If a function f is continuous and nonnegative on the interval $[a, \infty)$, and $\int_a^\infty f(x)\,dx$ converges, then $\int_a^\infty f(x)\,dx$ equals the area under the graph of $y = f(x)$ for $x \geq a$.

5. *True or False* If a function f is continuous for all x, then the improper integral $\int_{-\infty}^\infty f(x)\,dx = \lim_{a\to\infty} \int_{-a}^a f(x)\,dx$.

6. To determine whether the improper integral $\int_a^b f(x)\,dx$ converges or diverges, where f is continuous on $[a, b)$, but is not defined at b, requires finding what limit?

Skill Building

In Problems 7–14, determine whether each integral is improper. For those that are improper, state the reason.

7. $\int_0^\infty x^2\,dx$ **8.** $\int_0^5 x^3\,dx$ **9.** $\int_2^3 \frac{dx}{x-1}$

10. $\int_1^2 \frac{dx}{x-1}$ **11.** $\int_0^1 \frac{1}{x}\,dx$ **12.** $\int_{-1}^1 \frac{x\,dx}{x^2+1}$

13. $\int_0^1 \frac{x}{x^2-1}\,dx$ **14.** $\int_0^\infty e^{-2x}\,dx$

In Problems 15–24, determine whether each improper integral converges or diverges. If it converges, find its value.

15. $\int_1^{\infty} \frac{dx}{x^3}$ **16.** $\int_{-\infty}^{-10} \frac{dx}{x^2}$ PAGE 518 **17.** $\int_0^{\infty} e^{2x}\,dx$

18. $\int_0^{\infty} e^{-x}\,dx$ **19.** $\int_{-\infty}^{-1} \frac{4}{x}\,dx$ **20.** $\int_1^{\infty} \frac{4}{x}\,dx$

21. $\int_3^{\infty} \frac{dx}{(x-1)^4}$ **22.** $\int_{-\infty}^{0} \frac{dx}{(x-1)^4}$ PAGE 519 **23.** $\int_{-\infty}^{\infty} \frac{dx}{x^2+4}$

24. $\int_{-\infty}^{\infty} \frac{dx}{x^2+1}$

In Problems 25–32, determine whether each improper integral converges or diverges. If it converges, find its value.

25. $\int_0^1 \frac{dx}{x^2}$ **26.** $\int_0^1 \frac{dx}{x^3}$ PAGE 521 **27.** $\int_0^1 \frac{dx}{x}$

28. $\int_4^6 \frac{dx}{x-4}$ **29.** $\int_0^4 \frac{dx}{\sqrt{4-x}}$ **30.** $\int_1^5 \frac{x\,dx}{\sqrt{5-x}}$

PAGE 522 **31.** $\int_{-1}^1 \frac{dx}{\sqrt[3]{x}}$ **32.** $\int_0^3 \frac{dx}{(x-2)^2}$

In Problems 33–62, determine whether each improper integral converges or diverges. If it converges, find its value.

33. $\int_0^{\infty} \cos x\,dx$ **34.** $\int_0^{\infty} \sin(\pi x)\,dx$

35. $\int_{-\infty}^{0} e^x\,dx$ **36.** $\int_{-\infty}^{0} e^{-x}\,dx$

37. $\int_0^{\pi/2} \frac{x\,dx}{\sin x^2}$ **38.** $\int_0^1 \frac{\ln x\,dx}{x}$

39. $\int_0^1 \frac{dx}{1-x^2}$ **40.** $\int_1^2 \frac{dx}{\sqrt{x^2-1}}$

41. $\int_0^1 \frac{x}{\sqrt{1-x^2}}\,dx$ **42.** $\int_0^4 \frac{dx}{\sqrt{4-x}}$

PAGE 521 **43.** $\int_0^{\pi/4} \tan(2x)\,dx$ **44.** $\int_0^{\pi/2} \csc x\,dx$

45. $\int_0^{\infty} \frac{x\,dx}{(x+1)^{5/2}}$ **46.** $\int_2^{\infty} \frac{dx}{x\sqrt{x^2-1}}$

47. $\int_{-\infty}^{\infty} \frac{dx}{x^2+4x+5}$ **48.** $\int_{-\infty}^{\infty} \frac{dx}{e^x+e^{-x}}$

49. $\int_{-\infty}^{2} \frac{dx}{\sqrt{4-x}}$ **50.** $\int_{-\infty}^{1} \frac{x\,dx}{\sqrt{2-x}}$

51. $\int_2^4 \frac{2x\,dx}{\sqrt[3]{x^2-4}}$ **52.** $\int_0^{\pi} \frac{1}{1-\cos x}\,dx$

53. $\int_{-1}^1 \frac{1}{x^3}\,dx$ **54.** $\int_0^2 \frac{dx}{x-1}$

55. $\int_0^2 \frac{dx}{(x-1)^{1/3}}$ **56.** $\int_{-1}^1 \frac{dx}{x^{5/3}}$ **57.** $\int_1^2 \frac{dx}{(2-x)^{3/4}}$

58. $\int_0^4 \frac{dx}{\sqrt{8x-x^2}}$ **59.** $\int_1^3 \frac{2x\,dx}{(x^2-1)^{3/2}}$ **60.** $\int_0^3 \frac{x\,dx}{(9-x^2)^{3/2}}$

61. $\int_0^{\infty} xe^{-x^2}\,dx$ **62.** $\int_0^{\infty} e^{-x}\sin x\,dx$

In Problems 63–70:

(a) Use the Comparison Test for Improper Integrals to determine whether each improper integral converges or diverges. Hint: Use the fact that $\int_1^{\infty} \frac{dx}{x^p}$ *converges if* $p>1$ *and diverges if* $p \le 1$.

CAS *(b) If the integral converges, use a CAS to find its value.*

63. $\int_1^{\infty} \frac{1}{\sqrt{x^2-1}}\,dx$ **64.** $\int_2^{\infty} \frac{2}{\sqrt{x^2-4}}\,dx$

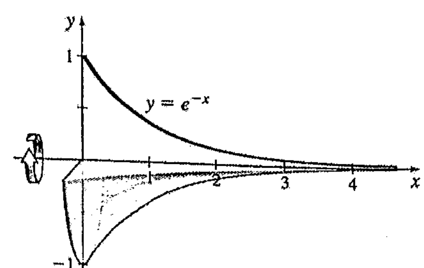
65. $\int_1^{\infty} \frac{1+e^{-x}}{x}\,dx$ **66.** $\int_1^{\infty} \frac{3e^{-x}}{x}\,dx$

PAGE 523 **67.** $\int_1^{\infty} \frac{\sin^2 x}{x^2}\,dx$ **68.** $\int_1^{\infty} \frac{\cos^2 x}{x^2}\,dx$

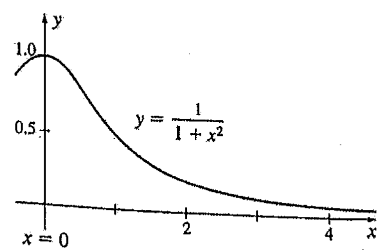
69. $\int_1^{\infty} \frac{dx}{(x+1)\sqrt{x}}$ **70.** $\int_1^{\infty} \frac{dx}{x\sqrt{1+x^2}}$

Applications and Extensions

71. Volume of a Solid of Revolution The volume V, if it is defined, of the solid of revolution generated by revolving the region bounded by the graph of $y=e^{-x}$ and the x-axis to the right of $x=0$ about the x-axis is given by the improper integral $V=\pi\int_0^{\infty}(e^{-x})^2dx$. Find the volume V of the solid, if it exists.



72. Area under a Graph Find the area, if it is defined, under the graph of $y=\frac{1}{1+x^2}$ to the right of $x=0$. See the figure below.



Solution

Begin the first approximation using the boundary condition $x_0 = 1$ and $y_0 = 2$, with $h = 0.25$. Then $x_1 = x_0 + h = 1.25$ and

$$\begin{aligned} y(x_1) = y(1.25) \approx y_1 &= y_0 + hf(x_0, y_0) \\ &= 2 + 0.25(1 \cdot 2 + 1^2) \qquad f(x, y) = xy + x^2 \\ &= 2.75 \end{aligned}$$

For the second approximation, we use $x_1 = 1.25$ and $y_1 = 2.75$ with $h = 0.25$. Then $x_2 = x_1 + h = x_1 + 0.25 = 1.5$, and

$$\begin{aligned} y(x_2) = y(1.5) \approx y_2 &= y_1 + hf(x_1, y_1) \\ &= 2.75 + 0.25[(1.25)(2.75) + 1.25^2] \\ &= 4 \end{aligned}$$

For the third approximation y_3, we use $x_2 = 1.5$ and $y_2 = 4$ with $h = 0.25$. Then $x_3 = x_2 + h = x_2 + 0.25 = 1.75$, and

$$\begin{aligned} y(x_3) = y(1.75) \approx y_3 &= y_2 + hf(x_2, y_2) \\ &= 4 + 0.25[(1.5)(4) + 1.5^2] \\ &= 6.0625 \end{aligned}$$

■

NOW WORK Problem 3 and AP® Practice Problems 1 and 2.

7.4 Assess Your Understanding

Concepts and Vocabulary

1. *True or False* Euler's method is used to approximate a particular solution of a first-order differential equation.
2. *True or False* When using Euler's method, the increment h used remains constant for each approximation.

Skill Building

In Problems 3–8, use Euler's method to approximate the particular solution to the differential equation with the given initial condition. Assume $y = y(x)$ is a solution to the differential equation.

3. Approximate $y(0.4)$ if $\frac{dy}{dx} = x^2 - y$ and $y = 1$ when $x = 0$. Use $h = 0.2$ as the increment.
4. Approximate $y(0.2)$ if $\frac{dy}{dx} = y^2 - x$ and $y = 1$ when $x = 0$. Use $h = 0.1$ as the increment.
5. Approximate $y(1.4)$ if $\frac{dy}{dx} = xy^2 - x$ and $y = 2$ when $x = 1$. Use $h = 0.2$ as the step size.
6. Approximate $y(1.2)$ if $\frac{dy}{dx} = xy^2 - x$ and $y = 2$ when $x = 1$. Use $h = 0.1$ as the step size.
7. Approximate $y(0.6)$ if $\frac{dy}{dx} = 1 - x - 2y$ and $y = 3$ when $x = 1$. Use $h = -0.2$ as the step size.
8. Approximate $y(0.7)$ if $\frac{dy}{dx} = 1 - x - 2y$ and $y = 3$ when $x = 1$. Use $h = -0.1$ as the step size.

Applications and Extensions

9. Suppose $\frac{dy}{dx} = \cos(\pi x)$ with the boundary condition, if $x = 0$, then $y = 1$.
   (a) Use Euler's method with step size 0.2 to approximate $y(1)$.
   (b) Find the particular solution to the differential equation.
   (c) Use the solution from (b) to evaluate $y(1)$.

Preparing for the AP® Exam

AP® Practice Problems

1. Suppose $y = f(x)$ is the solution of the differential equation $\frac{dy}{dx} = x + 2y$ with the initial condition $y(0) = 1$. Approximate $f(0.3)$ using Euler's method with $(x_0, y_0) = (0, 1)$ and using $h = 0.1$ as the increment.
   (A) 1.20 (B) 1.76 (C) 2.03 (D) 2.78
2. Suppose $y = f(x)$ is the solution to the differential equation $\frac{dy}{dx} = 3x - 2y$ with the boundary condition $f(1) = 5$. What is the approximation for $f(1.2)$ using Euler's method starting at $x = 1$ and using two steps of equal size?

7.5 Assess Your Understanding

Concepts and Vocabulary

1. *True or False* An uninhibited growth model can be used to approximate a logistic growth model when the population P is close to the carrying capacity M.

2. *Multiple Choice* In a logistic differential equation, $\frac{dy}{dx} = \frac{k}{M}P(M - P)$, k represents the
(a) carrying capacity (b) average growth rate
(c) maximum population growth rate
(d) location of the inflection point

3. *True or False* All logistic curves have two horizontal asymptotes.

4. *Multiple Choice* In a logistic model, the population P is growing most rapidly when
(a) $t = 0$ (b) P is close to 0
(c) P is close to the carrying capacity M (d) $P = \frac{M}{2}$

Skill Building

In Problems 5–10, for each logistic function P, identify

(a) the carrying capacity M.
(b) the maximum population growth rate k.
(c) the initial population $P_0 = P(0)$.
(d) the population P when P is growing most rapidly.

5. $P(t) = \frac{5500}{1 + 99e^{-0.02t}}$
6. $P(t) = \frac{120{,}000}{1 + 4999e^{-0.3t}}$
7. $P(t) = \frac{660}{1 + 32e^{-0.05t}}$
8. $P(t) = \frac{6800}{1 + 16e^{-0.2t}}$
9. $P(t) = \frac{150}{1 + 2e^{-0.2t}}$
10. $P(t) = \frac{8000}{1 + 63e^{-0.04t}}$

In Problems 11–20, the given logistic differential equation models the rate of change of a population P with respect to time t. For each differential equation, find

(a) the carrying capacity M.
(b) the maximum population growth rate k.
(c) the population when P is growing most rapidly.

11. $\frac{dP}{dt} = 0.12P\left(1 - \frac{P}{1000}\right)$
12. $\frac{dP}{dt} = 0.03P\left(1 - \frac{P}{570}\right)$
13. $\frac{dP}{dt} = 0.25P\left(1 - \frac{P}{4000}\right)$
14. $\frac{dP}{dt} = 0.08P\left(1 - \frac{P}{1600}\right)$
15. $\frac{dP}{dt} = 0.002P\left(1 - \frac{P}{2800}\right)$
16. $\frac{dP}{dt} = 0.005P\left(1 - \frac{P}{15{,}000}\right)$
17. $\frac{dP}{dt} = 0.00003P(1000 - P)$
18. $\frac{dP}{dt} = 0.0005P(800 - P)$
19. $\frac{dP}{dt} = 0.0002P(1200 - P)$
20. $\frac{dP}{dt} = 0.00005P(4000 - P)$

In Problems 21 and 22, write the logistic differential equation $\frac{dP}{dt}$ whose solution is P.

21. $P(t) = \frac{2100}{1 + 29e^{-0.15t}}$
22. $P(t) = \frac{6000}{1 + 499e^{-0.8t}}$

In Problems 23–28, solve each logistic differential equation for the initial condition P(0).

23. $\frac{dP}{dt} = 0.2P\left(1 - \frac{P}{800}\right)$; $P(0) = 40$
24. $\frac{dP}{dt} = 0.03P\left(1 - \frac{P}{1200}\right)$; $P(0) = 25$
25. $\frac{dP}{dt} = 0.4P\left(1 - \frac{P}{680}\right)$; $P(0) = 20$
26. $\frac{dP}{dt} = 0.02P\left(1 - \frac{P}{500}\right)$; $P(0) = 8$
27. $\frac{dP}{dt} = 0.00075P(400 - P)$; $P(0) = 5$
28. $\frac{dP}{dt} = 0.00025P(1000 - P)$; $P(0) = 200$

Applications and Extensions

In Problems 29 and 30, for each logistic function find the population when it is growing most rapidly.

29. $P(t) = \frac{100}{1 + 2e^{-0.2t}}$
30. $P(t) = \frac{60}{1 + 4e^{-0.05t}}$

31. (a) Verify that $P(t) = \frac{500}{1 + 4e^{-0.2t}}$ is a solution to the logistic differential equation $\frac{dP}{dt} = 0.2P\left(1 - \frac{P}{500}\right)$.
(b) Find the initial condition $P_0 = P(0)$ that yields the solution to (a).

32. **Farmers** The number W of farm workers in the United States t years after 1910 follows the logistic decay model

$$\frac{dW}{dt} = -0.057W\left(1 - \frac{W}{14{,}656{,}248}\right)$$

where $W = 13{,}839{,}705$ when $t = 0$.
(a) What is the carrying capacity?
(b) What is the maximum yearly decay rate?
(c) What is the number of farm workers in the United States at the inflection point?

Source: U.S. Department of Agriculture

33. **Spread of Flu** Suppose one person with the flu is placed in a group of 49 people without the flu. The rate of change of those with the flu with respect to time t (in days) is proportional to the product of the number of those with the flu and the number without flu.
(a) If P is the number of people with the flu and the maximum daily population growth rate is 15%, write a differential equation that models the experiment.
(b) Write the logistic function that satisfies the differential equation.
(c) Find the time it takes for $\frac{1}{2}$ the population to become infected.
(d) How long does it take for 80% of the population to become infected?

Use the substitution $u=4+9y$. Then $du=9\,dy$ or, equivalently, $dy=\frac{du}{9}$. The limits of integration are $u=4$ when $y=0$, and $u=13$ when $y=1$.

$$s_1=\frac{1}{2}\int_4^{13}\sqrt{u}\,\frac{du}{9}=\frac{1}{18}\left[\frac{u^{3/2}}{\frac{3}{2}}\right]_4^{13}=\frac{1}{27}(13\sqrt{13}-8)$$

Now we investigate $x=g_2(y)=y^{3/2}$. Since $g_2'(y)=\frac{3}{2}y^{1/2}$ is continuous for all $y\geq 0$, we can use arc length formula (2) to find the arc length s_2 of g_2 from $y=0$ to $y=4$.

$$s_2=\int_0^4\sqrt{1+[g_2'(y)]^2}\,dy=\int_0^4\sqrt{1+\left(\frac{3}{2}y^{1/2}\right)^2}\,dy$$

$$=\int_0^4\sqrt{1+\frac{9}{4}y}\,dy=\frac{1}{2}\int_0^4\sqrt{4+9y}\,dy \qquad \text{Let } u=4+9y,\ du=9\,dy$$

$$=\frac{1}{2}\int_4^{40}\sqrt{u}\,\frac{du}{9}=\frac{1}{18}\left[\frac{u^{3/2}}{\frac{3}{2}}\right]_4^{40}=\frac{1}{27}(80\sqrt{10}-8)$$

The arc length s of $y=f(x)=x^{2/3}$ from $x=-1$ to $x=8$ is the sum

$$s=s_1+s_2=\frac{1}{27}(13\sqrt{13}-8)+\frac{1}{27}(80\sqrt{10}-8)$$
$$=\frac{1}{27}(80\sqrt{10}+13\sqrt{13}-16)$$

■

NOW WORK Problem 23 and AP® Practice Problem 4.

Summary

Suppose $y=f(x)$ is a function that is continuous on a closed interval $[a, b]$, and suppose the derivative f' of f is continuous on some interval containing a and b.

- **Arc Length:** The arc length s of the graph of f from a to b is

$$s=\int_a^b\sqrt{1+[f'(x)]^2}\,dx$$

8.5 Assess Your Understanding

Concepts and Vocabulary

1. *True or False* If a function f has a derivative that is continuous on an interval containing a and b, the arc length s of the graph of $y=f(x)$ from $x=a$ to $x=b$ is given by the formula $s=\int_a^b\sqrt{1+[f(x)]^2}\,dx$.
2. *True or False* If the derivative of a function $y=f(x)$ is not continuous at some number in the interval $[a, b]$, its arc length from $x=a$ to $x=b$ can sometimes be found by partitioning the y-axis.

Skill Building

In Problems 3–6, use the arc length formula to find the length of each line between the points indicated. Verify your answer by using the distance formula.

3. $y=3x-1$, from $(1, 2)$ to $(3, 8)$
4. $y=-4x+1$, from $(-1, 5)$ to $(1, -3)$
5. $2x-3y+4=0$, from $(1, 2)$ to $(4, 4)$
6. $3x+4y-12=0$, from $(0, 3)$ to $(4, 0)$

In Problems 7–22, find the arc length of each graph by partitioning the x-axis.

7. $y=x^{2/3}+1$, from $x=1$ to $x=8$
8. $y=x^{2/3}+6$, from $x=1$ to $x=8$
9. (PAGE 615) $y=x^{3/2}$, from $x=0$ to $x=4$
10. $y=x^{3/2}+4$, from $x=1$ to $x=4$
11. $9y^2=4x^3$, from $x=0$ to $x=1$; $y\geq 0$
12. $y=\frac{x^3}{6}+\frac{1}{2x}$, from $x=1$ to $x=3$
13. $y=\frac{2}{3}(x^2+1)^{3/2}$, from $x=1$ to $x=4$

14. $y=\frac{1}{3}(x^2+2)^{3/2}$, from $x=2$ to $x=4$
15. $y=\frac{2}{9}\sqrt{3}(3x^2+1)^{3/2}$, from $x=-1$ to $x=2$
16. $y=(1-x^{2/3})^{3/2}$, from $x=\frac{1}{8}$ to $x=1$
17. $8y=x^4+\frac{2}{x^2}$, from $x=1$ to $x=2$
18. $9y^2=4(1+x^2)^3$, $y\geq 0$, from $x=0$ to $x=2\sqrt{2}$
19. $y=\ln(\sin x)$, from $x=\frac{\pi}{6}$ to $x=\frac{\pi}{3}$
20. $y=\ln(\cos x)$, from $x=\frac{\pi}{6}$ to $x=\frac{\pi}{3}$
21. $(x+1)^3=4y^2$, $y\geq 0$, from $x=-1$ to $x=16$
22. $y=x^{3/2}+8$, from $x=0$ to $x=4$

In Problems 23–26, find the arc length of each graph by partitioning the y-axis.

23. $y=x^{2/3}$, from $x=0$ to $x=1$
24. $y=x^{2/3}$, from $x=-1$ to $x=0$
25. $(x+1)^2=4y^3$, $x\geq -1$, from $y=0$ to $y=1$
26. $x=\frac{2}{3}(y-5)^{3/2}$, from $y=5$ to $y=6$

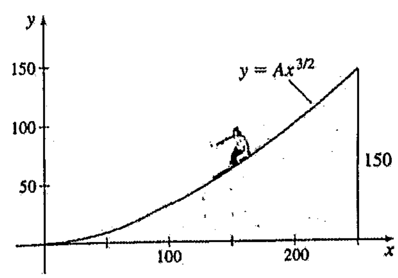
In Problems 27–32, (a) use the arc length formula (1) to set up the integral for arc length.

(b) Use technology to find the arc length.

27. $y=x^2$, from $x=0$ to $x=2$
28. $x=y^2$, from $y=1$ to $y=3$
29. $y=\sqrt{25-x^2}$, from $x=0$ to $x=4$
30. $x=\sqrt{4-y^2}$, from $y=0$ to $y=1$
31. $y=\sin x$, from $x=0$ to $x=\frac{\pi}{2}$
32. $x=y+\ln y$, from $y=1$ to $y=4$

Applications and Extensions

33. Find the arc length of the graph of $F(x)=\int_0^x \sqrt{16t^2-1}\,dt$ from $x=0$ to $x=2$.
34. Find the arc length of the graph of $F(x)=\int_0^x \sqrt{4t-1}\,dt$ from $x=0$ to $x=2$.
35. Find the arc length of the graph of $F(x)=\int_0^{3x}\sqrt{e^t-\frac{1}{9}}\,dt$ from $x=0$ to $x=4$.
36. Find the arc length of the graph of $F(x)=\int_0^{4x}\sqrt{t^4-\frac{1}{4}}\,dt$ from $x=1$ to $x=2$.
37. **Length of a Hypocycloid** Find the total length of the hypocycloid $x^{2/3}+y^{2/3}=a^{2/3}$, $a>0$.
38. **Distance Along a Curved Path** Find the distance between $(1, 1)$ and $(3, 3\sqrt{3})$ along the graph of $y^2=x^3$.
39. **Perimeter** Find the perimeter of the region bounded by the graphs of $y^3=x^2$ and $y=x$.
40. **Perimeter** Find the perimeter of the region bounded by the graphs of $y=3(x-1)^{3/2}$ and $y=3(x-1)$.
41. **Length of a Graph** Find the arc length of the graph of $6xy=y^4+3$ from $y=1$ to $y=2$.
42. **Length of a Graph** Find the arc length of the graph of the function $y=\frac{e^x+e^{-x}}{2}$ from $x=0$ to $x=2$.
43. **Length of an Graph** Find the arc length of $y=\ln(\csc x)$ from $x=\frac{\pi}{4}$ to $x=\frac{\pi}{2}$.
44. **Length of an Elliptical Arc** Set up, but do not attempt to evaluate, the integral for the arc length of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ from $x=0$ to $x=\frac{a}{2}$ in quadrant I. This integral, which is approximated by numerical techniques (see Chapter 6, Part 2), is called an **elliptical integral of the second kind.**
45. (a) Set up, but do not attempt to evaluate, the integral for the arc length of the ellipse $x^2+4y^2=4$ in the first quadrant from the point of intersection of the ellipse and the line $y=x$ to the point $(2, 0)$.
    (b) Use technology to evaluate the integral found in (a).
    (c) Use the result obtained in Example 3, p. 616, to find the perimeter of the ellipse $x^2+4y^2=4$.
46. **Modeling a Ski Slope** A ski slope is built on a mountainside and curves upward from ground level to a height h. The shape of the ski slope is modeled by the equation $y=Ax^{3/2}$, where x is the horizontal distance from the bottom of the ski slope measured along the base of the mountain and y is the vertical height of the ski slope at the distance x. See the figure below.
    (a) Find an expression, in terms of A and h, for the length of the ski slope.
    (b) Find A if the ski slope is 150 m high and has a horizontal distance of 250 m along the base.
    (c) If a skier skis directly downhill from the top of the ski slope to the bottom, how far does she travel?
    (d) Describe a simple way to check if the distance obtained in part (c) is reasonable.



47. **Arc Length** Find the arc length of the graph of the parabola $y=5x-x^2$ that lies above the x-axis.
48. **Arc Length** Find the arc length of the graph of the part of the parabola $x=6y-3y^2$ that lies in the first quadrant.
49. **Arc Length** Approximate the arc length of the graph of $y=\sin x$ from $x=0$ to $x=\frac{\pi}{2}$ using trapezoidal sums with three subintervals of equal width.
50. **Arc Length** Approximate the arc length of the graph of $y=e^x$ from $x=0$ to $x=4$ using trapezoidal sums with four subintervals of equal width.

(42)

6.6 - Integration by Parts - AP Practice Problems (p. 473)

1. An object in rectilinear motion is moving along the x-axis. Its acceleration at any time $t > 0$ is given by $a(t) = \ln(t+1)$. If the velocity $v = v(t)$ of the object at time $t = 1$ is $v(1) = 2$, then what is its velocity v at time $t = 3$?

   (A) $3\ln 4 + 8$ (B) $3\ln 4$ (C) $4\ln 4 + 8$ (D) $5\ln 4$

2. If $\int x^2 e^{2x}\,dx = f(x)e^{2x} + C$, then $f(x)$ equals

   (A) $x^2 + 2x + 2$ (B) $\frac{1}{2}x^2 - \frac{1}{4}x + \frac{1}{8}$

   (C) $\frac{1}{2}x^2 - \frac{1}{2}x - \frac{1}{4}$ (D) $\frac{1}{2}x^2 - \frac{1}{2}x + \frac{1}{4}$

3. $\int \cos^{-1} x\,dx =$

   (A) $\ln|\tan x + \sec x| + C$

   (B) $x\cos^{-1} x + \sqrt{1 - x^2} + C$

   (C) $x\cos^{-1} x - \sqrt{1 - x^2} + C$

   (D) $x\cos^{-1} x - 2\sqrt{1 - x^2} + C$

4. $\int_0^2 xe^{-x}\,dx =$

(A) $1 - 3e^{-2}$ (B) $-1 - e$
(C) $1 + e^{-2}$ (D) $3e^{-2}$

5. $\int (3x^2 + 2)\sin x\,dx =$

(A) $-6x\cos x + 6\int x\cos x\,dx$
(B) $-(3x^2 + 2)\cos x + 6\int x\cos x\,dx$
(C) $-6x\cos x - \int (3x^2 + 2)\sin x\,dx$
(D) $(3x^2 + 2)\cos x - 6\int x\cos x\,dx$

6. $\int xf'(x)\,dx =$

(A) $f(x) + C$ (B) $xf(x) - f(x) + C$
(C) $xf(x) - \int f(x)\,dx + C$ (D) $\dfrac{x^2}{2}f(x) + C$

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6.10 – Linear Partial Fractions - AP Practice Problems (p. 504)

1. $\int \frac{12}{x^2-9}\,dx =$

(A) $2\ln\left|\frac{x-3}{x+3}\right| + C$ (B) $2\ln\left|\frac{x+3}{x-3}\right| + C$

(C) $\frac{4}{3}\tan^{-1}\frac{x}{3} + C$ (D) $\ln\left|\frac{x-3}{x+3}\right| + C$

2. $\int \frac{3x}{(x-2)(x+1)}\,dx =$

(A) $\frac{x^2}{2}[2\ln|x-2| + \ln|x+1|] + C$

(B) $\ln|x+1| - 2\ln|x-2| + C$

(C) $2\ln|x+1| + \ln|x-2| + C$

(D) $2\ln|x-2| + \ln|x+1| + C$

3. $\displaystyle\int \frac{x+6}{x(x+2)}\,dx =$

(A) $3x - 12\ln|x+2| + C$

(B) $3x - 2\ln|x+2| + C$

(C) $3\ln|x| - 2\ln|x+2| + C$

(D) $3\ln|x| + 2\ln|x+2| + C$

4. $\displaystyle\int_0^1 \frac{2x-1}{x^2+3x+2}\,dx =$

(A) $2\ln 2 + 3\ln 3$

(B) $2\ln 2 + 5\ln 3$

(C) $-4\ln 2 + 3\ln 3$

(D) $-8\ln 2 + 5\ln 3$

(46)

6.12 – Evaluating Improper Integrals - AP Practice Problems (p. 526)

1. $\displaystyle\int_0^2 \frac{x+2}{x^2+4x-12}\,dx =$

(A) $-\dfrac{\ln 12}{2}$ (B) $\dfrac{1-\ln 12}{2}$

(C) $\dfrac{\ln 12 - \ln 2}{2}$ (D) diverges

2. Determine whether $\displaystyle\int_1^\infty \frac{2}{x^3}\,dx$ converges or diverges.

If it converges, find its value.

(A) $\dfrac{1}{2}$ (B) 1 (C) 2 (D) diverges

3. Determine whether $\int_{3}^{\infty} \frac{8x}{\sqrt[3]{8-x^2}}\,dx$ converges or diverges.
If it converges, find its value.

(A) 0 (B) 18 (C) $6(9^{2/3})$ (D) diverges

4. Determine whether $\int_{-\infty}^{0} xe^{x^2}\,dx$ converges or diverges.
If it converges, find its value.

(A) $\frac{1}{2}$ (B) 1 (C) $-e$ (D) diverges

(48)

5. Determine whether $\int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx$ converges or diverges. If it converges, find its value.

(A) $\frac{1}{\ln 2}$ (B) $\frac{1}{2}$ (C) $-\frac{1}{\ln 2}$ (D) diverges

6. Determine whether $\int_0^{1} \frac{1}{\sqrt[4]{x}}\,dx$ converges or diverges. If it converges, find its value.

(A) $\frac{3}{4}$ (B) 4 (C) $\frac{4}{3}$ (D) diverges

7. When the region bounded by the graph of $y = \frac{1}{x^2}$ and the x-axis to the right of the line $x = 1$ is revolved about the x-axis, the volume V, if it is defined, of the solid of revolution that is generated is given by the improper integral

$$\int_1^\infty \pi \left(\frac{1}{x^2}\right)^2 dx.$$

Determine whether the improper integral converges or diverges. If it converges, find the volume of the solid of revolution.

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4.4 AP Practice Problems (p. 301) – L'Hopital's Rule

1. $\lim_{x \to 0} \frac{e^{4x}-1}{\sin(2x)} =$

   (A) 0 (B) 2 (C) 4 (D) does not exist

2. $\lim_{x \to 0} \frac{1-\cos^2(3x)}{x^2} =$

   (A) 18 (B) 9 (C) 0 (D) 3

3. Find $\lim_{x \to \infty} \frac{x^{-3/2}}{\sin \frac{1}{x}}$.

   (A) $\frac{3}{2}$ (B) 1 (C) 0 (D) ∞

4. $\lim_{x \to 1} \frac{\ln x^3}{x^2-1} =$

   (A) 0 (B) 1 (C) $\frac{3}{2}$ (D) 3

5. For any positive integer k, $\lim_{x\to\infty} \frac{\ln x}{x^k} =$

(A) 0 (B) 1 (C) $k+1$ (D) ∞

6. $\lim_{\theta\to 0} \frac{1-\cos(2\theta)}{3\sin\theta} =$

(A) -2 (B) $\frac{2}{3}$ (C) 0 (D) $-\frac{1}{3}$

7. $\lim_{x\to \frac{\pi}{2}^-} \frac{\ln(\cos x)}{\tan x} =$

(A) $-\infty$ (B) 0 (C) 1 (D) ∞

8. Which of the following are indeterminate forms at 0?

I $\frac{x}{\ln(x+1)}$ II $\frac{e^x}{x^2-2x}$ III $\frac{x}{1-\cos(\pi x)}$

(A) I only (B) I and III only

(C) II and III only (D) I, II, and III

7.4 – Euler's Method - AP Practice Problems (p. 559)

1. Suppose $y = f(x)$ is the solution of the differential equation $\frac{dy}{dx} = x + 2y$ with the initial condition $y(0) = 1$. Approximate $f(0.3)$ using Euler's method with $(x_0, y_0) = (0, 1)$ and using $h = 0.1$ as the increment.

   (A) 1.20 (B) 1.76 (C) 2.03 (D) 2.78

2. Suppose $y = f(x)$ is the solution to the differential equation $\frac{dy}{dx} = 3x - 2y$ with the boundary condition $f(1) = 5$. What is the approximation for $f(1.2)$ using Euler's method starting at $x = 1$ and using two steps of equal size?

7.5 – Logistic Models - AP Practice Problems (p.566)

1. A population grows according to the logistic differential equation $\frac{dP}{dt} = 0.01P\left(3 - \frac{P}{2000}\right)$. According to the model, the carrying capacity is

   (A) 2000 (B) 4000 (C) 6000 (D) 10,000

2. A population P grows according to the logistic differential equation $\frac{dP}{dt} = 0.08P\left(1 - \frac{P}{2000}\right)$, where t is measured in years. What is the population when it is increasing most rapidly?

   (A) 40 (B) 80 (C) 1000 (D) 2000

3. A population P grows according to the logistic function $P(t) = \frac{1200}{1 + 64e^{-0.4t}}$. The maximum population growth rate is

   (A) −0.4 (B) 0.4 (C) 0.533 (D) 0.64

(54) 4. Suppose $P(t)$ is the particular solution of the differential equation $\frac{dP}{dt} = 0.3P(50 - P)$ with the initial condition if $t = 0$, then $P = 5$. Then $\lim_{t \to \infty} P(t)$ equals

(A) ∞ (B) 0.3 (C) 15 (D) 50

5. A virus spreads among a population of N people at a rate proportional to the product of the number of people infected with the virus and the number of people not infected with the virus. Which differential equation can be used to model the situation with respect to time t? Assume k is positive.

(A) $\frac{dP}{dt} = kP$ (B) $\frac{dP}{dt} = kN(N - P)$

(C) $\frac{dP}{dt} = kP(P - N)$ (D) $\frac{dP}{dt} = kP(N - P)$

6. An invasive breed of insect was introduced into a swamp in Florida. Initially 100 insects were present, and their daily maximum population growth rate is 20%. Entomologists have determined that the swamp can support a colony of 600,000 insects and that the population P follows a logistic growth model. Which logistic differential equation models the insect population?

(A) $\frac{dP}{dt} = 20\left(1 - \frac{P}{600{,}000}\right)$ (B) $\frac{dP}{dt} = 100\left(1 - \frac{P}{600{,}000}\right)$

(C) $\frac{dP}{dt} = 0.20P(600{,}000 - P)$ (D) $\frac{dP}{dt} = 0.20P\left(1 - \frac{P}{600{,}000}\right)$

7. In an attempt to regenerate the American bald eagle population, eagles were captured and released in a federal park in Montana. From previous studies, environmentalists know that the population grows according to the logistic function $P(t) = \dfrac{540}{1 + 89e^{-0.162t}}$, where t is measured in years.

   (a) How many American bald eagles were released in Montana?
   (b) What is the maximum population growth rate of the American bald eagle?
   (c) What is the carrying capacity of the park?
   (d) What is the American bald eagle population when it is growing most rapidly?

8. A population P grows according to the logistic differential equation

$$\frac{dP}{dt} = 0.0005P(800 - P)$$

   (a) What is the carrying capacity of the population?
   (b) What is the maximum population growth rate of the population?
   (c) How large is the population when it is increasing most rapidly?

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9. The population of hyenas in a game preserve increases according to the logistic function $P(t) = \dfrac{150}{1 + 14e^{-0.125t}}$, where t is measured in years.

(a) How many hyenas were originally in the preserve?
(b) How many hyenas can the preserve maintain?
(c) What is the maximum population growth rate?
(d) How many hyenas are there when the population is growing most rapidly?

8.5 – Arc Length of Curve & Distance Traveled - AP Practice Problems (p. 620)

1. The arc length of the graph of $f(x) = -x^4 + 2$ from $x = 0$ to $x = 10$ is given by

(A) $\int_0^{10} \sqrt{1 + 16x^3}\, dx$ (B) $\int_0^{10} \sqrt{1 + 4x^3}\, dx$

(C) $\int_0^{10} \sqrt{1 + (-x^4 + 2)^2}\, dx$ (D) $\int_0^{10} \sqrt{1 + 16x^6}\, dx$

2. The arc length of the graph of $y = \tan x$ from $x = a$ to $x = b$, where $-\frac{\pi}{2} < a < b < \frac{\pi}{2}$ is given by

(A) $\int_{-\pi/2}^{\pi/2} \sqrt{1 + \sec^2 x}\, dx$

(B) $\int_a^b \sqrt{1 + \sec^4 x}\, dx$

(C) $\int_a^b \sqrt{1 + \sec^2 x}\, dx$

(D) $\int_a^b \sqrt{1 + \sec^2 x \tan^2 x}\, dx$

3. What is the length of the graph of $f(x) = \ln \cos x$ from $x = 0$ to $x = \frac{\pi}{3}$?

(A) $\ln|\sqrt{3} - 2|$ (B) $\ln(2 + \sqrt{3})$ (C) $2\sqrt{3}$ (D) $\ln 4$

4. The arc length of $y = (x - 8)^{2/3}$ from $x = 0$ to $x = 16$ is given by

(A) $\displaystyle\int_0^{16} \sqrt{1 + \frac{4}{9(x-8)^{2/3}}}\, dx$ (B) $\displaystyle\int_0^{4} \sqrt{1 + \frac{9}{4}y}\, dy$

(C) $\displaystyle 2\int_0^{4} \sqrt{1 + \frac{9}{4}y}\, dy$ (D) $\displaystyle\int_0^{16} \sqrt{1 + \frac{2}{3(y-8)^{1/3}}}\, dy$

5. The region R bounded by the graph of $f(x) = -x^3 + 4x^2$ and the x-axis is shown in the figure below.

(a) Find the area under the graph of f from 0 to 4.

(b) Write, but do not evaluate, an integral for the arc length of the graph of f from $x = 0$ to $x = 4$.

(c) Write, but do not evaluate, an integral to find the volume of the solid of revolution generated by revolving the region R about the x-axis.