Name: ________________________________ Period: ____________

BC Calculus

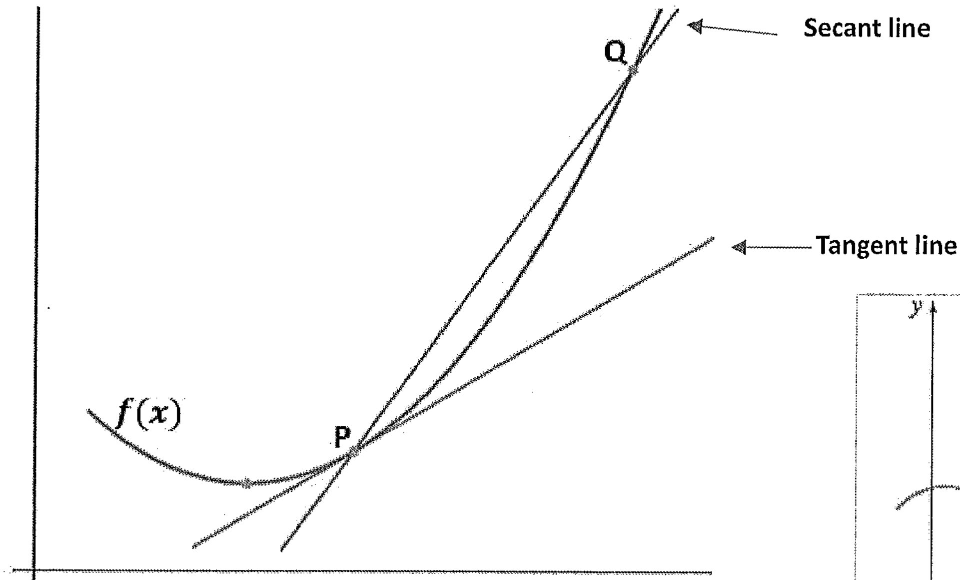
Unit 2

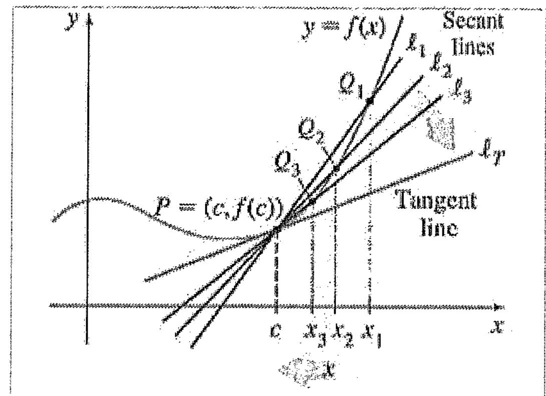
Differentiation

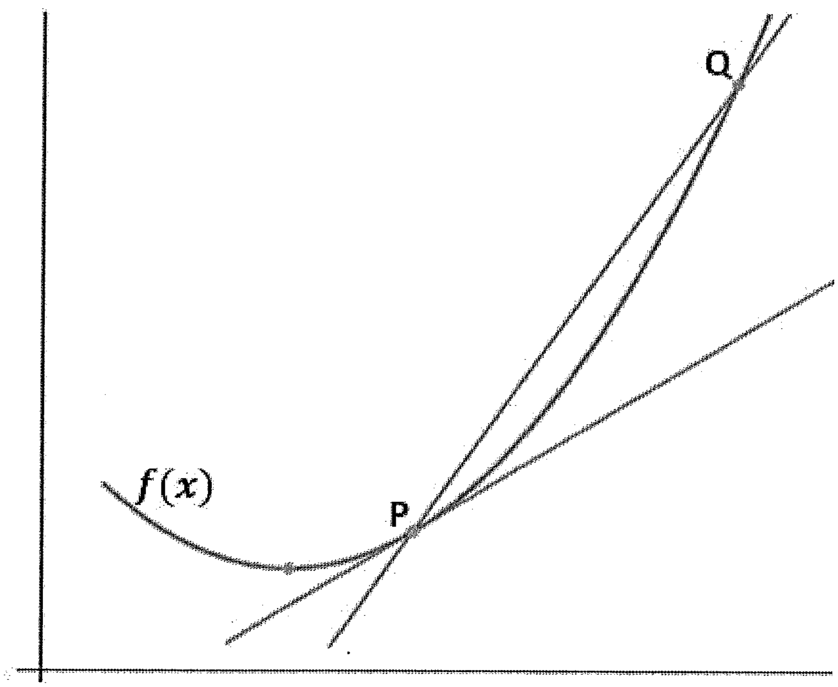
(Derivatives & Definition)

AP Calculus – 2.2 Notes - Limit Definition of a Derivative

Goal: To discover a formula to calculate the slope (steepness) of all tangent lines to a curved graph.







General Limit Definition of the Derivative:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

Alternate Limit Definition of a derivative

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

<u>*f '(x) is "f prime of x"*:</u> This is the notation for the derivative function.

<u>Derivative</u> is the slope (steepness) of a curve at a single point

*The <u>derivative function</u> is a **slope-finding formula** for a curved graph, where the slope is of the curve is ever-changing.

②

General Limit Definition of the Derivative:

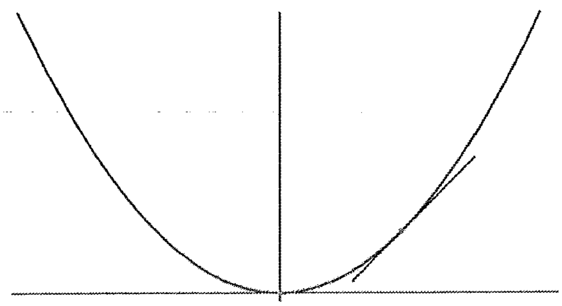
$$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

Alternate Limit Definition of a derivative:

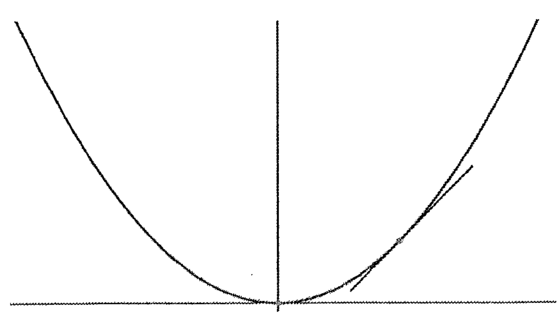
$$f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c}$$

Example 1: (a) Find the general derivative of $f(x) = x^2$

(b) Write the equation of the tangent line to $f(x)$ at $x = 1$ (point-slope form: $\boldsymbol{y - y_1 = m(x - x_1)}$)



(c) Write the equation of the tangent line to $f(x)$ at $x = -5$



To Recap:

* $\boldsymbol{f(x)}$ is the **height-finding formula** (finds the y-value of graph at that point)

*Since $f(1) = 1$, this tells us that when x = 1, the height of the graph has a y-value of 1

*$\boldsymbol{f'(x)}$ is the **slope-finding formula** for the $f(x)$ graph.

*Since $f'(1) = 2$, this tells us that when x = 1, the slope of the tangent line to $f(x)$ has a slope (steepness) of 2.

General Limit Definition of the Derivative:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Alternate Limit Definition of a derivative:

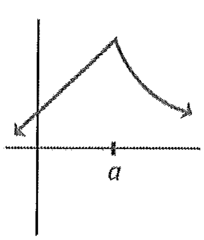
$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

Example 2: (a) Find the general derivative of $f(x) = \sqrt{x}$

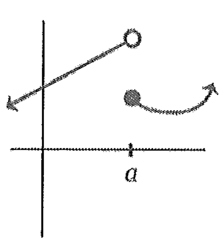
(b) Write the equation of the tangent line to $f(x)$ at $x = 2$ (point-slope form: $\boldsymbol{y - y_1 = m(x - x_1)}$)

Example 3: Use the alternative derivative definition to find slope of $f(x) = \sqrt{x}$ at x = 2.

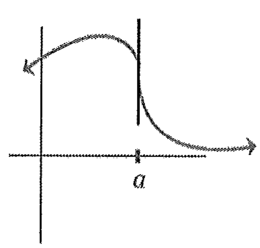
Differentiability: In order for a function to be **differentiable** (smooth curve) at a point a, then the graph must be continuous at that point, cannot contain a sharp turn & cannot have a vertical tangent at the point.



Cusp / Corner



Discontinuous



Vertical Tangent

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General Limit Definition of the Derivative:

Classwork Examples:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Find the derivative using limits

1. $f(x) = 7 - 6x$

2. $y = 5x^2 - x$

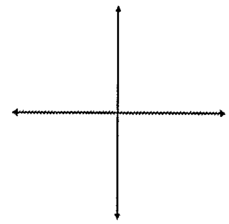
3. $y = \sqrt{5x + 2}$

4. $f(x) = \frac{1}{x-2}$

AP Calculus – 2.3 Notes – Derivatives of Polynomials (Power Rule)

1. Constant Rule: If $f(x) = c$, then $f'(x) = 0$

Example: $f(x) = 5$



2. Power Rule: If $f(x) = x^n$, then $f'(x) = n * x^{n-1}$

Steps $a)$ Bring Exponent down as coefficient in front of the variable

$b)$ Subtract 1 from the original exponent value

Power Rule Conditions:

$i)$ Convert radicals to rational exponents (ex: $\sqrt{x^5} = x^{\frac{5}{2}}$)

$ii)$ Bring variable to the numerator before applying power rule

$iii)$ Expand terms: resolve parentheses & fractional terms before applying Power Rule

**Important Note*: Be sure the function is in the appropriate form (all conditions met!) before applying Power Rule

Example 1: Find Derivatives of the following:

a) $y = x^7$

$b)$ $g(x) = \sqrt[3]{x}$

$c)$ $y = \dfrac{4}{x^5}$

$d)$ $y = 8x^{2/3} - \sqrt[5]{x} + \dfrac{2}{\sqrt{x}} + 0.875$

⑥

Example 2: If $f(x) = \frac{1}{x^2}$ $\quad find\ f'(2)$

Example 3: $If\ f(x) = \sqrt[3]{x^2}$ $,write\ the\ tangent\ line\ equation\ to\ f(x) at\ x = 1$

Example 4: Find $f'(x)$ if $f(x) = \frac{x^4 - 3x^2 + 4(\sqrt[3]{x})}{\sqrt{x}}$

Example 5: Find $f'(x)$ if $f(x) = 3x(x+1)^2$

AP Calculus – 2.3b Notes Derivatives of e^x, $\ln(x)$, $\sin x$, *and* $\cos x$ functions

Recall:

$\ln 1 =$ $\ln 0 =$ $e^0 =$ $e^{\ln a} =$ $\ln e^a =$

Derivatives of Exponential Functions

$$\frac{d}{dx}e^x =$$

Derivatives of Logarithmic Functions

$$\frac{d}{dx}\ln x =$$

Find the value of the derivative at the given point.

4. If $f(x) = 3\ln x + e^x$, find $f'(5)$

5) What is the slope of the line tangent to the graph of $y = 2\ln(x)$ at the point $x = 8$?

(A) $\frac{1}{16}$ (B) $\frac{1}{8}$ (C) $\frac{1}{4}$ (D) 16 (E) 4

6) If $f(x) = 4\ln x - 3e^x + e$, find $f'(1)$

⑧

Derivatives of $\cos x$ and $\sin x$

$$\frac{d}{dx}\cos x = \qquad\qquad \frac{d}{dx}\sin x =$$

Example: Find $f'(x)$ if $f(x) = 2\sin x - 5\cos x$

13. If $f(x) = 4e^x + 5\sin x$, find $f'(0)$

14. If $f(x) = 2\cos x + e^x$, find $f'(\pi)$

Find the equation of the tangent line at the given x-value.

15. $f(x) = 3\cos x + x$ at $x = \frac{\pi}{2}$

16. $f(x) = 4e^x - 3\sin x + x^2$ at $x = 0$

AP Calculus – 2.4 Notes - Product Rule and Quotient Rule

Product Rule

$$h(x) = f \cdot g$$

$$h'(x) =$$

Find the derivative of each function.

1. $f(x) = 8x \sin x$
2. $g(x) = 2e^x(\sqrt{x})$
3. $h(x) = \left(\frac{1}{x} + 1\right)(2x^2 - 5)$

The table below shows values of two differentiable functions f and g, as well as their derivatives.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
2	4	−2	−1	2
−5	3	4	−2	5

4. $h(x) = 3f(x)g(x)$
   Find $h'(2)$.
5. $r(x) = \left(\frac{f(x)}{2} + 2\right)(3 - g(x))$
   Find $r'(-5)$.

Find the derivative of each function.

1. $f(x) = (2x - 3)\sin x$
2. $g(x) = 2x^3e^x$
3. $h(x) = 4\sqrt{x}\ln x$
4. $f(x) = (4 - 5x)\cos x$
5. $g(x) = 6\ln x \sin x$
6. $h(x) = 2e^x(x^2 + x)$

(10)

Quotient Rule

$$h(x) = \frac{f}{g}$$

$$h'(x) =$$

Find the derivative of each function.

1. $y = \frac{2x^2}{3x+1}$

2. $g(x) = \frac{3e^x}{2x}$

3. $h(x) = \frac{\sin x}{2x^2-5}$

4. $h(x) = \frac{3x+1}{2x^2}$

The table below shows values of two differentiable functions f and g, as well as their derivatives.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
2	4	−2	−1	2

5. $h(x) = \frac{f(x)}{3g(x)}$
Find $h'(2)$.

6. $r(x) = -\frac{g(x)}{1-f(x)}$
Find $r'(2)$.

Find the derivative of each function.

1. $h(x) = \frac{4x-1}{3x+2}$

2. $g(x) = \frac{\sin x}{x}$

3. $h(x) = \frac{x^3+2x^2-x}{2x}$

Product and Quotient Rule Practice Problems:

13.

x	$d(x)$	$d'(x)$	$h(x)$	$h'(x)$
1	-3	-2	4	3

a. $a(t) = d(x)h(x)$
Find $a'(1)$.

b. $b(x) = -d(x)h(x)$
Find $b'(1)$.

c. $c(x) = \left(2 - \frac{d(x)}{2}\right)(6 - h(x))$
Find $c'(1)$.

Find the equation of the tangent line at the given x-value.

14. $f(x) = 8\sin x\cos x$ at $x = \frac{\pi}{3}$

14b) $g(x) = -2xe^x$ at $x = 0$

15. What is the instantaneous rate of change at $x = 4$ of the function $f(x) = \frac{x^2-1}{x-2}$?

(A) $-\frac{15}{2}$ (B) $\frac{1}{4}$ (C) $\frac{1}{2}$ (D) $\frac{15}{2}$

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16. Let f and g be differentiable functions with the following properties:
I. $f(x) < 0$ for all x
II. $g(5) = 2$

If $h(x) = \frac{f(x)}{g(x)}$ and $h'(x) = \frac{f'(x)}{g(x)}$, then $g(x) =$

(A) $\frac{1}{f'(x)}$ (B) $f(x)$ (C) $-f(x)$ (D) 0 (E) 2

(12)

17. The function f is defined by $f(x) = \frac{x}{x+4}$. What points (x, y) on the graph of f have the property that the line tangent to f at (x, y) has a slope of $\frac{1}{9}$?

(A) $\left(2, \frac{1}{3}\right)$ only

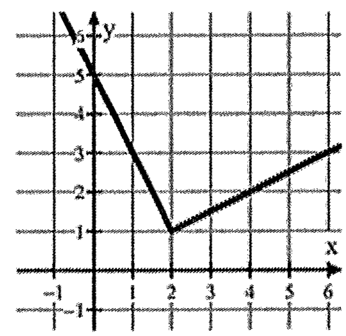
(B) $\left(\frac{1}{9}, \frac{1}{13}\right)$ only

(C) $\left(2, \frac{1}{3}\right)$ and $\left(-10, \frac{5}{3}\right)$

(D) $\left(2, \frac{1}{3}\right)$ and $(-2, -1)$

(E) There are no such points.

18. The graph of a function f is shown to the right. Let $g(x) = \frac{x^2-1}{f(x)}$. What is the value of $g'(4)$?



Graph of f

19. The graphs of f and g are shown to the right. If $h(x) = 4f(x)g(x)$, then $h'(1) =$

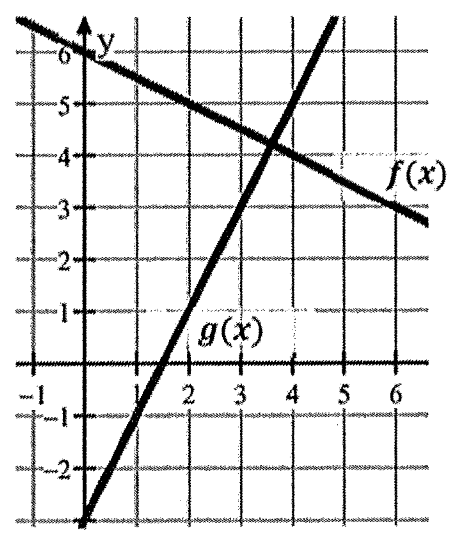
(A) -22

(B) -4

(C) 0

(D) 4

(E) 46



AP Calculus – 2.5 Notes - Derivatives of Trig Functions

Trig Derivatives

$\frac{d}{dx}\sin x =$	$\frac{d}{dx}\cos x =$
$\frac{d}{dx}\tan x =$	$\frac{d}{dx}\cot x =$
$\frac{d}{dx}\sec x =$	$\frac{d}{dx}\csc x =$

Common struggles for students dealing with trig derivatives:

- Memorizing.
- Unit Circle values.
- Simplifying/manipulating trig expressions.
- Trig reciprocals in a calculator.

1. Find the derivative of $y = \sin x \tan x$

2. Find $f'\left(\frac{\pi}{6}\right)$ if $f(x) = \frac{x}{\sec x}$

Find the derivative of each function

3. $h(x) = 2x\tan(x)$

4. $f(x) = \frac{1}{2\cos x}$

14

Find the derivative at the given x-value. Show your work!

5. $f(x) = 2\sec x$ at $x = \frac{\pi}{4}$.

6. $f(x) = x\cot x$ at $x = \frac{\pi}{6}$.

Find the equations of both the normal line and the tangent line.

7. $y = \sec x$ at $x = \pi$

Tangent: ____________________

Normal: ____________________

8. $y = \tan x$ at $x = \frac{\pi}{3}$

Tangent: ____________________

Normal: ____________________

Find the equation of the tangent line at the given x-value.

15. $f(x) = 3\cos x + x$ at $x = \frac{\pi}{2}$

16. $f(x) = 4e^x - 3\sin x + x^2$ at $x = 0$

15

Chapter 2.1-2.5 Quiz Review

(Limit Definition of Derivative , Derivative Rules, Product & Quotient Rule)

No Calculators (answers can be left unsimplified)

Find the average rate of change of each function on the given interval. Use appropriate units if necessary.

1. $w(x) = \ln x; \quad 1 \le x \le 7$	2. $s(t) = -t^2 - t + 4;\ [1, 5]$ t represents seconds s represents feet

3. Find the derivative of $y = 2x^2 + 3x - 1$ by using the definition of the derivative. $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

4. For the function $h(t)$, h is the temperature of the oven in Fahrenheit, and t is the time measured in minutes.
   a. Explain the meaning of the equation $h(15) = 420$.

   b. Explain the meaning of the equation $h'(43) = -11$.

Find the derivative of each function.

5. $f(x) = 4 - \frac{1}{2x^2}$	6. $g(x) = 3\sqrt{x} - \frac{6}{x^2} + 5\pi^3$	7. $h(x) = 4e^x - 2\cos x$

(16)

15

Find the derivative of each function.

8. $s(t) = t^2\sin(t)$

9. $d(t) = 3\sqrt{t}\ln t$

10. $y = \frac{4}{x} - \sec x$

11. $h(x) = \frac{2-x}{x+2}$

Find the equation of the tangent line of the function at the given x-value.

12. $f(x) = -2x^3 + 3x$ at $x = -1$.

13. $f(x) = 4\sin x - 2$ at $x = \pi$

14. Find the equation for the normal line of $y = \frac{1}{2}x^2 + \frac{3}{4}x - 4$ at $x = -3$

15. If $f(x) = 3\sin x - 2e^x$ find $f'(0)$. No calculator!

16. Use the table below to estimate the value of $d'(120)$. Indicate units of measures.

Explain the meaning of d'(120) within context of this table.

t seconds	2	13	60	180	500
$d(t)$ feet	10	81	412	808	2,105

17. Is the function differentiable at $x = 2$?

$$f(x) = \begin{cases} 3x - 3x^2 - 5, & x < 2 \\ 7 - 9x, & x \geq 2 \end{cases}$$

18. What values of a and b would make the function differentiable at $x = 4$?

$$f(x) = \begin{cases} a\sqrt{x} + bx^2 - 1, & x < 4 \\ \dfrac{16}{x} + bx, & x \geq 4 \end{cases}$$

Each limit represents the instantaneous rate of change of a function. Identify the original function, and the x-value of the instantaneous rate of change.

19. $\lim_{x \to 4} \frac{(x^2-3x)-(4)}{x-4}$

Function: $f(x) =$

Instantaneous rate at $x =$

20. $\lim_{h \to 0} \frac{9(5+h)-10(5+h)^2+(205)}{h}$

Function: $f(x) =$

Instantaneous rate at $x =$

Use the table to find the value of the derivatives of each function.

21.

x	$h(x)$	$h'(x)$	$r(x)$	$r'(x)$
-2	-3	2	-2	4

a. $f(x) = -h(x)r(x)$
Find $f'(-2)$.

b. $g(x) = \frac{h(x)+r(x)}{r(x)}$
Find $g'(-2)$.

c. $w(x) = (4 - 2h(x))(1 - r(x))$
Find $w'(-2)$.

22. At what x-value(s) does the function $f(x) = \frac{x^4}{4} - 3x^3 + 9x^2 + 7$ have a horizontal tangent?

23. If $f(x) = \cos x + \sin x$, find $f'\left(\frac{\pi}{3}\right)$

24. $S(x)$ is the number of students in Mr. Kelly's class and x is the number of years since 2015.

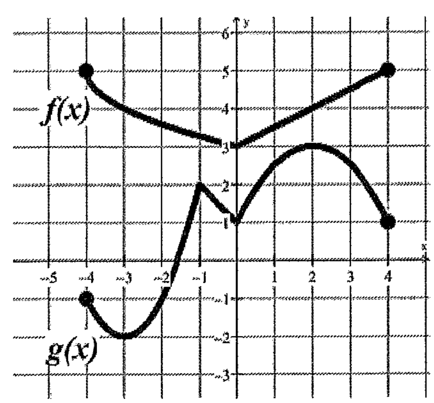
a. Explain the meaning of $S(3) = 127$.

b. Explain the meaning of $S'(3) = 4$.

25. **Use the graphs of *f* and *g* to find the following.**

a. $h(x) = f(g(x))$. Find the average rate of change on the interval [2,4].

b. $j(x) = g(f(x))$. Find the average rate of change on the interval [−3,2].



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Chapter 2.1-2.5 Quiz Review

Key

(Limit Definition of Derivative , Derivative Rules, Product & Quotient Rule)

No Calculators (answers can be left unsimplified)

Find the average rate of change of each function on the given interval. Use appropriate units if necessary. (slope b/t end points)

1. $w(x) = \ln x;\quad 1 \le x \le 7$

$\frac{w(7)-w(1)}{7-1} \to \frac{\ln 7 - \ln 1}{7-1} \to \frac{\ln 7 - 0}{6} = \boxed{\frac{\ln 7}{6}}$

2. $s(t) = -t^2 - t + 4;\ [1,5]$
t represents seconds
s represents feet

$s(1) = 2$, $s(5) = -26$

$\frac{s(5)-s(1)}{5-1} \to \frac{-26-2}{4} \to \frac{-28}{4} \to \boxed{-7\ \text{ft/sec}}$

3. Find the derivative of $y = 2x^2 + 3x - 1$ by using the definition of the derivative. $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$f(x) = 2x^2 + 3x - 1$
$f(\ ) = 2(\ )^2 + 3(\ ) - 1$
$f(x+h) = 2(x+h)^2 + 3(x+h) - 1$

$f'(x) = \lim_{h\to 0} \frac{2(x+h)^2 + 3(x+h) - 1 - (2x^2+3x-1)}{h}$

$f'(x) = \lim_{h\to 0} \frac{2(x^2+2xh+h^2)+3x+3h-1-2x^2-3x+1}{h}$

$f'(x) = \lim_{h\to 0} \frac{2x^2+4xh+2h^2+3x+3h-1-2x^2-3x+1}{h}$

$f'(x) = \lim_{h\to 0} \frac{h(4x+2h+3)}{h}$

$f'(x) = 4x + 2(0) + 3$

$\boxed{f'(x) = 4x+3}$

4. For the function $h(t)$, h is the temperature of the oven in Fahrenheit, and t is the time measured in minutes.
   a. Explain the meaning of the equation $h(15) = 420$.

   The oven is 420°F after 15 minutes.

   b. Explain the meaning of the equation $h'(43) = -11$.

   The temperature in the oven is cooling by 11 degrees per minute on the 43rd minute.

Find the derivative of each function.

5. $f(x) = 4 - \frac{1}{2x^2}$

$f(x) = 4 - \frac{1}{2}x^{-2}$

$f'(x) = 0 - \frac{1}{2} \cdot -2x^{-3}$

$f'(x) = x^{-3}$

$\boxed{f'(x) = \frac{1}{x^3}}$

6. $g(x) = 3\sqrt{x} - \frac{6}{x^2} + 5\pi^3$

$g(x) = 3x^{1/2} - 6x^{-2} + 5\pi^3$

$g'(x) = \frac{1}{2}\cdot 3x^{-1/2} - 2\cdot 6x^{-3} + 0$

$\boxed{g'(x) = \frac{3}{2x^{1/2}} + \frac{12}{x^3}}$ OR $\boxed{g'(x) = \frac{3}{2\sqrt{x}} + \frac{12}{x^3}}$

7. $h(x) = 4e^x - 2\cos x$

$h'(x) = 4e^x - 2(-\sin x)$

$\boxed{h'(x) = 4e^x + 2\sin x}$

Find the derivative of each function.

8. $s(t) = t^2 \sin(t)$ ← *product Rule

$s'(t) = 2t \cdot \sin(t) + t^2 \cdot \cos(t)$

$\boxed{s'(t) = 2t\sin(t) + t^2\cos(t)}$

9. $d(t) = 3\sqrt{t}\ln t$

$d(t) = 3t^{1/2} \cdot \ln(t)$

$d'(t) = \frac{1}{2}\cdot 3t^{-1/2} \cdot \ln(t) + 3t^{1/2} \cdot \frac{1}{t}$

$\boxed{d'(t) = \frac{3\ln(t)}{2t^{1/2}} + \frac{3}{t^{1/2}} \quad \text{or} \quad \frac{3\ln(t)}{2\sqrt{t}} + \frac{3}{\sqrt{t}}}$

10. $y = \frac{4}{x} - \sec x$

$f(x) = 4x^{-1} - \sec x$

$f'(x) = -4x^{-2} - \sec x \tan x$

$\boxed{f'(x) = \frac{-4}{x^2} - \sec x \tan x}$

11. $h(x) = \frac{2-x}{x+2}$ *quotient Rule

$h'(x) = \frac{(-1)(x+2) - (2-x)(1)}{(x+2)^2} \to \frac{-x-2-2+x}{(x+2)^2}$

$\boxed{h'(x) = \frac{-4}{(x+2)^2}}$

Find the equation of the tangent line of the function at the given x-value.

12. $f(x) = -2x^3 + 3x$ at $x = -1$.

$f(-1) = -2(-1)^3 + 3(-1) = 2 - 3 = -1$

$f'(x) = -6x^2 + 3$

$f'(-1) = -6(-1)^2 + 3 = -3$

point: $(-1,-1)$
slope: $m = -3$

$\boxed{y + 1 = -3(x+1)}$

13. $f(x) = 4\sin x - 2$ at $x = \pi$

$f(\pi) = 4\sin\pi - 2 \to 4(0) - 2 = -2$

$f'(x) = 4\cos x$

$f'(\pi) = 4\cos\pi = 4(-1) = -4$

point: $(\pi, -2)$
slope: $m = -4$

$\boxed{y + 2 = -4(x - \pi)}$

14. Find the equation for the normal line of $y = \frac{1}{2}x^2 + \frac{3}{4}x - 4$ at $x = -3$

$y(-3) = \frac{1}{2}(-3)^2 + \frac{3}{4}(-3) - 4 = \frac{9}{2} - \frac{9}{4} - \frac{16}{4} = \frac{-7}{4}$

point: $(-3, -7/4)$

$y'(x) = 2\cdot\frac{1}{2}x + \frac{3}{4}$

$y'(-3) = -3 + \frac{3}{4} = -\frac{9}{4}$

slope (normal line): $m_2 = \frac{4}{9}$

$\boxed{y + 7/4 = \frac{4}{9}(x+3)}$

15. If $f(x) = 3\sin x - 2e^x$ find $f'(0)$. No calculator!

$f'(x) = 3\cos x - 2e^x$

$f'(0) = 3\cos 0 - 2e^0$

$f'(0) = 3(1) - 2(1)$

$\boxed{f'(0) = 1}$

16. Use the table below to estimate the value of $d'(120)$. Indicate units of measures.

Explain the meaning of d'(120) within context of this table.

t seconds	2	13	60	180	500
$d(t)$ feet	10	81	412	808	2,105

$d'(120) \approx \frac{d(180)-d(60)}{180-60} = \frac{808-412}{120} = 3.3$ ft/sec

d'(120) means that the approximated rate of change of particle at t=120sec is 3.3 ft/sec.

17. Is the function differentiable at $x = 2$?

$f(x) = \begin{cases} 3x - 3x^2 - 5, & x < 2 \\ 7 - 9x, & x \geq 2 \end{cases}$

* f(x) is differentiable if $\lim_{x\to c^-} f'(x) = \lim_{x\to c^+} f'(x)$

$f'(x) = \begin{cases} 3-6x, & x<2 \\ -9, & x \geq 2 \end{cases}$

$\lim_{x\to 2^-} 3-6x = -9$, $\lim_{x\to 2^+} -9 = -9$ ✓

* and continuous at x=2: $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x)$

$\lim_{x\to 2^-} 3x-3x^2-5 = -11$, $\lim_{x\to 2^+} 7-9x = -11$ ✓

yes ✓

18. What values of a and b would make the function differentiable at $x = 4$?

$f(x) = \begin{cases} a\sqrt{x} + bx^2 - 1, & x < 4 \\ \frac{16}{x} + bx, & x \geq 4 \end{cases}$

* set equations equal
* set derivatives equal

$a\sqrt{4}+b(16)-1 = \frac{16}{4}+4b$ → $2a = 5-12b$

$f'(x) = \begin{cases} \frac{1}{2}ax^{-1/2}+2bx, & x<4 \\ -16x^{-2}+b, & x \geq 4 \end{cases}$

$\frac{a}{2\sqrt{4}} + 8b = -\frac{16}{4^2}+b$ → $4(\frac{a}{4}+8b = -1+b)$

$a + 32b = -4+4b$ → $a = -4-28b$

$2(-4-28b) = 5-12b$, $-8-56b = 5-12b$, $b = -\frac{13}{44}$

$a = \frac{47}{11}$

Each limit represents the instantaneous rate of change of a function. Identify the original function, and the x-value of the instantaneous rate of change.

19. $\lim_{x\to 4} \frac{(x^2-3x)-(4)}{x-4}$

Function: $f(x) = x^2 - 3x$

Instantaneous rate at $x = 4$

$f'(c) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c}$

$f'(x) = 2x-3$

$f'(4) = 2(4)-3 = 5$

20. $\lim_{h\to 0} \frac{9(5+h)-10(5+h)^2+(205)}{h}$

Function: $f(x) = 9x - 10x^2$

Instantaneous rate at $x = 5$

$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$f(x) = 9x-10x^2$

$f'(x) = 9-20x$

$f'(5) = 9-20(5) = -91$

Use the table to find the value of the derivatives of each function.

21.

x	$h(x)$	$h'(x)$	$r(x)$	$r'(x)$
−2	−3	2	−2	4

a. $f(x) = -h(x)r(x)$
Find $f'(-2)$.

$f'(x) = -h'(x)\cdot r(x) + -h(x)\cdot r'(x)$

$f'(-2) = -h'(-2)\cdot r(-2) - h(-2)\cdot r'(-2)$

$f'(-2) = -2(-2) - (-3)(4) = 4 + 12$

$f'(-2) = 16$

b. $g(x) = \frac{h(x)+r(x)}{r(x)}$
Find $g'(-2)$.

* quotient Rule

$g'(x) = \frac{(h'(x)+r'(x))\,r(x) - (h(x)+r(x))\,r'(x)}{r(x)^2}$

$g'(-2) = \frac{[h'(-2)+r'(-2)]\,r(-2) - [h(-2)+r(-2)]\,r'(-2)}{[r(-2)]^2}$

$g'(-2) = \frac{[2+4](-2) - (-3-2)(4)}{(-2)^2} \to \frac{-12+20}{4} \to \frac{8}{4} = 2$

c. $w(x) = (4 - 2h(x))(1 - r(x))$
Find $w'(-2)$.

$w'(x) = -2h'(x)[1-r(x)] + (4-2h(x))\cdot -r'(x)$

$w'(-2) = -2h'(-2)[1-r(-2)] + (4-2h(-2))\cdot -r'(-2)$

$= -2(2)(1+2) + (4+6)(-4)$

$= -4(3) + 10(-4)$

$= -52$

$w'(-2) = -52$

22. At what x-value(s) does the function $f(x) = \frac{x^4}{4} - 3x^3 + 9x^2 + 7$ have a horizontal tangent?

* find horizontal tangent by setting (numerator) of $f'(x) = 0$.

$f'(x) = 4\cdot\frac{1}{4}x^3 - 9x^2 + 18x$

$f'(x) = x^3 - 9x^2 + 18x$

$0 = x(x^2-9x+18)$

$0 = x(x-6)(x-3)$

$x=0,\ x=3,\ x=6$

23. If $f(x) = \cos x + \sin x$, find $f'\left(\frac{\pi}{3}\right)$

$f'(x) = -\sin x + \cos x$

$f'(\frac{\pi}{3}) = -\sin(\frac{\pi}{3}) + \cos(\frac{\pi}{3})$

$f'(\frac{\pi}{3}) = -(\frac{\sqrt{3}}{2}) + \frac{1}{2} = \frac{-\sqrt{3}+1}{2}$

$f'(\frac{\pi}{3}) = \frac{1-\sqrt{3}}{2}$

24. $S(x)$ is the number of students in Mr. Kelly's class and x is the number of years since 2015.

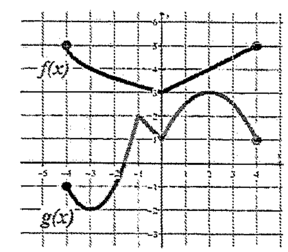
a. Explain the meaning of $S(3) = 127$.

In 2018, there are 127 students in Mr. Kelly's class.

b. Explain the meaning of $S'(3) = 4$.

In 2018, the number of students in Mr. Kelly's class is increasing by 4 students per year.

25. Use the graphs of *f* and *g* to find the following.



a. $h(x) = f(g(x))$. Find the average rate of change on the interval [2,4].

$g(4) = 1$, $g(2) = 3$

$\frac{f(g(4)) - f(g(2))}{4-2} \to \frac{f(1)-f(3)}{4-2} \to \frac{3.5-4.5}{2} = \frac{-1}{2}$

b. $j(x) = g(f(x))$. Find the average rate of change on the interval [−3,2].

$f(2) = 4$, $f(-3) = 4$

$\frac{g(f(2)) - g(f(-3))}{2-(-3)} \to \frac{g(4)-g(4)}{5} \to \frac{1-1}{5} \to \frac{0}{5} \to 0$

20

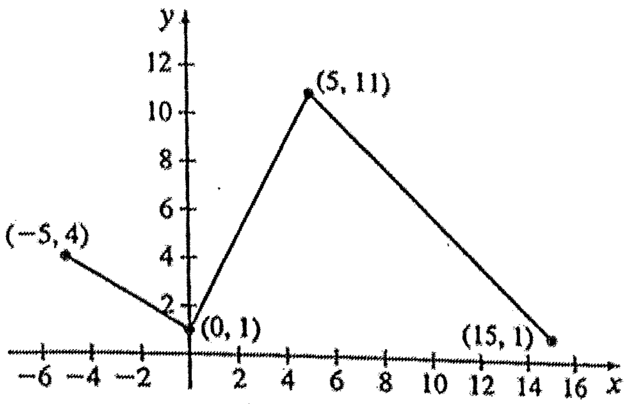
2.1 AP Practice Problems (p.171) – Rates of Change and the Derivative

1. The line $x + y = 5$ is tangent to the graph of $y = f(x)$ at the point where $x = 2$. The values $f(2)$ and $f'(2)$ are:

   (A) $f(2) = 2;\ f'(2) = -1$ (B) $f(2) = 3;\ f'(2) = -1$

   (C) $f(2) = 2;\ f'(2) = 1$ (D) $f(2) = 3;\ f'(2) = 2$

2. The graph of the function f, given below, consists of three line segments. Find $f'(3)$.



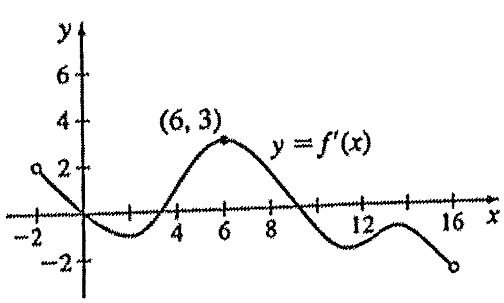
Graph of f

   (A) 1 (B) 2 (C) 3 (D) $f'(3)$ does not exist

3. What is the instantaneous rate of change of the function $f(x) = 3x^2 + 5$ at $x = 2$?

   (A) 5 (B) 7 (C) 12 (D) 17

4. The function f is defined on the closed interval $[-2, 16]$. The graph of the derivative of f, $y = f'(x)$, is given below.



The point $(6, -2)$ is on the graph of $y = f(x)$. An equation of the tangent line to the graph of f at $(6, -2)$ is

   (A) $y = 3$ (B) $y + 2 = 6(x + 3)$

   (C) $y + 2 = 6x$ (D) $y + 2 = 3(x - 6)$

(22)

5. If $x - 3y = 13$ is an equation of the normal line to the graph of f at the point (2, 6), then $f'(2) =$

(A) $-\frac{1}{3}$ (B) $\frac{1}{3}$ (C) -3 (D) $-\frac{13}{3}$

6. If f is a function for which $\lim_{x \to -3} \frac{f(x) - f(-3)}{x+3} = 0$, then which of the following statements must be true?

(A) $x = -3$ is a vertical asymptote of the graph.
(B) The derivative of f at $x = -3$ exists.
(C) The function f is continuous at $x = 3$.
(D) f is not defined at $x = -3$.

7. If the position of an object on the x-axis at time t is $4t^2$, then the average velocity of the object over the interval $0 \leq t \leq 5$ is

(A) 5 (B) 20 (C) 40 (D) 100

8. A tank is filled with 80 liters of water at 7 a.m. ($t = 0$). Over the next 12 hours the water is continuously used and no water is added to replace it. The table below gives the amount of water $A(t)$ (in liters) remaining in the tank at selected times t, where t measures the number of hours after 7 a.m.

t	0	2	5	7	9	12
$A(t)$	80	71	66	60	54	50

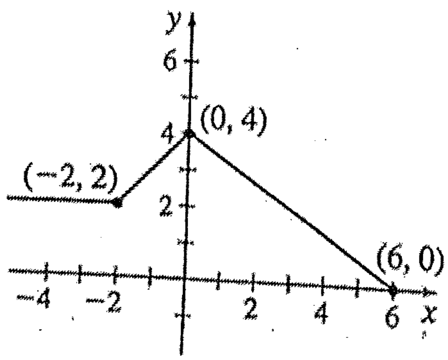
Use the table to approximate $A'(5)$.

2.2 AP Practice Problems (p.182) – Derivative as a function & differentiability

1. The function $f(x) = \begin{cases} x^2 - ax & \text{if } x \le 1 \\ ax + b & \text{if } x > 1 \end{cases}$, where a and b are constants. If f is differentiable at $x = 1$, then $a + b =$

(A) -3 (B) -2 (C) 0 (D) 2

2. The graph of the function f, given below, consists of three line segments. Find $\lim_{h \to 0} \frac{f(3+h) - f(3)}{h}$.



(A) -1 (B) $-\frac{2}{3}$ (C) $-\frac{3}{2}$ (D) does not exist

3. If $f(x) = \begin{cases} \dfrac{x^2 - 25}{x - 5} & \text{if } x \ne 5 \\ 5 & \text{if } x = 5 \end{cases}$

which of the following statements about f are true?

I. $\lim_{x \to 5} f$ exists.

II. f is continuous at $x = 5$.

III. f is differentiable at $x = 5$.

(A) I only (B) I and II only

(C) I and III only (D) I, II, and III

4. Suppose f is a function that is differentiable on the open interval $(-2, 8)$. If $f(0) = 3$, $f(2) = -3$, and $f(7) = 3$, which of the following must be true?

I. f has at least 2 zeros.

II. f is continuous on the closed interval $[-1, 7]$.

III. For some c, $0 < c < 7$, $f(c) = -2$.

(A) I only (B) I and II only

(C) II and III only (D) I, II, and III

5. If $f(x) = |x|$, which of the following statements about f are true?

I. f is continuous at 0.

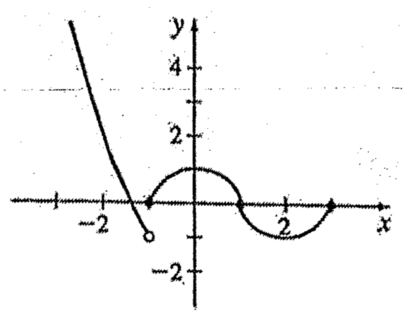
II. f is differentiable at 0.

III. $f(0) = 0$.

(A) I only (B) III only

(C) I and III only (D) I, II, and III

6. The graph of the function f shown in the figure has horizontal tangent lines at the points $(0, 1)$ and $(2, -1)$ and a vertical tangent line at the point $(1, 0)$. For what numbers x in the open interval $(-2, 3)$ is f not differentiable?



(A) -1 only (B) -1 and 1 only

(C) $-1, 0$, and 2 only (D) $-1, 0, 1$, and 2

7. Let f be a function for which $\lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = -3$.
Which of the following must be true?
I. f is continuous at 1.
II. f is differentiable at 1.
III. f' is continuous at 1.

(A) I only (B) II only

(C) I and II only (D) I, II, and III

8. At what point on the graph of $f(x) = x^2 - 4$ is the tangent line parallel to the line $6x - 3y = 2$?

(A) $(1, -3)$ (B) $(1, 2)$ (C) $(2, 0)$ (D) $(2, 4)$

9. At $x = 2$, the function $f(x) = \begin{cases} 4x + 1 & \text{if } x \leq 2 \\ 3x^2 - 3 & \text{if } x > 2 \end{cases}$ is

(A) Both continuous and differentiable.
(B) Continuous but not differentiable.
(C) Differentiable but not continuous.
(D) Neither continuous nor differentiable.

(26)

10. Oil is leaking from a tank. The amount of oil, in gallons, in the tank is given by $G(t) = 4000 - 3t^2$, where t, $0 \le t \le 24$ is the number of hours past midnight.
   (a) Find $G'(5)$ using the definition of the derivative.
   (b) Using appropriate units, interpret the meaning of $G'(5)$ in the context of the problem.

11. A rod of length 12 cm is heated at one end. The table below gives the temperature $T(x)$ in degrees Celsius at selected numbers x cm from the heated end.

x	0	2	5	7	9	12
$T(x)$	80	71	66	60	54	50

   (a) Use the table to approximate $T'(8)$.
   (b) Using appropriate units, interpret $T'(8)$ in the context of the problem.

2.3 AP Practice Problems (p. 193) – Derivative Power Rule & exponential e^x

1. If $g(x) = x$, then $g'(7) =$

   (A) 0 (B) 1 (C) 7 (D) $\frac{49}{2}$

2. The line $x + y = k$, where k is a constant, is a tangent line to the graph of the function $f(x) = x^2 - 5x + 2$. What is the value of k?

   (A) -1 (B) 2 (C) -2 (D) -4

3. An object moves along the x-axis so that its position at time t is $x(t) = 3t^2 - 9t + 7$. For what time t is the velocity of the object zero?

   (A) -3 (B) 3 (C) $\frac{3}{2}$ (D) 7

4. If $f(x) = e^x$, then $\ln(f'(3)) =$

   (A) 3 (B) 0 (C) e^3 (D) $\ln 3$

5. An equation of the normal line to the graph of $g(x) = x^3 + 2x^2 - 2x + 1$ at the point where $x = -2$ is

   (A) $x + 2y = 12$ (B) $x - 2y = 8$
   (C) $2x + y = -9$ (D) $x + 2y = 8$

6. The line $9x - 16y = 0$ is tangent to the graph of $f(x) = 3x^3 + k$, where k is a constant, at a point in the first quadrant. Find k.

   (A) $\frac{3}{32}$ (B) $\frac{3}{16}$ (C) $\frac{3}{64}$ (D) $\frac{9}{64}$

(28)

7. If $f(x) = 1 + |x - 4|$, find $f'(4)$.

(A) -1 (B) 0 (C) 1 (D) $f'(4)$ does not exist.

8. The cost C (in dollars) of manufacturing x units of a product is $C(x) = 0.3x^2 + 4.02x + 3500$.
What is the rate of change of C when $x = 1000$ units?

(A) 307.52 (B) 0.60402 (C) 604.02 (D) 1020

9. $\frac{d}{dx}(5 \ln x) =$

(A) $\frac{1}{5x}$ (B) $5e^x$ (C) $-\frac{5}{\ln x}$ (D) $\frac{5}{x}$

10. For the function $f(x) = x^2 + 4$

(a) Find $f'(1)$.

(b) Find an equation of the tangent line to the graph of f at $x = 1$.

(c) Find $f'(-4)$.

(d) Find an equation of the tangent line to the graph of f at $x = -4$.

(e) Find the point of intersection of the two tangent lines found in (b) and (d).

---

11. Which is an equation of the tangent line to the graph of $f(x) = x^4 + 3x^2 + 2$ at the point where $f'(x) = 2$?

(A) $y = 2x + 2$ (B) $y = 2x + 2.929$

(C) $y = 2x + 1.678$ (D) $y = 2x - 2.929$

2.4 AP Practice Problems (p. 206) – Product & Quotient Rule & Higher order derivatives

1. What is the instantaneous rate of change at $x = -2$ of the function $f(x) = \dfrac{x-1}{x^2+2}$?

(A) $-\dfrac{1}{6}$ (B) $\dfrac{1}{9}$ (C) $\dfrac{1}{2}$ (D) -1

2. An equation of the tangent line to the graph of $f(x) = \dfrac{5x-3}{3x-6}$ at the point $(3, 4)$ is

(A) $7x + 3y = 37$ (B) $7x + 3y = 33$
(C) $7x - 3y = 9$ (D) $13x + 3y = 51$

3. If f, g, and h are nonzero differentiable functions of x, then $\dfrac{d}{dx}\left(\dfrac{gh}{f}\right) =$

(A) $\dfrac{fgh' + fg'h - f'gh}{f^2}$ (B) $\dfrac{g'h' - ghf'}{f^2}$
(C) $\dfrac{gh' + g'h}{f'}$ (D) $\dfrac{fgh' + fg'h + f'gh}{f^2}$

4. If $y = x^3e^x$, then $\dfrac{dy}{dx} =$

(A) $3x^2e^x$ (B) $3x^2 + e^x$
(C) $3x^2e^x(x+1)$ (D) $x^2e^x(x+3)$

5. $\dfrac{d}{dt}\left(t^2 - \dfrac{1}{t^2} + \dfrac{1}{t}\right)$ at $t = 2$ is

(A) $\dfrac{7}{2}$ (B) $\dfrac{9}{2}$ (C) $\dfrac{9}{4}$ (D) 4

6. The position of an object moving along a straight line at time t, in seconds, is given by $s(t) = 16t^2 - 5t + 20$ meters. What is the acceleration of the object when $t = 2$?

(A) 32 m/s (B) 0 m/s² (C) 32 m/s² (D) 64 m/s²

7. If $y = \frac{x-3}{x+3}$, $x \neq -3$, the instantaneous rate of change of y with respect to x at $x = 3$ is

(A) $-\frac{1}{6}$ (B) $\frac{1}{6}$ (C) $\frac{1}{36}$ (D) 1

8. Find an equation of the normal line to the graph of the function $f(x) = \frac{x^2}{x+1}$ at $x = 1$.

(A) $8x + 6y = 11$ (B) $-8x + 6y = -5$
(C) $-3x + 4y = -1$ (D) $3x + 4y = 5$

9. If $y = xe^x$, then the nth derivative of y is

(A) e^x (B) $(x+n)e^x$ (C) ne^x (D) $x^n e^x$

2.5 AP Practice Problems (p. 214) – Derivatives of Trig Functions

1. If $y = x\sin x$, then $\frac{dy}{dx} =$

   (A) $x\cos x + \sin x$ (B) $x\cos x - \sin x$
   (C) $\cos x + \sin x$ (D) $(x+1)\cos x$

2. What is $\lim_{h\to 0} \frac{\cos\left(\frac{\pi}{3}+h\right) - \cos\frac{\pi}{3}}{h}$?

   (A) 0 (B) $\frac{1}{2}$ (C) $\frac{\sqrt{3}}{2}$ (D) $-\frac{\sqrt{3}}{2}$

3. If $f(x) = \tan x$, then $f'\left(\frac{\pi}{3}\right)$ equals

   (A) $2\sqrt{3}$ (B) 4 (C) 2 (D) $\frac{1}{4}$

4. The position s (in meters) of an object moving along a horizontal line at time t, $0 \le t \le \frac{\pi}{2}$, (in seconds) is given by $s(t) = 6\sin t + \frac{3}{2}t^2 + 8$. What is the velocity of the object when its acceleration is zero?

   (A) 6 m/s (B) $3 + \pi$ m/s
   (C) $\frac{6\sqrt{3}+\pi}{2}$ m/s (D) $\left(3\sqrt{3} - \frac{\pi}{2}\right)$ m/s

5. If $y = \sin x$, then $\frac{d^{50}}{dx^{50}} \sin x$ equals

   (A) $\sin x$ (B) $-\sin x$ (C) $\cos x$ (D) $-\cos x$

6. If $f(x) = \dfrac{x}{\cos x}$, find $f'\left(\dfrac{\pi}{3}\right)$.

(A) $2 - \dfrac{2\sqrt{3}}{3}\pi$ (B) $1 + \dfrac{\sqrt{3}}{3}\pi$

(C) $1 - \dfrac{\sqrt{3}}{3}\pi$ (D) $2 + \dfrac{2\sqrt{3}}{3}\pi$

7. If $y = x - \tan x$, then $\dfrac{dy}{dx}$ equals

(A) $1 - \sec x \tan x$ (B) $-\tan^2 x$

(C) $\tan^2 x$ (D) $-\sec^2 x$

8. If $g(x) = e^x \cos x + 2\pi$, then $g'(x) =$

(A) $e^x - \sin x$ (B) $e^x \cos x - e^x \sin x + 3\pi$

(C) $e^x \cos x - e^x \sin x$ (D) $e^x \cos x + e^x \sin x$

9. At which of the following numbers x, $0 \le x \le 2\pi$, does the graph of $y = x + \cos x$ have a horizontal tangent line?

(A) 0 only (B) $\dfrac{\pi}{2}$ only

(C) $\dfrac{3\pi}{2}$ only (D) 0 and $\dfrac{\pi}{2}$ only

10. An equation of the tangent line to the graph of $f(x) = \sin x$ at $x = \dfrac{2\pi}{3}$ is

(A) $3x + 6y = 4\pi - 3\sqrt{3}$ (B) $3x + 6y = 2\pi + 3\sqrt{3}$

(C) $6y - 3x = 2\pi - 3\sqrt{3}$ (D) $6y - 3x = 4\pi - 3\sqrt{3}$

Ch.2 Review AP Practice Problems (p. 220) – Derivative definition and properties

1. If $f(x) = \sec x$, then $f'\left(\frac{\pi}{4}\right) =$

(A) $\frac{\sqrt{2}}{2}$ (B) 2 (C) 1 (D) $\sqrt{2}$

2. If a function f is differentiable at c, then $f'(c)$ is given by

I. $\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$

II. $\lim_{x \to c} \frac{f(x+h) - f(x)}{h}$

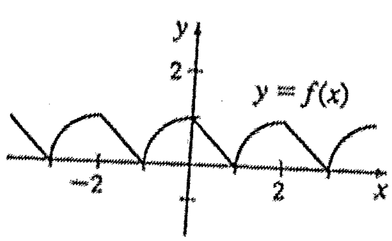
III. $\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$

(A) I only (B) III only
(C) I and II only (D) I and III only

3. If $y = \frac{3}{x^2 - 5}$, then $\frac{dy}{dx} =$

(A) $\frac{6x}{(x^2-5)^2}$ (B) $-\frac{6x}{(x^2-5)^2}$
(C) $\frac{6x}{x^2-5}$ (D) $\frac{2x}{(x^2-5)^2}$

4. The graph of the function f is shown below. Which statement about the function is true?



(A) f is differentiable everywhere.
(B) $0 \le f'(x) \le 1$, for all real numbers.
(C) f is continuous everywhere.
(D) f is an even function.

5. The table displays select values of a differentiable function f. What is an approximate value of $f'(2)$?

x	1.996	1.998	2	2.002	2.004
$f(x)$	3.168	3.181	3.194	3.207	3.220

(A) 6.5 (B) 1.154 (C) 0.013 (D) 0.0016

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6. If $y = \sin x + xe^x + 6$, what is the instantaneous rate of change of y with respect to x at $x = 5$?

(A) $\cos 5 + 6e^5$ (B) 2

(C) $\cos 5 + 5e^5$ (D) $6e^5 - \cos 5$

7. An equation of the normal line to the graph of $f(x) = 3xe^x + 5$ at $x = 0$ is

(A) $y = 3x + 5$ (B) $y = -\frac{1}{3}x + 5$

(C) $y = \frac{1}{3}x + 5$ (D) $y = -3x + 5$

8. An object moves along a horizontal line so that its position at time t is $s(t) = t^4 - 6t^3 - 2t - 1$. At what time t is the acceleration of the object zero?

(A) at 0 only (B) at 1 only

(C) at 3 only (D) at 0 and 3 only

9. If $f(x) = e^x(\sin x + \cos x)$, then $f'(x) =$

(A) $2e^x(\cos x + \sin x)$ (B) $e^x \cos x$

(C) $2e^x \cos x$ (D) $e^x(\cos^2 x - \sin^2 x)$

10. Find an equation of the tangent line to the graph of $f(x) = \dfrac{x+3}{x^2+2}$ at $x = 1$.

(A) $5x + 9y = 17$ (B) $9y - 5x = 7$

(C) $5x + 3y = 9$ (D) $5x + 9y = 7$

11. $\displaystyle\lim_{x \to \frac{\pi}{4}} \frac{\tan x - 1}{x - \frac{\pi}{4}} =$

(A) 0 (B) -1 (C) 2 (D) Does not exist.