Name: ______________________________ Period: __________

BC Calculus

Unit 3

Differentiation Part 2

(Composite, Implicit, & Inverse Functions)

AP Calculus – 3.1 Notes - Chain Rule

Composite Functions:

$f(g(x))$

$\sin(x^2)$	$\sqrt{\ln x}$	$\cos(\sin(5x))$

The Chain Rule (derivative of a composite function)

$$\frac{d}{dx}f(g(x)) =$$

Find the derivative

1. $f(x) = (x^2 - 5)^4$	2. $g(x) = \sqrt{4x - 3}$	3. $h(x) = \sin^2 5x$

4. $y = \ln(x^3)$

5. $y = \ln(x^3)$

6. $f(x) = \left(\frac{t^2+1}{2t-5}\right)^3$

②

7. If $g(x) = 2x\sqrt{1-x}$ find $g'(-3)$.

8. Given the following table of values, find $f'(4)$ for each function.

x	$g(x)$	$g'(x)$	$h(x)$	$h'(x)$
3	-1	7	-2	-3
4	3	-2	9	5

$f(x) = (g(x))^2$

$f(x) = \sqrt{h(x)}$

$f(x) = h(g(x))$

Practice Problems:

Find the derivative of each function.

1. $g(x) = (3x^2 - 1)^5$
2. $y = \sin 2x$
3. $h(r) = \sqrt[3]{5r^2 - 2r + 1}$
4. $y = \sqrt{4 - \cos(x^2)}$
5. $h(x) = \ln(5^x)$
6. $g(x) = \ln(2x^3)$
7. $f(x) = \sqrt{\tan(2x)}$
8. $y = \cos^2 x$
9. $y = \frac{1}{(7x^2-1)^2}$

3.1 Assess Your Understanding

Concepts and Vocabulary

1. The derivative of a composite function $(f \circ g)(x)$ can be found using the ______ Rule.
2. *True or False* If $y = f(u)$ and $u = g(x)$ are differentiable functions, then $y = f(g(x))$ is differentiable.
3. *True or False* If $y = f(g(x))$ is a differentiable function, then $y' = f'(g'(x))$.
4. To find the derivative of $y = \tan(1 + \cos x)$, using the Chain Rule, begin with $y =$ ______ and $u =$ ______.
5. If $y = (x^3 + 4x + 1)^{100}$, then $y' =$ ______.
6. If $f(x) = e^{3x^2+5}$, then $f'(x) =$ ______.
7. *True or False* The Chain Rule can be applied to multiple composite functions.
8. $\dfrac{d}{dx} \sin x^2 =$ ______.

Skill Building

In Problems 9–14, write y as a function of x. Find $\dfrac{dy}{dx}$ using the Chain Rule.

PAGE 224 9. $y = u^5$, $u = x^3 + 1$

10. $y = u^3$, $u = 2x + 5$

11. $y = \dfrac{u}{u+1}$, $u = x^2 + 1$

12. $y = \dfrac{u-1}{u}$, $u = x^2 - 1$

13. $y = (u+1)^2$, $u = \dfrac{1}{x}$

14. $y = (u^2 - 1)^3$, $u = \dfrac{1}{x+2}$

In Problems 15–32, find the derivative of each function using the Power Rule for Functions.

15. $f(x) = (3x+5)^2$

16. $f(x) = (2x-5)^3$

17. $f(x) = (6x-5)^{-3}$

18. $f(t) = (4t+1)^{-2}$

19. $g(x) = (x^2+5)^4$

20. $F(x) = (x^3-2)^5$

PAGE 228 21. $f(u) = \left(u - \dfrac{1}{u}\right)^3$

22. $f(x) = \left(x + \dfrac{1}{x}\right)^3$

23. $g(x) = (4x + e^x)^3$

24. $F(x) = (e^x - x^2)^2$

25. $f(x) = \tan^2 x$

26. $f(x) = \sec^3 x$

27. $f(z) = (\tan z + \cos z)^2$

28. $f(z) = (e^z + 2\sin z)^3$

PAGE 229 29. $y = (x^2+4)^2(2x^3-1)^3$

30. $y = (x^2-2)^3(3x^4+1)^2$

31. $y = \left(\dfrac{\sin x}{x}\right)^2$

32. $y = \left(\dfrac{x + \cos x}{x}\right)^5$

In Problems 33–54, find y′.

33. $y = \sin(4x)$

34. $y = \cos(5x)$

35. $y = 2\sin(x^2 + 2x - 1)$

36. $y = \dfrac{1}{2}\cos(x^3 - 2x + 5)$

PAGE 224 37. $y = \sin\dfrac{1}{x}$

38. $y = \sin\dfrac{3}{x}$

39. $y = \sec(4x)$

40. $y = \cot(5x)$

PAGE 225 41. $y = e^{1/x}$

42. $y = e^{1/x^2}$

43. $y = \dfrac{1}{x^4 - 2x + 1}$

44. $y = \dfrac{3}{x^5 + 2x^2 - 3}$

45. $y = \dfrac{100}{1 + 99e^{-x}}$

46. $y = \dfrac{1}{1 + 2e^{-x}}$

PAGE 228 47. $y = 2^{\sin x}$

48. $y = (\sqrt{3})^{\cos x}$

49. $y = 6^{\sec x}$

50. $y = 3^{\tan x}$

51. $y = 5xe^{3x}$

52. $y = x^3 e^{2x}$

53. $y = x^2 \sin(4x)$

54. $y = x^2 \cos(4x)$

In Problems 55–58, find y′. Treat a and b as constants.

55. $y = e^{-ax}\sin(bx)$

56. $y = e^{ax}\cos(-bx)$

57. $y = \dfrac{e^{ax} - 1}{e^{ax} + 1}$

58. $y = \dfrac{e^{-ax} + 1}{e^{bx} - 1}$

In Problems 59–62, write y as a function of x. Find $\dfrac{dy}{dx}$ using the Chain Rule.

PAGE 230 59. $y = u^3$, $u = 3v^2 + 1$, $v = \dfrac{4}{x^2}$

60. $y = 3u$, $u = 3v^2 - 4$, $v = \dfrac{1}{x}$

61. $y = u^2 + 1$, $u = \dfrac{4}{v}$, $v = x^2$

62. $y = u^3 - 1$, $u = -\dfrac{2}{v}$, $v = x^3$

In Problems 63–70, find y′.

63. $y = e^{-2x}\cos(3x)$

64. $y = e^{\pi x}\tan(\pi x)$

65. $y = \cos(e^{x^2})$

66. $y = \tan(e^{x^2})$

67. $y = e^{\cos(4x)}$

68. $y = e^{\csc^2 x}$

PAGE 230 69. $y = 4\sin^2(3x)$

70. $y = 2\cos^2(x^2)$

In Problems 71 and 72, find the derivative of each function by:

(a) Using the Chain Rule.
(b) Using the Power Rule for Functions.
(c) Expanding and then differentiating.
(d) Verify the answers from parts (a)–(c) are equal.

71. $y = (x^3 + 1)^2$

72. $y = (x^2 - 2)^3$

In Problems 73–78:

(a) Find an equation of the tangent line to the graph of f at the given point.
(b) Find an equation of the normal line to the graph of f at the given point.
(c) Use technology to graph f, the tangent line, and the normal line on the same screen.

73. $f(x)=(x^2-2x+1)^5$ at $(1, 0)$

74. $f(x)=(x^3-x^2+x-1)^{10}$ at $(0, 1)$

75. $f(x)=\dfrac{x}{(x^2-1)^3}$ at $\left(2, \dfrac{2}{27}\right)$

76. $f(x)=\dfrac{x^2}{(x^2-1)^2}$ at $\left(2, \dfrac{4}{9}\right)$

PAGE 226 77. $f(x)=\sin(2x)+\cos\dfrac{x}{2}$ at $(0, 1)$

78. $f(x)=\sin^2 x+\cos^3 x$ at $\left(\dfrac{\pi}{2}, 1\right)$

In Problems 79 and 80, find the indicated derivative.

79. $\dfrac{d^2}{dx^2}\cos(x^5)$

80. $\dfrac{d^3}{dx^3}\sin^3 x$

PAGE 227 81. Suppose $h=f\circ g$. Find $h'(1)$ if $f'(2)=6$, $f(1)=4$, $g(1)=2$, and $g'(1)=-2$.

82. Suppose $h=f\circ g$. Find $h'(1)$ if $f'(3)=4$, $f(1)=1$, $g(1)=3$, and $g'(1)=3$.

83. Suppose $h=g\circ f$. Find $h'(0)$ if $f(0)=3$, $f'(0)=-1$, $g(3)=8$, and $g'(3)=0$.

84. Suppose $h=g\circ f$. Find $h'(2)$ if $f(1)=2$, $f'(1)=4$, $f(2)=-3$, $f'(2)=4$, $g(-3)=1$, and $g'(-3)=3$.

85. If $y=u^5+u$ and $u=4x^3+x-4$, find $\dfrac{dy}{dx}$ at $x=1$.

86. If $y=e^u+3u$ and $u=\cos x$, find $\dfrac{dy}{dx}$ at $x=0$.

Applications and Extensions

In Problems 87–94, find the indicated derivative.

87. $\dfrac{d}{dx}f(x^2+1)$

Hint: Let $u=x^2+1$.

88. $\dfrac{d}{dx}f(1-x^2)$

89. $\dfrac{d}{dx}f\left(\dfrac{x+1}{x-1}\right)$

90. $\dfrac{d}{dx}f\left(\dfrac{1-x}{1+x}\right)$

91. $\dfrac{d}{dx}f(\sin x)$

92. $\dfrac{d}{dx}f(\tan x)$

93. $\dfrac{d^2}{dx^2}f(\cos x)$

94. $\dfrac{d^2}{dx^2}f(e^x)$

95. **Rectilinear Motion** An object is in rectilinear motion and its position s, in meters, from the origin at time t seconds is given by $s=s(t)=A\cos(\omega t+\phi)$, where A, ω, and ϕ are constants.
   (a) Find the velocity v of the object at time t.
   (b) When is the velocity of the object 0?
   (c) Find the acceleration a of the object at time t.
   (d) When is the acceleration of the object 0?

96. **Rectilinear Motion** A bullet is fired horizontally into a bale of paper. The distance s (in meters) the bullet travels into the bale of paper in t seconds is given by

$$s=s(t)=8-(2-t)^3 \quad 0\le t\le 2$$

   (a) Find the velocity v of the bullet at any time t.
   (b) Find the velocity of the bullet at $t=1$.
   (c) Find the acceleration a of the bullet at any time t.
   (d) Find the acceleration of the bullet at $t=1$.
   (e) How far into the bale of paper did the bullet travel?

97. **Rectilinear Motion** Find the acceleration a of a car if the distance s, in feet, it has traveled along a highway at time $t\ge 0$ seconds is given by

$$s(t)=\frac{80}{3}\left[t+\frac{3}{\pi}\sin\left(\frac{\pi}{6}t\right)\right]$$

98. **Rectilinear Motion** An object moves in rectilinear motion so that at time $t\ge 0$ seconds, its position from the origin is $s(t)=\sin e^t$, in feet.
   (a) Find the velocity v and acceleration a of the object at any time t.
   (b) At what time does the object first have zero velocity?
   (c) What is the acceleration of the object at the time t found in (b)?

PAGE 226 99. **Resistance** The resistance R (measured in ohms) of an 80-meter-long electric wire of radius x (in centimeters) is given by the formula $R=R(x)=\dfrac{0.0048}{x^2}$. The radius x is given by $x=0.1991+0.000003T$ where T is the temperature in Kelvin. How fast is R changing with respect to T when $T=320$ K?

100. **Pendulum Motion in a Car** The motion of a pendulum swinging in the direction of motion of a car moving at a low, constant speed can be modeled by

$$s=s(t)=0.05\sin(2t)+3t \qquad 0\le t\le\pi$$

where s is the distance in meters and t is the time in seconds.
   (a) Find the velocity v at $t=\dfrac{\pi}{8}$, $t=\dfrac{\pi}{4}$, and $t=\dfrac{\pi}{2}$.
   (b) Find the acceleration a at the times given in (a).
   (c) Graph $s=s(t)$, $v=v(t)$, and $a=a(t)$ on the same screen.

Source: Mathematics students at Trine University, Angola, Indiana.

101. **Economics** The function $A(t)=102-90\,e^{-0.21t}$ represents the relationship between A, the percentage of the market penetrated by the latest generation smart phones, and t, the time in years, where $t=0$ corresponds to the year 2020.
   (a) Find $\lim\limits_{t\to\infty}A(t)$ and interpret the result.
   (b) Graph the function $A=A(t)$, and explain how the graph supports the answer in (a).

(c) Find the rate of change of A with respect to time.

(d) Evaluate $A'(5)$ and $A'(10)$ and interpret these results.

(e) Graph the function $A' = A'(t)$, and explain how the graph supports the answers in (d).

102. **Meteorology** The atmospheric pressure at a height of x meters above sea level is $P(x) = 10^4 e^{-0.00012x}$ kg/m^2. What is the rate of change of the pressure with respect to the height at $x = 500$ m? At $x = 750$ m?

103. **Hailstones** Hailstones originate at an altitude of about 3000 m, although this varies. As they fall, air resistance slows down the hailstones considerably. In one model of air resistance, the speed of a hailstone of mass m as a function of time t is given by $v(t) = \frac{mg}{k}(1 - e^{-kt/m})$ m/s, where $g = 9.8$ m/s^2 is the acceleration due to gravity and k is a constant that depends on the size of the hailstone and the conditions of the air.

(a) Find the acceleration $a(t)$ of a hailstone as a function of time t.

(b) Find $\lim_{t\to\infty} v(t)$. What does this limit say about the speed of the hailstone?

(c) Find $\lim_{t\to\infty} a(t)$. What does this limit say about the acceleration of the hailstone?

104. **Mean Earnings** The mean earnings E, in dollars, of workers 18 years and over are given in the table below:

Year	1975	1980	1985	1990	1995	2000	2005	2010	2015
Mean Earnings	8,552	12,665	17,181	21,793	26,792	32,604	41,231	49,733	48,000

Source: U.S. Bureau of the Census, Current Population Survey.

(a) Find the exponential function of best fit and show that it equals $E = E(t) = 9854(1.05)^t$, where t is the number of years since 1974.

(b) Find the rate of change of E with respect to t.

(c) Find the rate of change at $t = 26$ (year 2000).

(d) Find the rate of change at $t = 36$ (year 2010).

(e) Find the rate of change at $t = 41$ (year 2015).

(f) Compare the answers to (c), (d), and (e). Interpret each answer and explain the differences.

105. **Rectilinear Motion** An object moves in rectilinear motion so that at time $t > 0$ its position s from the origin is $s = s(t)$. The velocity v of the object is $v = \frac{ds}{dt}$, and its acceleration is $a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$. If the velocity $v = v(s)$ is expressed as a function of s, show that the acceleration a can be expressed as $a = v\frac{dv}{ds}$.

106. **Student Approval** Professor Miller's student approval rating is modeled by the function $Q(t) = 21 + \dfrac{10\sin\frac{2\pi t}{7}}{\sqrt{t} - \sqrt{20}}$, where $0 \le t \le 16$ is the number of weeks since the semester began.

(a) Find $Q'(t)$.

(b) Evaluate $Q'(1)$, $Q'(5)$, and $Q'(10)$.

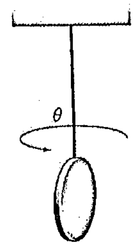
(c) Interpret the results obtained in (b).

(d) Use technology to graph $Q(t)$ and $Q'(t)$.

(e) How would you explain the results in (d) to Professor Miller?

Source: Mathematics students at Millikin University, Decatur, Illinois.

107. **Angular Velocity** If the disk in the figure is rotated about a vertical line through an angle θ, torsion in the wire attempts to turn the disk in the opposite direction. The motion θ at time t (assuming no friction or air resistance) obeys the equation



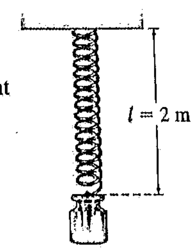
$$\theta(t) = \frac{\pi}{3}\cos\left(\frac{1}{2}\sqrt{\frac{2k}{5}}\,t\right)$$

where k is the coefficient of torsion of the wire.

(a) Find the angular velocity $\omega = \frac{d\theta}{dt}$ of the disk at any time t.

(b) What is the angular velocity at $t = 3$?

108. **Harmonic Motion** A weight hangs on a spring making it 2 m long when it is stretched out (see the figure). If the weight is pulled down and then released, the weight oscillates up and down, and the length l of the spring after t seconds is given by $l(t) = 2 + \cos(2\pi t)$.



(a) Find the length l of the spring at the times $t = 0, \frac{1}{2}, 1, \frac{3}{2}$, and $\frac{5}{8}$.

(b) Find the velocity v of the weight at time $t = \frac{1}{4}$.

(c) Find the acceleration a of the weight at time $t = \frac{1}{4}$.

109. Find $F'(1)$ if $f(x) = \sin x$ and $F(t) = f(t^2 - 1)$.

110. **Normal Line** Find the point on the graph of $y = e^{-x}$ where the normal line to the graph passes through the origin.

111. Use the Chain Rule and the fact that $\cos x = \sin\left(\frac{\pi}{2} - x\right)$ to show that $\frac{d}{dx}\cos x = -\sin x$.

112. If $y = e^{2x}$, show that $y'' - 4y = 0$.

113. If $y = e^{-2x}$, show that $y'' - 4y = 0$.

114. If $y = Ae^{2x} + Be^{-2x}$, where A and B are constants, show that $y'' - 4y = 0$.

115. If $y = Ae^{ax} + Be^{-ax}$, where A, B, and a are constants, show that $y'' - a^2 y = 0$.

116. If $y = Ae^{2x} + Be^{3x}$, where A and B are constants, show that $y'' - 5y' + 6y = 0$.

117. If $y = Ae^{-2x} + Be^{-x}$, where A and B are constants, show that $y'' + 3y' + 2y = 0$.

118. If $y = A\sin(\omega t) + B\cos(\omega t)$, where A, B, and ω are constants, show that $y'' + \omega^2 y = 0$.

119. Show that $\frac{d}{dx} f(h(x)) = 2xg(x^2)$, if $\frac{d}{dx} f(x) = g(x)$ and $h(x) = x^2$.

120. Find the nth derivative of $f(x) = (2x+3)^n$.

121. Find a general formula for the nth derivative of y.
(a) $y = e^{ax}$ (b) $y = e^{-ax}$

122. (a) What is $\frac{d^{10}}{dx^{10}} \sin(ax)$?
(b) What is $\frac{d^{25}}{dx^{25}} \sin(ax)$?
(c) Find the nth derivative of $f(x) = \sin(ax)$.

123. (a) What is $\frac{d^{11}}{dx^{11}} \cos(ax)$?
(b) What is $\frac{d^{12}}{dx^{12}} \cos(ax)$?
(c) Find the nth derivative of $f(x) = \cos(ax)$.

124. If $y = e^{-at}[A\sin(\omega t) + B\cos(\omega t)]$, where A, B, a, and ω are constants, find y' and y''.

125. Show that if a function f has the properties:

- $f(u+v) = f(u)f(v)$ for all choices of u and v
- $f(x) = 1 + xg(x)$, where $\lim_{x \to 0} g(x) = 1$,

then $f' = f$.

Challenge Problems

126. Find the nth derivative of $f(x) = \frac{1}{3x-4}$.

127. Let $f_1(x), \ldots, f_n(x)$ be n differentiable functions. Find the derivative of $y = f_1(f_2(f_3(\ldots(f_n(x)\ldots))))$.

128. Let

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Show that $f'(0)$ exists, but that f' is not continuous at 0.

129. Define f by

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Show that f is differentiable on $(-\infty, \infty)$ and find f' for each value of x.
Hint: To find $f'(0)$, use the definition of the derivative. Then show that $1 < x^2 e^{1/x^2}$ for $x \neq 0$. PAGE 229

130. Suppose $f(x) = x^2$ and $g(x) = |x-1|$. The functions f and g are continuous on their domains, the set of all real numbers.

(a) Is f differentiable at all real numbers? If not, where does f' not exist? PAGE 224
(b) Is g differentiable at all real numbers? If not, where does g' not exist?
(c) Can the Chain Rule be used to differentiate the composite function $(f \circ g)(x)$ for all x? Explain. PAGE 225
(d) Is the composite function $(f \circ g)(x)$ differentiable? If so, what is its derivative?

131. Suppose $f(x) = x^4$ and $g(x) = x^{1/3}$. The functions f and g are continuous on their domains, the set of all real numbers.

(a) Is f differentiable at all real numbers? If not, where does f' not exist? PAGE 225
(b) Is g differentiable at all real numbers? If not, where does g' not exist?
(c) Can the Chain Rule be used to differentiate the composite function $(f \circ g)(x)$ for all x? Explain.
(d) Is the composite function $(f \circ g)(x)$ differentiable? If so, what is its derivative?

132. The function $f(x) = e^x$ has the property $f'(x) = f(x)$. Give an example of another function $g(x)$ such that $g(x)$ is defined for all real x, $g'(x) = g(x)$, and $g(x) \neq f(x)$.

133. Harmonic Motion The motion of the piston of an automobile engine is approximately simple harmonic. If the stroke of a piston (twice the amplitude) is 10 cm and the angular velocity is 60 revolutions per second, then the motion of the piston is given by $s(t) = 5\sin(120\pi t)$ cm.

(a) Find the acceleration a of the piston at the end of its stroke $\left(t = \frac{1}{240} \text{ second}\right)$.
(b) If the mass of the piston is 1 kg, what resultant force must be exerted on it at this point?
Hint: Use Newton's Second Law, that is, $F = ma$.

3.1 AP Practice Problems (p. 235) – The Chain Rule

1. If $f(x) = (x+3)(x^2-2)^4$, then $f'(x) =$

   (A) $8x(x^2-2)^3$

   (B) $8x(x+3)(x^2-2)^3$

   (C) $8x(x^2+x+1)(x^2-2)^3$

   (D) $(9x^2+24x-2)(x^2-2)^3$

2. If $f(x) = \sec(4x)$, then $f'\left(\frac{\pi}{6}\right) =$

   (A) $\frac{\sqrt{3}}{2}$ (B) $2\sqrt{3}$ (C) $8\sqrt{3}$ (D) $-8\sqrt{3}$

3. $\frac{d}{dx}e^{-3/x} =$

   (A) $\frac{3e^{-3/x}}{x^2}$ (B) $-3e^{-3/x}$

   (C) $-\frac{3e^{-3/x}}{x^2}$ (D) $-3x^2e^{-3/x}$

4. $\frac{d}{dx}\tan e^{-x} =$

   (A) $\sec^2 e^{-x}$ (B) $-x\sec^2 e^{-x}$

   (C) $e^{-x}\sec^2 e^{-x}$ (D) $-e^{-x}\sec^2 e^{-x}$

5. An equation of the normal line to the graph of $f(x) = 2(10-x)^2$ at the point $(9, 2)$ is

   (A) $y-2 = \frac{1}{4}(x-9)$ (B) $y-2 = -\frac{1}{4}(x-9)$

   (C) $y-2 = -4(x-9)$ (D) $y-2 = 4(x-9)$

(8)

6. If $y = \left(\dfrac{e^{4x}}{2x}\right)^2$, the instantaneous rate of change of y with respect to x is:

(A) $\dfrac{(4x-1)e^{8x}}{2x^3}$ (B) $\dfrac{(2x-1)e^{8x}}{x^3}$

(C) $\dfrac{(4x-1)e^{8x}}{x^3}$ (D) $\dfrac{(x-1)e^{8x}}{x^3}$

7. An equation of the tangent line to the graph of $g(x) = \sin(2x)$ at $x = \dfrac{\pi}{3}$ is

(A) $x + y = \dfrac{2\pi - 3\sqrt{3}}{6}$ (B) $x + y = \dfrac{3\sqrt{3} + 2\pi}{6}$

(C) $x + y = \dfrac{2\sqrt{3} + 3\pi}{6}$ (D) $y - x = \dfrac{2\pi - 3\sqrt{3}}{6}$

8. If $y = \cos^3(x^2)$, then $\dfrac{dy}{dx} =$

(A) $-6x\cos^2(x^2)$ (B) $-6x\cos^2(x^2)\sin x^2$

(C) $6x\cos^2(x^2)\sin x^2$ (D) $-3\cos^2(x^2)\sin x$

9. If $f(x) = e^{\sin^2 x}$, then $f'(x) =$

(A) $2e^{\sin^2 x}\sin x\cos x$ (B) $e^{\sin x(\sin x + 2\cos x)}$

(C) $2e^x \sin x \cos x$ (D) $e^x \sin^2 x$

10. An object is moving along the x-axis. Its position (in kilometers) at time $t \geq 0$ (in hours) is given by $s(t) = \sin(3t) - \cos(4t)$. What is the acceleration of the object at time $t = \dfrac{\pi}{2}$?

(A) 7 km/h^2 (B) 9 km/h^2

(C) -7 km/h^2 (D) 25 km/h^2

11. $\frac{d}{dx}5^x =$

(A) $5^x \ln 5$ (B) $(5^{x-1})x$ (C) 5^{x-1} (D) $\frac{5^x}{\ln 5}$

12. If $y = e^{kx}$, then $\frac{d^k}{dx^k}(e^{kx}) =$

(A) k^2e^{kx} (B) k^ke^x (C) k^ke^{kx} (D) $k!\,e^{kx}$

13. If the functions f and g are both twice differentiable and if $h(x) = (f \circ g)(x) = f(g(x))$, then $h''(x) =$

(A) $\left[f'(g(x)) \cdot g'(x)\right]^2$

(B) $f'(g(x)) \cdot g''(x) + g'(x)f''(g(x))$

(C) $f'(g(x)) \cdot g''(x) + [g'(x)]^2 f''(g(x))$

(D) $f'(g(x)) \cdot g'(x)$

14. The velocity v (in meters/second) of an object moving on a line is given by $v(t) = 3 - 1.5^{-t^2}$, $t \geq 0$. What is the acceleration of the object at $t = 4$ seconds?

(A) $0.005\ \text{m/s}^2$ (B) $-0.005\ \text{m/s}^2$

(C) $0.012\ \text{m/s}^2$ (D) $2.998\ \text{m/s}^2$

AP Calculus – 3.2 Notes – Implicit Differentiation

Recall:

Explicit equation

Implicit equation

Chain Rule and Implicit Differentiation

In terms of x	In terms of y
$\frac{d}{dx}x =$	$\frac{d}{dx}y =$
$\frac{d}{dx}x^2 =$	$\frac{d}{dx}y^2 =$
$\frac{d}{dx}e^{5x} =$	$\frac{d}{dx}e^{5y} =$

Implicit Differentiation Example: Find $\frac{dy}{dx}$ for $y^2 - 5x^3 = 3y$

Step 1: Take the derivative. Each time the derivative of "y" is involved, include a $\frac{dy}{dx}$.

Step 2: Gather all terms with $\frac{dy}{dx}$ on the left side, everything else on the right.

Step 3: Factor out the $\frac{dy}{dx}$ if necessary, to create only one $\frac{dy}{dx}$ term.

Step 4. Solve for $\frac{dy}{dx}$.

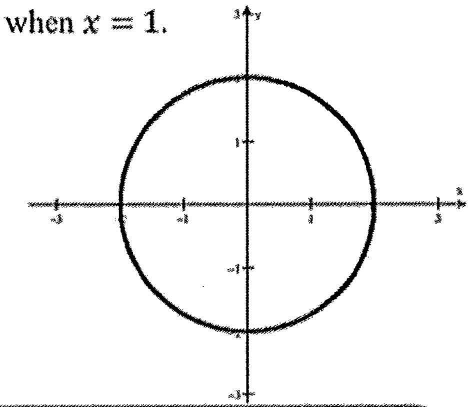
1. $y^3 - 2x = x^4 + 2y$

2. $\sin(xy) = 10x$

12

Derivative at a point – implicit differentiation.

3. Find the equation of all tangent lines for $x^2 + y^2 = 4$ when $x = 1$.



Horizontal and Vertical Tangent Lines

Horizontal tangent lines exist when the slope, $\frac{dy}{dx} =$

Vertical tangent lines exist when the slope, $\frac{dy}{dx}$ is

4. Find all ***horizontal*** tangent lines of the graph $3x^2 + 2y^2 = 16$.

5. Find all ***vertical*** tangent lines of the graph $3x^2 + 2y^2 = 16$.

Practice Problems:

Find $\frac{dy}{dx}$.

1. $5x^2 + 2y^3 = 4$	2. $5y^2 + 3 = x^2$	3. $\sin(x + y) = 2x$

4. $4x + 1 = \cos y^2$

5. $5x^2 - e^{4y^2} = -6$

6. $\ln(y^3) = 5x + 3$

7. $x^2 = 4y^3 + 5y^2$

8. $5x^3 - 2y = 5y^3$

9. $\ln y^2 + \cos^2 x = 1 - y$

10. $\sin\left(\frac{y}{2}\right) + e^y = 4x$

11. $x^3 + y^3 = 6xy$

12. $\frac{x}{\sin y} = 5$

13. $\ln x\, e^{3y} = 2y^2$

(14)

Find the slope of the tangent line at the given point. Show work.

14. $2 = 3x^4 + xy^4$ at $(-1, 1)$

15. $x \ln y = 4 - 2x$ at $(2, 1)$

Find the equation of the tangent line at the given point.

16. $x^2 + y^2 + 19 = 2x + 12y$ at $(4, 3)$

17. $x \sin 2y = y \cos 2x$ at $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

Find the equations of all horizontal and vertical tangent lines. Calculator allowed. Round to three decimals.

18. $x^2 + x + 2y^2 = 8$

Horizontal: ____________________

Vertical: ____________________

19. $x + y = y^2$

Horizontal: ____________________

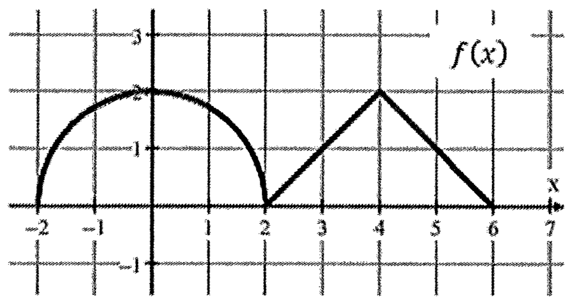
Vertical: ____________________

20. Find the slope of the normal line to $y = x + \cos(xy)$ at $(0,1)$.

(A) 1 (B) -1 (C) 0 (D) 2 (E) Undefined

---

21. The graph of $f(x)$, shown below, consists of a semicircle and two-line segments. $f'(1) =$



(A) -1 (B) $-\frac{1}{\sqrt{3}}$ (C) $\frac{1}{\sqrt{3}}$ (D) 1 (E) $\sqrt{3}$

---

22. Find the value(s) of $\frac{dy}{dx}$ of $x^2y + y^2 = 5$ at $y = 1$.

(A) $-\frac{3}{2}$ only (B) $-\frac{2}{3}$ only (C) $\frac{2}{3}$ only (D) $\pm\frac{2}{3}$ (E) $\pm\frac{3}{2}$

(16)

17

EXAMPLE 7 Differentiating Functions Using the Power Rule

(a) $\frac{d}{ds}(s^3-2s+1)^{5/3} = \frac{5}{3}(s^3-2s+1)^{2/3}\frac{d}{ds}(s^3-2s+1) = \frac{5}{3}(s^3-2s+1)^{2/3}(3\ldots$ PAGE 238 5...

(b) $\frac{d}{dx}\sqrt[3]{x^4-3x+5} = \frac{d}{dx}(x^4-3x+5)^{1/3} = \frac{1}{3}(x^4-3x+5)^{-2/3}\frac{d}{dx}(x^4-3x+\ldots$

$$= \frac{4x^3-3}{3(x^4-3x+5)^{2/3}}$$

(c) $\frac{d}{d\theta}[\tan(3\theta)]^{-3/4} = -\frac{3}{4}[\tan(3\theta)]^{-7/4}\frac{d}{d\theta}\tan(3\theta) = -\frac{3}{4}[\tan(3\theta)]^{-7/4}\cdot\sec^2(3\theta)\ldots$

$$= -\frac{9\sec^2(3\theta)}{4[\tan(3\theta)]^{7/4}}$$

NOW WORK Problem 39 and AP® Practice Problems 2, 4, and 7.

3.2 Assess Your Understanding

Concepts and Vocabulary

1. *True or False* Implicit differentiation is a technique for finding the derivative of an implicitly defined function.
2. *True or False* If $y^q = x^p$ for integers p and q, then $qy^{q-1} = px^{p-1}$.
3. $\frac{d}{dx}(3x^{1/3}) =$ ________.
4. If $y = (x^2+1)^{3/2}$, then $y' =$ ________.

Skill Building

In Problems 5–30, find $y' = \frac{dy}{dx}$ using implicit differentiation.

5. $3x+2y=3$
6. $2x-5y=7$
7. $x^2+y^2=4$
8. $y^4-4x^2=4$
9. $e^y=\sin x$
10. $e^y=\tan x$
11. $e^{x+y}=y$
12. $e^{x+y}=x^2$
13. $x^2y=5$
14. $x^3y=8$
15. $x^2-y^2-xy=2$ PAGE 238
16. $x^2-4xy+y^2=y$
17. $\frac{1}{x}+\frac{1}{y}=1$ PAGE 237
18. $\frac{1}{x}-\frac{1}{y}=4$
19. $x^2+y^2=\frac{2y}{x}$
20. $x^2+y^2=\frac{2y^2}{x^2}$
21. $e^x\sin y+e^y\cos x=4$
22. $e^y\cos x+e^{-x}\sin y=10$
23. $(x^2+y)^3=y$
24. $(x+y^2)^3=3x$
25. $y=\tan(x-y)$ PAGE 239
26. $y=\cos(x+y)$
27. $y=x\sin y$
28. $y=x\cos y$
29. $x^2y=e^{xy}$
30. $ye^x=y-x$

In Problems 31–46, find y'.

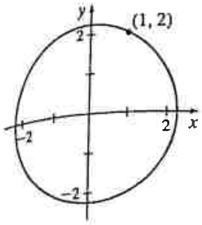
31. $y=x^{2/3}+4$ PAGE 241
32. $y=x^{1/3}-1$
33. $y=\sqrt[3]{x^2}$
34. $y=\sqrt[4]{x^5}$
35. $y=\sqrt[3]{x}-\frac{1}{\sqrt[3]{x}}$
36. $y=\sqrt{x}+\frac{1}{\sqrt{x}}$
37. $y=(x^3-1)^{1/2}$
38. $y=(x^2-1)^{1/3}$
39. $y=x\sqrt{x^2-1}$ PAGE 242
40. $y=x\sqrt{x^3+1}$
41. $y=e^{\sqrt{x^2-9}}$
42. $y=\sqrt{e^x}$
43. $y=(x^2\cos x)^{3/2}$
44. $y=(x^2\sin x)^{3/2}$
45. $y=(x^2-3)^{3/2}(6x+1)^{5/3}$
46. $y=\frac{(2x^3-1)^{4/3}}{(3x+4)^{5/2}}$

In Problems 47–52, find y' and y''.

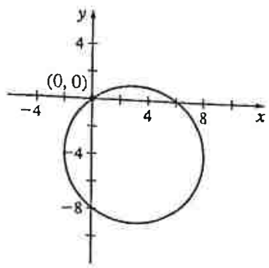
47. $x^2+y^2=4$
48. $x^2-y^2=1$
49. $x^2-y^2=4+5x$ PAGE 240
50. $4xy=x^2+y^2$
51. $y=\sqrt{x^2+1}$
52. $y=\sqrt{4-x^2}$

In Problems 53–58, for each implicitly defined equation:
(a) Find the slope of the tangent line to the graph of the equation at the indicated point.
(b) Write an equation for this tangent line.
(c) Graph the tangent line on the same axes as the graph of the equation.

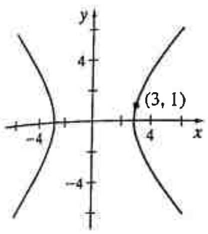
53. $x^2+y^2=5$ at $(1, 2)$



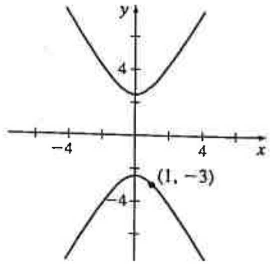
54. $(x-3)^2+(y+4)^2=25$ at $(0, 0)$



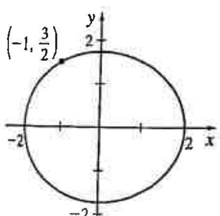
55. $x^2-y^2=8$ at $(3, 1)$



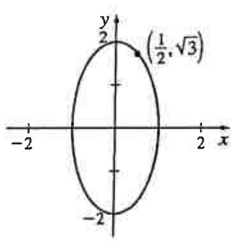
56. $y^2-3x^2=6$ at $(1, -3)$



57. $\frac{x^2}{4}+\frac{y^2}{3}=1$ at $\left(-1, \frac{3}{2}\right)$



58. $x^2+\frac{y^2}{4}=1$ at $\left(\frac{1}{2}, \sqrt{3}\right)$



59. Find y' and y'' at the point $(-1, 1)$ on the graph of

$$3x^2y+2y^3=5x^2$$

60. Find y' and y'' at the point $(0, 0)$ on the graph of

$$4x^3+2y^3=x+y$$

Applications and Extensions

In Problems 61–68, find y'.

Hint: Use the fact that $|x|=\sqrt{x^2}$.

61. $y=|3x|$

62. $y=|x^5|$

63. $y=|2x-1|$

64. $y=|5-x^2|$

65. $y=|\cos x|$

66. $y=|\sin x|$

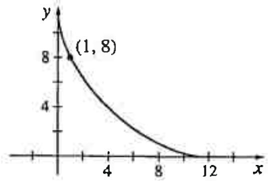
67. $y=\sin|x|$

68. $|x|+|y|=1$

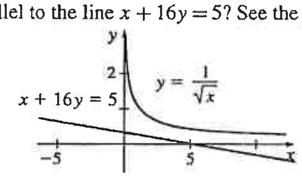
69. **Tangent Line to a Hypocycloid** The graph of

$$x^{2/3}+y^{2/3}=5$$

is called a **hypocycloid**. Part of its graph is shown in the figure. Find an equation of the tangent line to the hypocycloid at the point $(1, 8)$.



70. **Tangent Line** At what point does the graph of $y=\frac{1}{\sqrt{x}}$ have a tangent line parallel to the line $x+16y=5$? See the figure.

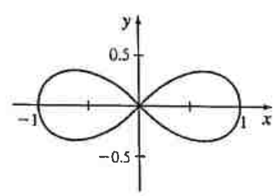


71. **Tangent Line** For the equation $x+xy+2y^2=6$:
(a) Find an expression for the slope of the tangent line at any point (x, y) on the graph.
(b) Write an equation for the line tangent to the graph at the point $(2, 1)$.
(c) Find the coordinates of any other point on this graph with slope equal to the slope at $(2, 1)$.
CAS (d) Graph the equation and the tangent lines found in parts (b) and (c) on the same screen.

72. **Tangent Line to a Lemniscate** The graph of

$$(x^2+y^2)^2=x^2-y^2$$

called a **lemniscate**, is shown in the figure. There are exactly four points at which the tangent line to the lemniscate is horizontal. Find them.



73. **Rectilinear Motion** An object of mass m moves in rectilinear motion so that at time $t>0$ its position s from the origin and its velocity $v=\frac{ds}{dt}$ satisfy the equation

$$m\left(v^2-v_0^2\right)=k\left(s_0^2-s^2\right)$$

where k is a positive constant and v_0 and s_0 are the initial velocity and position, respectively, of the object. Show that if $v>0$, then

$$ma=-ks$$

where $a=\frac{d^2s}{dt^2}$ is the acceleration of the object.

Hint: Differentiate the expression $m\left(v^2-v_0^2\right)=k\left(s_0^2-s^2\right)$ with respect to t.

(19)

74. Price Function It is estimated that t months from now the average price (in dollars) of a tablet will be given by

$$P(t)=\frac{300}{1+\frac{1}{6}\sqrt{t}}+100 \quad 0\le t\le 60$$

(a) Find $P'(t)$

(b) Find $P'(16)$ and $P'(49)$ and interpret the results.

(c) Graph $P=P(t)$, and explain how the graph supports the answers in (b).

75. Production Function The production of commodities sometimes requires several resources such as land, labor, and machinery. If there are two inputs that require amounts x and y, then the output z is given by the function of two variables: $z=f(x,y)$. Here, z is called a **production function.** For example, if x represents land, y represents capital, and z is the amount of a commodity produced, a possible production function is $z=x^{0.5}y^{0.4}$. Set z equal to a fixed amount produced and show that $\frac{dy}{dx}=-\frac{5y}{4x}$. This illustrates that the rate of change of capital with respect to land is always negative when the amount produced is fixed.

76. Learning Curve The psychologist L. L. Thurstone suggested the following function for the time T it takes to memorize a list of n words: $T=f(n)=Cn\sqrt{n-b}$, where C and b are constants depending on the person and the task.

(a) Find the rate of change of the time T with respect to the number n of words to be memorized.

(b) Suppose that for a certain person and a certain task, $C=2$ and $b=2$. Find $f'(10)$ and $f'(30)$.

(c) Interpret the results found in (c).

77. The Folium of Descartes The graph of the equation $x^3+y^3=2xy$ is called the **folium of Descartes.**

(a) Find y'.

(b) Find an equation of the tangent line to the folium of Descartes at the point $(1, 1)$.

(c) Find any points on the graph where the tangent line to the graph is horizontal. Ignore the origin.

CAS (d) Graph the equation $x^3+y^3=2xy$. Explain how the graph supports the answers to (b) and (c).

78. If n is an even positive integer, show that the tangent line to the graph of $y=\sqrt[n]{x}$ at $(1, 1)$ is perpendicular to the tangent line to the graph of $y=x^n$ at $(-1, 1)$.

79. At what point(s), if any, is the line $y=x-1$ parallel to the tangent line to the graph of $y=\sqrt{25-x^2}$?

80. What is wrong with the following?
If $x+y=e^{x+y}$, then $1+y'=e^{x+y}(1+y')$.
Since $e^{x+y}>0$, then $y'=-1$ for all x. Therefore, $x+y=e^{x+y}$ must be a line of slope -1.

81. Show that if a function y is differentiable, and x and y are related by the equation $x^ny^m+x^my^n=k$, where k is a constant, then

$$\frac{dy}{dx}=-\frac{y(nx^r+my^r)}{x(mx^r+ny^r)} \quad \text{where} \quad r=n-m$$

82. Physics For ideal gases, **Boyle's law** states that pressure is inversely proportional to volume. A more realistic relationship between pressure P and volume V is given by the van der Waals equation

$$P+\frac{a}{V^2}=\frac{C}{V-b}$$

where C is the constant of proportionality, a is a constant that depends on molecular attraction, and b is a constant that depends on the size of the molecules. Find $\frac{dV}{dP}$, which measures the compressibility of the gas.

83. Tangent Line Show that an equation for the tangent line at a point (x_0, y_0) on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\frac{xx_0}{a^2}+\frac{yy_0}{b^2}=1$.

84. Tangent Line Show that the slope of the tangent line to a hypocycloid $x^{2/3}+y^{2/3}=a^{2/3}$, $a>0$, at any point for which $x\neq 0$, is $-\frac{y^{1/3}}{x^{1/3}}$.

85. Tangent Line Use implicit differentiation to show that the tangent line to a circle $x^2+y^2=R^2$ at any point P on the circle is perpendicular to OP, where O is the center of the circle.

Challenge Problems

86. Let $A=(2, 1)$ and $B=(5, 2)$ be points on the graph of $f(x)=\sqrt{x-1}$. A line is moved upward on the graph so that it remains parallel to the secant line AB. Find the coordinates of the last point on the graph of f before the secant line loses contact with the graph.

Orthogonal Graphs *Problems 87 and 88 require the following definition:*

The graphs of two functions are said to be ***orthogonal*** *if the tangent lines to the graphs are perpendicular at each point of intersection.*

87. (a) Show that the graphs of $xy=c_1$ and $-x^2+y^2=c_2$ are orthogonal, where c_1 and c_2 are positive constants.

CAS (b) Graph each function on one coordinate system for $c_1=1, 2, 3$ and $c_2=1, 9, 25$.

88. Find $a>0$ so that the parabolas $y^2=2ax+a^2$ and $y^2=a^2-2ax$ are orthogonal.

89. Show that if p and $q>0$ are integers, then $y=x^{p/q}$ is a differentiable function of x.

90. We say that y is an **algebraic function** of x if it is a function that satisfies an equation of the form

$$P_0(x)y^n+P_1(x)y^{n-1}+\cdots+P_{n-1}(x)y+P_n(x)=0$$

where $P_k(x)$, $k=0, 1, 2, \ldots, n$, are polynomials. For example, $y=\sqrt{x}$ satisfies

$$y^2-x=0$$

Use implicit differentiation to obtain a formula for the derivative of an algebraic function.

Preparing for the AP® Exam

AP® Practice Problems

PAGE 239 1. If $\sin(x^2y) = x$, then $\dfrac{dy}{dx}$ equals

(A) $\cos(x^2y)(2xy + x^2)$ (B) $\dfrac{1}{\cos(x^2y)(2xy + x^2)}$

(C) $\dfrac{\sec(x^2y) - 2xy}{x^2}$ (D) $\dfrac{\sec(x^2y)}{2x}$

PAGE 242 2. If $\dfrac{dy}{dx} = \sqrt{1 - 2y^3}$, then $\dfrac{d^2y}{dx^2}$ equals

(A) $3y^2$ (B) $-3y^2$ (C) $-\dfrac{3y^2}{\sqrt{x - 2y^3}}$ (D) $-6y^2$

PAGE 238 3. The slope of the tangent line to the graph of $x^2y - xy^3 = 10$ at the point $(-1, 2)$ is

(A) -12 (B) $-\dfrac{4}{13}$ (C) $-\dfrac{4}{11}$ (D) $\dfrac{12}{13}$

PAGE 242 4. For $f(x) = (x^3 - 4x + 32)^{1/5}$, find $f'(0)$.

(A) $-\dfrac{1}{4}$ (B) $-\dfrac{1}{20}$ (C) $\dfrac{1}{80}$ (D) $\dfrac{1}{20}$

PAGE 238 5. Find y' if $x^4 + 4xy + 6y^2 = 12$.

(A) $-\dfrac{x^3 + y}{x + 3y}$ (B) $-\dfrac{x^3 + 3y}{x + 12y}$

(C) $\dfrac{x^3 + y}{x + 3y}$ (D) $\dfrac{-x^3 + y}{x + 3y}$

PAGE 241 6. Find $\dfrac{dy}{dx}$ at the point $(3, 8)$ on the graph of $5xy^{2/3} - x^2y = -12$.

(A) 2 (B) 17 (C) -7 (D) $-\dfrac{28}{3}$

PAGE 242 7. What is the domain of the derivative of $f(x) = 3x^{2/3}(x + 5)^{1/3}$?

(A) The set of all real numbers (B) $\{x | x \geq 0\}$

(C) $\{x | x \neq 0\}$ (D) $\{x | x \neq 0,\ x \neq -5\}$

PAGE 239 8. If $e^y = \tan x$, $0 < x < \dfrac{\pi}{2}$, what is $\dfrac{dy}{dx}$ in terms of x?

(A) $\sec x \csc x$ (B) $\sec^2 x$ (C) $\sec x$ (D) $\sin x \sec x$

PAGE 241 9. $\displaystyle\lim_{h \to 0} \frac{\sqrt[3]{x + h} - \sqrt[3]{x}}{h} =$

(A) $-\dfrac{3}{x^{2/3}}$, $x \neq 0$ (B) $\dfrac{1}{3x^{2/3}}$, $x \neq 0$

(C) $\dfrac{1}{3x^{2/3}}$, $x \geq 0$ (D) $\dfrac{x^{2/3}}{3}$

PAGE 239 10. Consider the equation $x^2y + 3y^3 = 24$.

(a) Find $\dfrac{dy}{dx}$.

(b) Determine the points, if any, where the tangent line to the graph of the equation is horizontal.

3.3 Derivatives of the Inverse Trigonometric Functions

OBJECTIVES *When you finish this section, you should be able to:*

1 Find the derivative of an inverse function (p. 245)

2 Find the derivative of the inverse trigonometric functions (p. 247)

NEED TO REVIEW? Inverse functions are discussed in Section P.4, pp. 37–40.

The Chain Rule is used to find the derivative of an inverse function.

1 Find the Derivative of an Inverse Function

Suppose f is a function and g is its inverse function. Then

$$g(f(x)) = x$$

for all x in the domain of f.

If both f and g are differentiable, we can differentiate both sides with respect to x and use the Chain Rule.

$$\frac{d}{dx}g(f(x)) = \frac{d}{dx}x$$

$$g'(f(x)) \cdot f'(x) = 1 \qquad \textbf{Use the Chain Rule on the left.}$$

Since the product of the derivatives is never 0, each function has a nonzero derivative on its domain.

3.2 AP Practice Problems (p. 245) – Implicit Differentiation

1. If $\sin(x^2y) = x$, then $\frac{dy}{dx}$ equals

(A) $\cos(x^2y)(2xy + x^2)$ (B) $\frac{1}{\cos(x^2y)(2xy + x^2)}$

(C) $\frac{\sec(x^2y) - 2xy}{x^2}$ (D) $\frac{\sec(x^2y)}{2x}$

2. If $\frac{dy}{dx} = \sqrt{1 - 2y^3}$, then $\frac{d^2y}{dx^2}$ equals

(A) $3y^2$ (B) $-3y^2$ (C) $-\frac{3y^2}{\sqrt{x - 2y^3}}$ (D) $-6y^2$

3. The slope of the tangent line to the graph of $x^2y - xy^3 = 10$ at the point $(-1, 2)$ is

(A) -12 (B) $-\frac{4}{13}$ (C) $-\frac{4}{11}$ (D) $\frac{12}{13}$

4. For $f(x) = (x^3 - 4x + 32)^{1/5}$, find $f'(0)$.

(A) $-\frac{1}{4}$ (B) $-\frac{1}{20}$ (C) $\frac{1}{80}$ (D) $\frac{1}{20}$

5. Find y' if $x^4 + 4xy + 6y^2 = 12$.

(A) $-\frac{x^3 + y}{x + 3y}$ (B) $-\frac{x^3 + 3y}{x + 12y}$

(C) $\frac{x^3 + y}{x + 3y}$ (D) $\frac{-x^3 + y}{x + 3y}$

22

6. Find $\frac{dy}{dx}$ at the point (3, 8) on the graph of $5xy^{2/3} - x^2y = -12$.

(A) 2 (B) 17 (C) -7 (D) $-\frac{28}{3}$

7. What is the domain of the derivative of $f(x) = 3x^{2/3}(x+5)^{1/3}$?

(A) The set of all real numbers (B) $\{x \mid x \geq 0\}$

(C) $\{x \mid x \neq 0\}$ (D) $\{x \mid x \neq 0,\ x \neq -5\}$

8. If $e^y = \tan x$, $0 < x < \frac{\pi}{2}$, what is $\frac{dy}{dx}$ in terms of x?

(A) $\sec x \csc x$ (B) $\sec^2 x$ (C) $\sec x$ (D) $\sin x \sec x$

9. $\lim_{h \to 0} \frac{\sqrt[3]{x+h} - \sqrt[3]{x}}{h} =$

(A) $-\frac{3}{x^{2/3}}$, $x \neq 0$ (B) $\frac{1}{3x^{2/3}}$, $x \neq 0$

(C) $\frac{1}{3x^{2/3}}$, $x \geq 0$ (D) $\frac{x^{2/3}}{3}$

10. Consider the equation $x^2y + 3y^3 = 24$.

(a) Find $\frac{dy}{dx}$.

(b) Determine the points, if any, where the tangent line to the graph of the equation is horizontal.

AP Calculus – 3.3a Notes – Derivative of Inverse at a Point

Inverse Function: A function's inverse is found by swapping the input (x) and output (y) values.

(Domains and Ranges are also swapped between a function and its inverse)

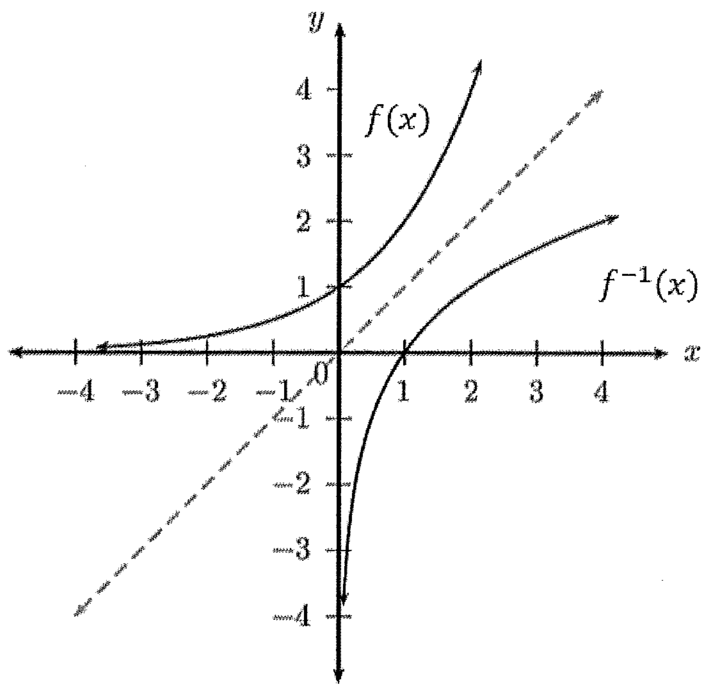
Example A:

Find the inverse of $f(x) = 6x + 2$

Three ways to say the same thing:

1. $g(x)$ is the inverse of $f(x)$
2. $g(x) = f^{-1}(x)$
3. $f(g(x)) = x$ and $g(f(x)) = x$

Ex: Graphical example of function $f(x)$ & its inverse $f^{-1}(x)$



Evaluate derivative of inverse at a point: (find $(f^{-1})'(a)$

$f(b) =$	$(f^{-1})(a) =$
$f'(b) =$	$(f^{-1})'(a) =$

Derivative of an Inverse Function:

$$\frac{d}{dx}[f^{-1}(x)] =$$

Example B: $f(x) = x^3 + 4x + 2$ $\quad$ *find* $(f^{-1})'(-3)$

Example C: $f(x) = \sqrt{x^3 - 7}$ $\quad$ *find* $(f^{-1})'(1)$

(24)

Example D:

The table below gives values of the differentiable functions f, g, and f' at selected values of x. Let $g(x) = f^{-1}(x)$.

x	$f(x)$	$f'(x)$
1	3	-3
2	1	-2
3	-5	-5
4	0	-6

Evaluate derivative of inverse at a point: (find $(f^{-1})'(a)$)

$f(b) = a$	$(f^{-1})(a) = b$
$f'(b) = n$	$(f^{-1})'(a) = \frac{1}{n}$

1. What is the value of $g'(1)$?

2. Write an equation for the line tangent to f^{-1} at $x = 1$.

3. Let g be a differentiable function such that $g(12) = 4$, $g(3) = 6$, $g'(12) = -5$, and $g'(3) = -2$. The function h is differentiable and $h(x) = g^{-1}(x)$ for all x. What is the value of $h'(6)$?

4. If $f(x) = 3x^3 + 1$ and g is the inverse function of f, what is the value of $g'(25)$?

For each function $g(x)$, its inverse $g^{-1}(x) = f(x)$. Evaluate the given derivative.

13. $g(x) = \cos(x) + 3x^2$
$g\left(\frac{\pi}{2}\right) = \frac{3\pi}{4}$. Find $f'\left(\frac{3\pi}{4}\right)$

14. $g(x) = 2x^3 - x^2 - 5x$
$g(-2) = -10$. Find $f'(-10)$

19. A function h satisfies $h(3) = 5$ and $h'(3) = 7$. Which of the following statements about the inverse of h must be true?

(A) $(h^{-1})'(5) = 3$ (B) $(h^{-1})'(7) = 3$ (C) $(h^{-1})'(5) = 7$

(D) $(h^{-1})'(5) = \frac{1}{7}$ (E) $(h^{-1})'(7) = \frac{1}{5}$

AP Calculus – 3.3b Notes – Derivatives of Inverse Trig Functions

Quote from the AP Exam:
"Notation: The inverse of a trigonometric function x may be indicated using the inverse function notation f^{-1} or with the prefix "arc" (e.g., $\sin^{-1} x = \arcsin x$)."

Inverse Trig Derivative Rules:

1) $\frac{d}{dx}\arcsin u =$	2) $\frac{d}{dx}\arccos u =$
3) $\frac{d}{dx}\arctan u =$	4) $\frac{d}{dx}\operatorname{arccot} u =$
5) $\frac{d}{dx}\operatorname{arcsec} u =$	6) $\frac{d}{dx}\operatorname{arccsc} u =$

Find the derivative.

1. $\frac{d}{dx}\sin^{-1}(3x)$	2. $\frac{d}{dx}\tan^{-1}(2x^2)$	3. $\frac{d}{dx}\operatorname{arcsec}(5x)$

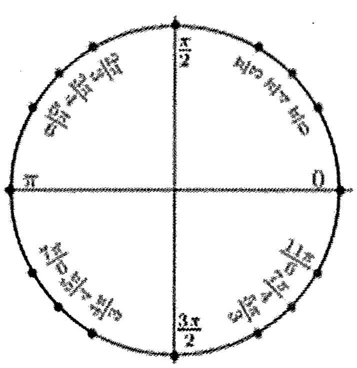
Simplifying $\sec^{-1} x$ Derivatives.

Simplify the following expressions.

4. $\frac{9x^2}{\lvert 3x^3 \rvert \sqrt{9x^6-1}}$	5. $\frac{4x}{\lvert 2x^2 \rvert \sqrt{4x^2-1}}$

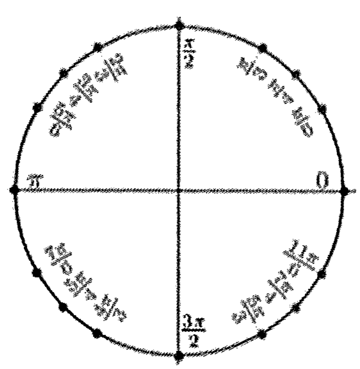
Domain of an inverse trig function.

$y = \sin^{-1}(x)$



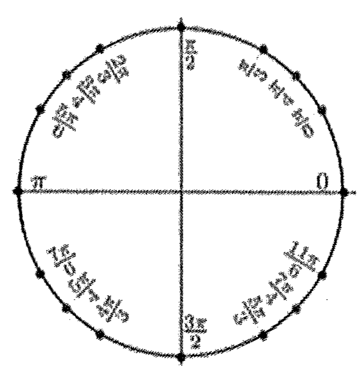
Domain: $-1 \le x \le 1$
Range: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

$y = \cos^{-1}(x)$



Domain: $-1 \le x \le 1$
Range: $0 \le y \le \pi$

$y = \tan^{-1}(x)$



Domain: $-\infty \le x \le \infty$
Range: $-\frac{\pi}{2} < y < \frac{\pi}{2}$

26

Evaluate each function at the given x-value.

6. $f(x) = \arcsin(x)$ at $x = \frac{\sqrt{3}}{2}$	7. $f(x) = \cos^{-1}\left(\frac{x}{4}\right)$ at $x = -2$	8. $f(x) = \arctan(x)$ at $x = \frac{1}{\sqrt{3}}$

Practice Problems:

Find the derivative of each expression.

1. $\frac{d}{dx}\sin^{-1}(5x)$	2. $\frac{d}{dx}\csc^{-1}(4x^5)$	3. $\frac{d}{dx}\arctan(2x)$	4. $\frac{d}{dx}\sec^{-1}(x^3)$

5. $\frac{d}{dx}\csc 6x$	6. $\frac{d}{dx}\arccos(3x^2)$	7. $\frac{d}{dx}\cot^{-1}(-x)$

Find the tangent line equation of the curve at the given point.

11. $y = \arcsin(x)$ at the point where $x = \frac{\sqrt{2}}{2}$	12. $y = \cos^{-1}(4x)$ at the point where $x = \frac{\sqrt{3}}{8}$

19. If $\arctan y = \ln x$, then $\frac{dy}{dx} =$

(A) $\tan\left(\frac{1}{x}\right)$ (B) $\tan(\ln x)$ (C) $\frac{1+y^2}{xy}$ (D) $\frac{x}{1+y^2}$ (E) $\frac{1+y^2}{x}$

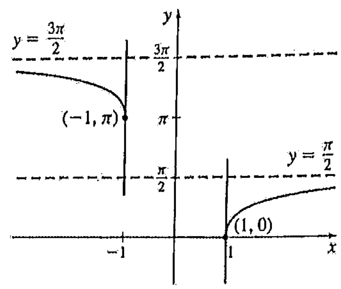


Figure 4 $y = \sec^{-1} x$, $|x| \geq 1$, $0 \leq y < \frac{\pi}{2}$ or $\pi \leq y < \frac{3\pi}{2}$

Notice that $y = \sec^{-1} x$ is not differentiable when $x = \pm 1$. In fact, as Figure 4 shows, the graph of $y = \sec^{-1} x$ has vertical tangent lines at the points $(-1, \pi)$ and $(1, 0)$.

NOW WORK Problem 19.

The formulas for the derivatives of the three remaining inverse trigonometric functions can be obtained using the following identities:

- Since $\cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x$, then $\frac{d}{dx} \cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}$, where $|x| < 1$.
- Since $\cot^{-1} x = \frac{\pi}{2} - \tan^{-1} x$, then $\frac{d}{dx} \cot^{-1} x = -\frac{1}{1+x^2}$, where $-\infty < x < \infty$.
- Since $\csc^{-1} x = \frac{\pi}{2} - \sec^{-1} x$, then $\frac{d}{dx} \csc^{-1} x = -\frac{1}{x\sqrt{x^2-1}}$, where $|x| > 1$.

3.3 Assess Your Understanding

Concepts and Vocabulary

1. *True or False* If f is a one-to-one differentiable function and if $f'(x) > 0$, then f has an inverse function whose derivative is positive.
2. *True or False* $\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}$, where $-\infty < x < \infty$.
3. *True or False* If f and g are inverse functions, then $$g'(y_0) = \frac{1}{f'(y_0)}$$
4. $\frac{d}{dx} \sin^{-1} x =$ ______ where $-1 < x < 1$.

Skill Building

In Problems 5–8, the functions f and g are inverse functions.

5. If $f(0) = 4$ and $f'(0) = -2$, find $g'(4)$.
6. If $f(1) = -2$ and $f'(1) = 4$, find $g'(-2)$.
7. If $g(3) = -2$ and $g'(3) = \frac{1}{2}$, find $f'(-2)$.
8. If $g(-1) = 0$ and $g'(-1) = -\frac{1}{3}$, find $f'(0)$.

In Problems 9–16, f and g are inverse functions. For each function f, find $g'(y_0)$.

9. $f(x) = x^5$; $y_0 = 32$
10. $f(x) = x^3$; $y_0 = 27$
11. $f(x) = x^2 + 2$, $x \geq 0$; $y_0 = 6$
12. $f(x) = x^2 - 5$, $x \geq 0$; $y_0 = 4$
13. $f(x) = x^{1/3}$; $y_0 = 2$
14. $f(x) = x^{2/5}$; $y_0 = 4$
15. $f(x) = 3x^{4/3} + 1$; $y_0 = 49$
16. $f(x) = x^{2/3} + 5$; $y_0 = 6$

In Problems 17–42, find the derivative of each function.

17. $f(x) = \sin^{-1}(4x)$
18. $f(x) = \sin^{-1}(3x - 2)$
19. $g(x) = \sec^{-1}(3x)$
20. $g(x) = \cos^{-1}(2x)$
21. $s(t) = \tan^{-1} \frac{t}{2}$
22. $s(t) = \sec^{-1} \frac{t}{3}$
23. $f(x) = \tan^{-1}(1 - 2x^2)$
24. $f(x) = \sin^{-1}(1 - x^2)$
25. $f(x) = \sec^{-1}(x^2 + 2)$
26. $f(x) = \cos^{-1} x^2$
27. $F(x) = \sin^{-1} e^x$
28. $F(x) = \tan^{-1} e^x$
29. $g(x) = \tan^{-1} \frac{1}{x}$
30. $g(x) = \sec^{-1} \sqrt{x}$
31. $g(x) = x \sin^{-1} x$
32. $g(x) = x \tan^{-1}(x + 1)$
33. $s(t) = t^2 \sec^{-1} t^3$
34. $s(t) = t^2 \sin^{-1} t^2$
35. $f(x) = \tan^{-1}(\sin x)$
36. $f(x) = \sin^{-1}(\cos x)$
37. $G(x) = \sin(\tan^{-1} x)$
38. $G(x) = \cos(\tan^{-1} x)$
39. $f(x) = e^{\tan^{-1}(3x)}$
40. $f(x) = e^{\sec^{-1} x^2}$
41. $g(x) = \frac{\sin^{-1} x}{\sqrt{1-x^2}}$
42. $g(x) = \frac{\sec^{-1} x}{\sqrt{x^2-1}}$

In Problems 43–46, use implicit differentiation to find $\frac{dy}{dx}$.

43. $y^2 + \sin^{-1} y = 2x$
44. $xy + \tan^{-1} y = 3$
45. $40 \tan^{-1} y^2 - \pi x^3 y = 2\pi$
46. $\sec^{-1} y - xy = \frac{\pi}{3}$
47. The function $f(x) = x^3 + 2x$ has an inverse function g. Find $g'(0)$ and $g'(3)$.
48. The function $f(x) = 2x^3 + x - 3$ has an inverse function g. Find $g'(-3)$ and $g'(0)$.

Applications and Extensions

49. **Tangent Line**
 (a) Find an equation for the tangent line to the graph of $y = \sin^{-1} \frac{x}{2}$ at the origin.
 (b) Use technology to graph $y = \sin^{-1} \frac{x}{2}$ and the tangent line found in (a).

50. **Tangent Line**
(a) Find an equation for the tangent line to the graph of $y=\tan^{-1}x$ at $x=1$.
(b) Use technology to graph $y=\tan^{-1}x$ and the tangent line found in (a).

51. **Tangent Line** The function $f(x)=2x^3-x$, $x\geq 1$, is one-to-one and has an inverse function g. Find an equation of the tangent line to the graph of g at the point $(14, 2)$ on g.

52. **Tangent Line** The function $f(x)=x^5-3x$ is one-to-one and has an inverse function g. Find an equation of the tangent line to the graph of g at the point $(-2, 1)$ on g.

53. **Normal Line** The function $f(x)=x+2x^{1/3}$ is one-to-one and has an inverse function g. Find an equation of the normal line to the graph of g at the point $(3, 1)$ on g.

54. **Normal Line** The function $f(x)=x^4+x$, $x>0$, is one-to-one and has an inverse function g. Find an equation of the normal line to the graph of g at the point $(2, 1)$ on g.

55. **Rectilinear Motion** An object moves along the x-axis so that its position x from the origin (in meters) is given by $x(t)=\sin^{-1}\frac{1}{t}$, $t>0$, where t is the time in seconds.
(a) Find the velocity of the object at $t=2$ s.
(b) Find the acceleration of the object at $t=2$ s.

56. If $g(x)=\cos^{-1}(\cos x)$, show that $g'(x)=\dfrac{\sin x}{|\sin x|}$.

57. Show that $\dfrac{d}{dx}\tan^{-1}(\cot x)=-1$.

58. Show that $\dfrac{d}{dx}\cot^{-1}x=\dfrac{d}{dx}\tan^{-1}\dfrac{1}{x}$ for all $x\neq 0$.

59. Show that $\dfrac{d}{dx}\left[\sin^{-1}x-x\sqrt{1-x^2}\right]=\dfrac{2x^2}{\sqrt{1-x^2}}$.

Challenge Problem

60. Another way of finding the derivative of $y=\sqrt[n]{x}$ is to use inverse functions. The function $y=f(x)=x^n$, n a positive integer, has the derivative $f'(x)=nx^{n-1}$. So, if $x\neq 0$, then $f'(x)\neq 0$. The inverse function of f, namely, $x=g(y)=\sqrt[n]{y}$, is defined for all y if n is odd and for all $y\geq 0$ if n is even. Since this inverse function is differentiable for all $y\neq 0$, we have

$$g'(y)=\frac{d}{dy}\sqrt[n]{y}=\frac{1}{f'(x)}=\frac{1}{nx^{n-1}}$$

Since $nx^{n-1}=n\left(\sqrt[n]{y}\right)^{n-1}=ny^{(n-1)/n}=ny^{1-(1/n)}$, we have

$$\frac{d}{dy}\sqrt[n]{y}=\frac{d}{dy}y^{1/n}=\frac{1}{ny^{1-(1/n)}}=\frac{1}{n}y^{(1/n)-1}$$

Use the result from above and the Chain Rule to prove the formula

$$\frac{d}{dx}x^{p/q}=\frac{p}{q}x^{(p/q)-1}$$

Preparing for the AP® Exam

AP® Practice Problems

PAGE 249 1. What is the slope of the normal line to the graph of $y=\tan^{-1}(2x)$ where $x=-1$?

(A) $\frac{2}{5}$ (B) $-\frac{2}{5}$ (C) -5 (D) $-\frac{5}{2}$

PAGE 248 2. $\dfrac{d}{dx}\sin^{-1}(e^{2x})=$

(A) $-\dfrac{2e^{2x}}{\sqrt{1-e^{2x}}}$ (B) $\dfrac{2e^{2x}}{\sqrt{1-e^{4x}}}$

(C) $\dfrac{2e^{2x}}{\sqrt{1-e^{4x^2}}}$ (D) $\dfrac{2e^{2x}}{\sqrt{1-e^{2x}}}$

PAGE 246 3. The functions f and g are differentiable and g is the inverse of f. If $g(-3)=2$ and $f'(2)=-\frac{1}{3}$, then $g'(-3)$ is

(A) 3 (B) $\frac{1}{2}$ (C) -3 (D) $\frac{1}{3}$

PAGE 249 4. If $y=\tan^{-1}(\cos x)$, then $y'=$

(A) $\dfrac{\sin x}{1+\cos^2 x}$ (B) $-\dfrac{\cos x}{1+\sin^2 x}$

(C) $-\dfrac{\sin x}{1+\cos^2 x}$ (D) $\dfrac{\cos x}{1+\sin^2 x}$

PAGE 246 5. The function g is given by $g(x)=x^3+x^2-3$, $x\geq 0$, so $g(2)=9$. If the function f is the inverse function of g, find $f'(9)$.

(A) $\frac{1}{261}$ (B) -261 (C) 16 (D) $\frac{1}{16}$

PAGE 249 6. An object is moving along the y-axis. Its position (in centimeters) at time $t>0$ seconds is given by $y(t)=\tan^{-1}t$. What is the acceleration of the object at $t=1$ second?

(A) $\frac{1}{2}$ cm/s^2 (B) $-\frac{1}{2}$ cm/s

(C) -2 cm/s^2 (D) $-\frac{1}{2}$ cm/s^2

PAGE 247 7. $F(x)=x^3-x^2+x-5$ and g are inverse functions. Find an equation of the tangent line to the graph of g at the point $(-4, 1)$ on g.

(A) $y=\frac{1}{2}x-\frac{9}{2}$ (B) $y=\frac{1}{57}x-\frac{53}{57}$

(C) $y=\frac{1}{2}x+3$ (D) $y=2x+9$

PAGE 246 8. The functions f and g are differentiable; $f(x)=g^{-1}(x)$. Find $f'(4)$ if $g(4)=3$, $g(-1)=4$, $g'(-1)=5$, and $g'(4)=-1$.

(A) -1 (B) $-\frac{1}{5}$ (C) $\frac{1}{5}$ (D) $\frac{1}{4}$

3.3 AP Practice Problems (p.251) – Derivatives of Inverse Trig Functions

1. What is the slope of the normal line to the graph of $y = \tan^{-1}(2x)$ where $x = -1$?

   (A) $\frac{2}{5}$ (B) $-\frac{2}{5}$ (C) -5 (D) $-\frac{5}{2}$

2. $\frac{d}{dx}\sin^{-1}(e^{2x}) =$

   (A) $-\frac{2e^{2x}}{\sqrt{1-e^{2x}}}$ (B) $\frac{2e^{2x}}{\sqrt{1-e^{4x}}}$

   (C) $\frac{2e^{2x}}{\sqrt{1-e^{4x^2}}}$ (D) $\frac{2e^{2x}}{\sqrt{1-e^{2x}}}$

3. The functions f and g are differentiable and g is the inverse of f. If $g(-3) = 2$ and $f'(2) = -\frac{1}{3}$, then $g'(-3)$ is

   (A) 3 (B) $\frac{1}{2}$ (C) -3 (D) $\frac{1}{3}$

4. If $y = \tan^{-1}(\cos x)$, then $y' =$

   (A) $\frac{\sin x}{1+\cos^2 x}$ (B) $-\frac{\cos x}{1+\sin^2 x}$

   (C) $-\frac{\sin x}{1+\cos^2 x}$ (D) $\frac{\cos x}{1+\sin^2 x}$

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5. The function g is given by $g(x) = x^3 + x^2 - 3, x \geq 0$, so $g(2) = 9$. If the function f is the inverse function of g, find $f'(9)$.

(A) $\frac{1}{261}$ (B) -261 (C) 16 (D) $\frac{1}{16}$

6. An object is moving along the y-axis. Its position (in centimeters) at time $t > 0$ seconds is given by $y(t) = \tan^{-1} t$. What is the acceleration of the object at $t = 1$ second?

(A) $\frac{1}{2}$ cm/s^2 (B) $-\frac{1}{2}$ cm/s

(C) -2 cm/s^2 (D) $-\frac{1}{2}$ cm/s^2

7. $F(x) = x^3 - x^2 + x - 5$ and g are inverse functions. Find an equation of the tangent line to the graph of g at the point $(-4, 1)$ on g.

(A) $y = \frac{1}{2}x - \frac{9}{2}$ (B) $y = \frac{1}{57}x - \frac{53}{57}$

(C) $y = \frac{1}{2}x + 3$ (D) $y = 2x + 9$

8. The functions f and g are differentiable; $f(x) = g^{-1}(x)$. Find $f'(4)$ if $g(4) = 3$, $g(-1) = 4$, $g'(-1) = 5$, and $g'(4) = -1$.

(A) -1 (B) $-\frac{1}{5}$ (C) $\frac{1}{5}$ (D) $\frac{1}{4}$

AP Calculus – 3.4 Notes – Log Derivatives and Log Differentiation

Recall:

$\ln 1 =$ $\ln 0 =$ $e^0 =$ $e^{\ln a} =$ $\ln e^a =$

Derivatives of Exponential Functions

$$\frac{d}{dx} a^u = \qquad \frac{d}{dx} e^u =$$

Example: Find $f'(x)$ if $f(x) = 2^x + 3e^x$

Derivatives of Logarithmic Functions

$$\frac{d}{dx} log_a u = \qquad \frac{d}{dx} \ln u =$$

Example: Find $f'(x)$ if $f(x) = \log_4 x - 4\ln x$

Find the derivative of each function.

1. $f(x) = 2\sin x + 5e^x$	2. $f(x) = 3^x - 4\cos x$	3. $f(x) = \log_2 x - \sin x$

32

Natural Log Properties:

$\ln 1 = 0$	$\ln e = 1$	$\ln a^n = n * \ln a$
$\ln(ab) = \ln a + \ln b$		$ln\left(\frac{a}{b}\right) = \ln a - \ln b$

Log Differentiation

Logarithmic Differentiation : Simplifying non-log functions using log properties to expand before differentiating.

Log differentiation steps:

1. Take the **ln** (natural log) of both sides
2. Simplify and expand using log properties
3. Use implicit differentiation
4. Substitute for y

Example 1: Find the derivative of $y = \frac{(x-2)^2}{\sqrt{x^2+1}}$

Example 2: Find the derivative of $y = x^{2x+3}$

3.4 Assess Your Understanding

Concepts and Vocabulary

1. $\frac{d}{dx}\ln x =$ ______.

2. *True or False* $\frac{d}{dx}x^e = e\,x^{e-1}$.

3. *True or False* $\frac{d}{dx}\ln[x\sin^2 x] = \frac{d}{dx}\ln x \cdot \frac{d}{dx}\ln\sin^2 x$.

4. *True or False* $\frac{d}{dx}\ln\pi = \frac{1}{\pi}$.

5. $\frac{d}{dx}\ln|x| =$ ____ for all $x \neq 0$.

6. $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n =$ ______.

Skill Building

In Problems 7–44, find the derivative of each function.

7. $f(x) = 5\ln x$

8. $f(x) = -3\ln x$

(PAGE 253) 9. $s(t) = \log_2 t$

10. $s(t) = \log_3 t$

11. $g(x) = (\cos x)(\ln x)$

12. $g(x) = (\sin x)(\ln x)$

13. $F(x) = \ln(3x)$

14. $F(x) = \ln\frac{x}{2}$

15. $s(t) = \ln(e^t - e^{-t})$

16. $s(t) = \ln(e^{3t} + e^{-3t})$

(PAGE 254) 17. $f(x) = x\ln(x^2+4)$

18. $f(x) = x\ln(x^2+5x+1)$

19. $f(x) = x^2\log_5(3x+5)$

20. $f(x) = x^3\log_4(2x+3)$

21. $f(x) = \frac{1}{2}\ln\frac{1+x}{1-x}$

22. $f(x) = \frac{1}{2}\ln\frac{1+x^2}{1-x^2}$

23. $f(x) = \ln(\ln x)$

24. $f(x) = \ln\left(\ln\frac{1}{x}\right)$

25. $g(x) = \ln\frac{x}{\sqrt{x^2+1}}$

26. $g(x) = \ln\frac{4x^3}{\sqrt{x^2+4}}$

(PAGE 255) 27. $f(x) = \ln\frac{(x^2+1)^2}{x\sqrt{x^2-1}}$

28. $f(x) = \ln\frac{x\sqrt{3x-1}}{(x^2+1)^3}$

29. $F(\theta) = \ln(\sin\theta)$

30. $F(\theta) = \ln(\cos\theta)$

31. $g(x) = \ln\left(x+\sqrt{x^2+4}\right)$

32. $g(x) = \ln\left(\sqrt{x+1}+\sqrt{x}\right)$

33. $f(x) = \log_2(1+x^2)$

34. $f(x) = \log_2(x^2-1)$

35. $f(x) = \tan^{-1}(\ln x)$

36. $f(x) = \sin^{-1}(\ln x)$

37. $s(t) = \ln(\tan^{-1} t)$

38. $s(t) = \ln(\sin^{-1} t)$

39. $G(x) = (\ln x)^{1/2}$

40. $G(x) = (\ln x)^{-1/2}$

41. $f(\theta) = \sin(\ln\theta)$

42. $f(\theta) = \cos(\ln\theta)$

43. $g(x) = (\log_3 x)^{3/2}$

44. $g(x) = (\log_2 x)^{4/3}$

In Problems 45–50, use implicit differentiation to find $y' = \frac{dy}{dx}$.

45. $x\ln y + y\ln x = 2$

46. $\frac{\ln y}{x} + \frac{\ln x}{y} = 2$

47. $\ln(x^2+y^2) = x+y$

48. $\ln(x^2-y^2) = x-y$

49. $\ln\frac{y}{x} = y$

50. $\ln\frac{y}{x} - \ln\frac{x}{y} = 1$

In Problems 51–72, use logarithmic differentiation to find y'. Assume that the variable is restricted so that all arguments of logarithm functions are positive.

(PAGE 255) 51. $y = (x^2+1)^2(2x^3-1)^4$

52. $y = (3x^2+4)^3(x^2+1)^4$

53. $y = \frac{x^2(x^3+1)}{\sqrt{x^2+1}}$

54. $y = \frac{\sqrt{x}(x^3+2)^2}{\sqrt[3]{3x+4}}$

55. $y = \frac{x\cos x}{(x^2+1)^3\sin x}$

56. $y = \frac{x\sin x}{(1+e^x)^3\cos x}$

(PAGE 256) 57. $y = (3x)^x$

58. $y = (x-1)^x$

59. $y = x^{\ln x}$

60. $y = (2x)^{\ln x}$

61. $y = x^{x^2}$

62. $y = (3x)^{\sqrt{x}}$

63. $y = x^{e^x}$

64. $y = (x^2+1)^{e^x}$

65. $y = x^{\sin x}$

66. $y = x^{\cos x}$

67. $y = (\sin x)^x$

68. $y = (\cos x)^x$

69. $y = (\sin x)^{\cos x}$

70. $y = (\sin x)^{\tan x}$

71. $x^y = 4$

72. $y^x = 10$

In Problems 73–76, find an equation of the tangent line to the graph of $y = f(x)$ at the given point.

73. $y = \ln(5x)$ at $\left(\frac{1}{5}, 0\right)$

74. $y = x\ln x$ at $(1, 0)$

(PAGE 256) 75. $y = \frac{x^2\sqrt{3x-2}}{(x-1)^2}$ at $(2, 8)$

76. $y = \frac{x(\sqrt[3]{x}+1)^2}{\sqrt{x+1}}$ at $(8, 24)$

In Problems 77–80, express each limit in terms of e.

(PAGE 258) 77. $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{2n}$

78. $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n/2}$

79. $\lim_{n\to\infty}\left(1+\frac{1}{3n}\right)^{n}$

80. $\lim_{n\to\infty}\left(1+\frac{4}{n}\right)^{n}$

Applications and Extensions

81. Find $\frac{d^{10}}{dx^{10}}(x^9\ln x)$.

82. If $f(x) = \ln(x-1)$, find $f^{(n)}(x)$.

83. If $y = \ln(x^2+y^2)$, find the value of $\frac{dy}{dx}$ at the point $(1, 0)$.

84. If $f(x) = \tan\left(\ln x - \frac{1}{\ln x}\right)$, find $f'(e)$.

85. Find y' if $y = x^x$, $x > 0$, by using $y = x^x = e^{\ln x^x}$ and the Chain Rule.

86. If $y = \ln(kx)$, where $x > 0$ and $k > 0$ is a constant, show that $y' = \frac{1}{x}$.

In Problems 87 and 88, find y'. Assume that a is a constant.

87. $y = x\tan^{-1}\frac{x}{a} - \frac{1}{2}a\ln(x^2 + a^2), \quad a \neq 0$

88. $y = x\sin^{-1}\frac{x}{a} + a\ln\sqrt{a^2 - x^2}, \quad |a| > |x|, \quad a \neq 0$

Continuously Compounded Interest *In Problems 89 and 90, use the following discussion:*

Suppose an initial investment, called the **principal** *P, earns an annual rate of interest r* (expressed as a decimal), *which is compounded n times per year. The interest earned on the principal P in the first compounding period is $P\left(\frac{r}{n}\right)$, and the resulting amount A of the investment after one compounding period is $A = P + P\left(\frac{r}{n}\right) = P\left(1 + \frac{r}{n}\right)$. After k compounding periods, the amount A of the investment is $A = P\left(1 + \frac{r}{n}\right)^k$. Since in t years there are nt compounding periods, the amount A after t years is*

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

When interest is compounded so that after t years the accumulated amount is $A = \lim_{n\to\infty} P\left(1 + \frac{r}{n}\right)^{nt}$, the interest is said to be ***compounded continuously.***

89. (a) Show that if the annual rate of interest r is compounded continuously, then the amount A after t years is $A = Pe^{rt}$, where P is the initial investment.

(b) If an initial investment of $P = \$5000$ earns 2% interest compounded continuously, how much is the investment worth after 10 years?

(c) How long does it take an investment of \$10,000 to double if it is invested at 2.4% compounded continuously?

(d) Show that the rate of change of A with respect to t when the interest rate r is compounded continuously is $\frac{dA}{dt} = rA$.

90. A bank offers a certificate of deposit (CD) that matures in 10 years with a rate of interest of 3% compounded continuously. (See Problem 89.) Suppose you purchase such a CD for \$2000 in your IRA.

(a) Write an equation that gives the amount A in the CD as a function of time t in years.

(b) How much is the CD worth at maturity?

(c) What is the rate of change of the amount A at $t = 3$? At $t = 5$? At $t = 8$?

(d) Explain the results found in (c).

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91. Sound Level of a Leaf Blower The loudness L, measured in decibels (dB), of a sound of intensity I is defined as $L(x) = 10\log\frac{I(x)}{I_0}$, where x is the distance in meters from the source of the sound and $I_0 = 10^{-12}$ W/m^2 is the least intense sound that a human ear can detect. The intensity I is defined as the power P of the sound wave divided by the area A on which it falls. If the wave spreads out uniformly in all directions, that is, if it is spherical, the surface area is $A(x) = 4\pi x^2$ m^2, and $I(x) = \frac{P}{4\pi x^2}$ W/m^2.

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(a) If you are 2.0 m from a noisy leaf blower and are walking away from it, at what rate is the loudness L changing with respect to distance x?

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(b) Interpret the sign of your answer.

92. Show that $\ln x + \ln y = 2x$ is equivalent to $xy = e^{2x}$. Use $xy = e^{2x}$ to find y'. Compare this result to the solution found in Example 1(c).

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93. If $\ln T = kt$, where k is a constant, show that $\frac{dT}{dt} = kT$.

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94. Graph $y = \left(1 + \frac{1}{x}\right)^x$ and $y = e$ on the same screen. Explain how the graph supports the fact that $\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = e$.

95. Power Rule for Functions Show that if u is a function of x that is differentiable and a is a real number, then

$$\frac{d}{dx}[u(x)]^a = a[u(x)]^{a-1}u'(x)$$

provided u^a and u^{a-1} are defined.
Hint: Let $|y| = |[u(x)]^a|$ and use logarithmic differentiation.

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96. Show that the tangent lines to the graphs of the family of parabolas $f(x) = -\frac{1}{2}x^2 + k$ are always perpendicular to the tangent lines to the graphs of the family of natural logarithms $g(x) = \ln(bx) + c$, where $b > 0$, k, and c are constants.
Source: Mathematics students at Millikin University, Decatur, Illinois.

97. Find the derivative of $y = \ln|x|$ by writing $y = \ln\sqrt{x^2}$ and using the Chain Rule.

Challenge Problem

98. If f and g are differentiable functions, and if $f(x) > 0$, show that

$$\frac{d}{dx}f(x)^{g(x)} = g(x)f(x)^{g(x)-1}f'(x) + f(x)^{g(x)}[\ln f(x)]g'(x)$$

3.4 AP Practice Problems (p. 261) – Derivatives of Log Functions

1. If $f(x) = e^x + x^2$, find $\frac{d}{dx}[f(\ln x)]$.

   (A) $\frac{1+2\ln x}{x}$ (B) $1 + \frac{2\ln x}{x}$

   (C) $1 + \frac{2}{x^2}$ (D) $x + 2x\ln x$

2. $\frac{d}{dx}(x^2 e^{\ln x^3}) =$

   (A) $2x + 3x^2$ (B) $5x^3$

   (C) $5x^4$ (D) $6x^4$

3. If $h(x) = \ln(x^2+4)$, then $h'(x)$ equals

   (A) $\left|\frac{2x}{x^2+4}\right|$ (B) $\frac{2x}{x^2+4}$

   (C) $\frac{x^2}{x^2+4}$ (D) $\frac{1}{x^2+4}$

4. Find the rate of change of y with respect to x when $x = 1$ if $\ln(xy) = x$.

   (A) e (B) 0 (C) 1 (D) $e - 1$

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5. If $f(x) = \ln(e^{x^2-3x})$, then $f'(x)$ equals

(A) $\dfrac{1}{e^{x^2-3x}}$ (B) $\dfrac{2x-3}{e^{x^2-3x}}$

(C) $x^2 - 3x$ (D) $2x - 3$

6. Find the slope of the tangent line to the graph of $y = \ln(\sec^2 x)$ at $x = \dfrac{\pi}{4}$.

(A) 2 (B) $\dfrac{\sqrt{2}}{2}$

(C) $\sqrt{2}$ (D) $2\sqrt{2}$

7. $\dfrac{d}{dx} \ln\left|\sin \dfrac{\pi}{x}\right| =$

(A) $\cot \dfrac{\pi}{x}$ (B) $-\dfrac{\pi}{x^2} \csc \dfrac{\pi}{x}$

(C) $-\dfrac{\pi}{x^2} \cot \dfrac{\pi}{x}$ (D) $\dfrac{\pi}{x} \cot \dfrac{\pi}{x}$

8. Find $\frac{d}{dx}(x^4+2)^x$.

(A) $4x^4(x^4+2)^{x-1}$

(B) $\frac{4x^4}{x^4+2}+\ln(x^4+2)$

(C) $(x^4+2)^x\left[\frac{4x^4}{x^4+2}+\ln(x^4+2)\right]$

(D) $(x^4+2)^x\ln(x^4+2)$

9. Find $\frac{d^2y}{dx^2}$ for $y=\ln(x\sqrt{x})$.

(A) $-\frac{3}{2x^2}$ (B) $\frac{3}{2x^2}$ (C) $\frac{3}{2x}$ (D) $-\frac{3}{x^3}$

10. $\frac{d}{dx}\frac{\log_2 x}{x}=$

(A) $\frac{1-\log_2 x}{x^2}$ (B) $-\frac{\ln 2+\log_2 x}{x^2}$

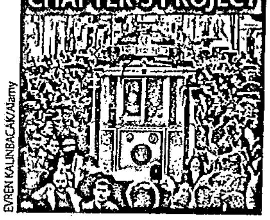
(C) $\frac{\ln 2-\log_2 x}{x^2\ln 2}$ (D) $\frac{1-\ln 2\log_2 x}{x^2\ln 2}$

11. If $f(x) = x \ln x$, then $f'(x)$ equals

(A) $x + \ln x$ (B) 1 (C) $1 + \ln x$ (D) $\frac{1}{x} + \ln x$

12. Suppose $g(x) = \ln(f(x))$, where $f(x) > 0$ for all real numbers and f is differentiable for all real numbers. If $f(4) = 2$ and $f'(4) = -\frac{1}{5}$, find $g'(4)$. Show the computations that lead to the answer.

CHAPTER 3 PROJECT



EVREN KALINBACAK/Alamy

World Population Growth

The Law of Uninhibited Growth states that, under certain conditions, the rate of change of a population is proportional to the size of the population at that time. One consequence of this law is that the time it takes for a population to double remains constant. For example, suppose a culture of bacteria obeys the Law of Uninhibited Growth. Then if it takes five hours for the culture to double from 100 organisms to 200 organisms, in the next five hours it will double again from 200 to 400. We can model this mathematically using the formula

$$P(t) = P_0\, 2^{t/D}$$

where $P(t)$ is the population at time t, P_0 is the population at time $t = 0$, and D is the doubling time. If we use this formula to model population growth, a few observations are in order. For example, the model is continuous, but actual population growth is discrete. That is, an actual population would change from 100 to 101 individuals in an instant, as opposed to a model that has a continuous flow from 100 to 101. The model also produces fractional answers, whereas an actual population is counted in whole numbers. For large populations, however, the growth is continuous enough for the model to match real-world conditions, at least for a short time. In general, as growth continues, there are obstacles to growth at which point the model will fail to be accurate. Situations that follow the model of the Law of Uninhibited Growth vary from the introduction of invasive species into a new environment to the spread of a deadly virus for which there is no immunization. Here, we investigate how accurately the model predicts world population.

1. The world population on July 1, 1959, was approximately 2.983435×10^9 persons and had a doubling time of $D = 40$ years. Use these data and the Law of Uninhibited Growth to write a formula for the world population $P = P(t)$. Use this model to solve Problems 2 through 4.
2. Find the rate of change of the world population $P = P(t)$ with respect to time t.
3. Find the rate of change on July 1, 2020 of the world population with respect to time. (Note that $t = 0$ is July 1, 1959.) Round the answer to the nearest whole number.
4. Approximate the world population on July 1, 2020. Round the answer to the nearest person.
5. According to the United Nations, the world population on July 1, 2015 was 7.349472×10^9. Use $t_0 = 2015$, $P_0 = 7.349472 \times 10^9$, and $D = 40$ and find a new formula to model the world population $P = P(t)$.
6. Use the new model from Problem 5 to find the rate of change of the world population on July 1, 2020.
7. Compare the results from Problems 3 and 6. Interpret and explain any discrepancy between the two rates of change.
8. Use the new model to approximate the world population on July 1, 2020. Round the answer to the nearest person.
9. Discuss possible reasons for the discrepancies in the approximations of the 2020 population. Do you think one set of data gives better results than the other?

Source: *UN World Population 2015* © 2016. https://esa.un.org/unpd/wpp/Publications/Files/World_Population_2015

Chapter Review

THINGS TO KNOW

3.1 The Chain Rule

Theorems:

- Chain Rule: $(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$ (p. 223) $\quad \dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$
- Power Rule for functions: $\dfrac{d}{dx}[g(x)]^n = n[g(x)]^{n-1} g'(x)$, where n is an integer (p. 228)

Basic Derivative Formulas:

- $\dfrac{d}{dx} e^{u(x)} = e^{u(x)} \dfrac{du}{dx}$ (p. 225)
- Derivatives of trigonometric functions (p. 225)
- $\dfrac{d}{dx} a^x = a^x \ln a \quad a > 0$ and $a \neq 1$ (p. 227)

3.2 Implicit Differentiation

To differentiate an implicit function (p. 237):

- Assume y is a differentiable function of x.
- Differentiate both sides of the equation with respect to x.
- Solve the resulting equation for $y' = \dfrac{dy}{dx}$.

Theorems:

- Power Rule for rational exponents: $\dfrac{d}{dx} x^{p/q} = \dfrac{p}{q} \cdot x^{(p/q)-1}$, provided $x^{p/q}$ and $x^{p/q-1}$ are defined. (p. 241)
- Power Rule for functions: $\dfrac{d}{dx}[u(x)]^r = r[u(x)]^{r-1} u'(x)$, r a rational number; provided u^r and u^{r-1} are defined. (p. 241)

3.3 Derivatives of the Inverse Trigonometric Functions

Theorem: The derivative of an inverse function (p. 246)

Basic Derivative Formulas:

- $\dfrac{d}{dx} \sin^{-1} x = \dfrac{1}{\sqrt{1-x^2}} \quad -1 < x < 1$ (p. 248)
- $\dfrac{d}{dx} \tan^{-1} x = \dfrac{1}{1+x^2}$ (p. 249)
- $\dfrac{d}{dx} \sec^{-1} x = \dfrac{1}{x\sqrt{x^2-1}} \quad |x| > 1$ (p. 249)

3.4 Derivatives of Logarithmic Functions

Basic Derivative Formulas:

- $\frac{d}{dx}\log_a x = \frac{1}{x \ln a}$, $x > 0$, $a > 0$, $a \neq 1$ (p. 252)
- $\frac{d}{dx}\ln x = \frac{1}{x}$, $x > 0$ (p. 252)
- $\frac{d}{dx}\ln u(x) = \frac{u'(x)}{u(x)}$, $u(x) > 0$ (p. 253)
- $\frac{d}{dx}\ln |x| = \frac{1}{x}$, $x \neq 0$ (p. 254)

Steps for Using Logarithmic Differentiation (p. 255):

- **Step 1** If the function $y = f(x)$ consists of products, quotients, and powers, take the natural logarithm of each side. Then simplify using properties of logarithms.
- **Step 2** Differentiate implicitly, and use $\frac{d}{dx}\ln y = \frac{y'}{y}$.
- **Step 3** Solve for y', and replace y with $f(x)$.

Theorems:

- **Power Rule** If a is a real number, then $\frac{d}{dx}x^a = ax^{a-1}$ (p. 257)
- The number e can be expressed as $\lim_{h\to 0}(1+h)^{1/h} = e$ or $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$ (p. 257)

OBJECTIVES

Section	You should be able to ...	Examples	Review Exercises	Preparing for the AP® Exam: AP® Review Problems
3.1	1 Differentiate a composite function (p. 223)	1–5	1, 13, 24	
	2 Differentiate $y = a^x$, $a > 0$, $a \neq 1$ (p. 227)	6	19, 22	1, 3
	3 Use the Power Rule for functions to find a derivative (p. 228)	7, 8	1, 11, 12, 14, 17	
	4 Use the Chain Rule for multiple composite functions (p. 229)	9	15, 18, 57	2, 4
3.2	1 Find a derivative using implicit differentiation (p. 236)	1–4	39–48, 58, 59	5, 6
	2 Find higher-order derivatives using implicit differentiation (p. 240)	5	45–48	
	3 Differentiate functions with rational exponents (p. 240)	6, 7	2–8, 15, 16, 53–56	
3.3	1 Find the derivative of an inverse function (p. 245)	1, 2	49, 50	7
	2 Find the derivative of the inverse trigonometric functions (p. 247)	3, 4	32–38	9
3.4	1 Differentiate logarithmic functions (p. 252)	1–3	20, 21, 23, 25–30, 48	8
	2 Use logarithmic differentiation (p. 255)	4–7	9, 10, 31, 57	
	3 Express e as a limit (p. 257)	8	51, 52	

REVIEW EXERCISES

In Problems 1–38, find the derivative of each function. When a, b, or n appear, they are nonzero constants.

1. $y = (ax+b)^n$
2. $y = \sqrt{2ax}$
3. $y = x\sqrt{1-x}$
4. $y = \frac{1}{\sqrt{x^2+1}}$
5. $y = (x^2+4)^{3/2}$
6. $F(x) = \frac{x^2}{\sqrt{x^2-1}}$
7. $z = \frac{\sqrt{2ax-x^2}}{x}$
8. $y = \sqrt{x} + \sqrt[3]{x}$
9. $y = (e^x - x)^{5x}$
10. $\phi(x) = \frac{(x^2-a^2)^{3/2}}{\sqrt{x+a}}$
11. $f(x) = \frac{x^2}{(x-1)^2}$
12. $u = (b^{1/2} - x^{1/2})^2$
13. $y = x\sec(2x)$
14. $u = \cos^3 x$
15. $y = \sqrt{a^2 \sin\frac{x}{a}}$
16. $\phi(z) = \sqrt{1+\sin z}$
17. $u = \sin v - \frac{1}{3}\sin^3 v$
18. $y = \tan\sqrt{\frac{\pi}{x}}$
19. $y = 1.05^x$
20. $v = \ln(y^2+1)$
21. $z = \ln(\sqrt{u^2+25} - u)$
22. $y = x^2 + 2^x$
23. $y = \ln[\sin(2x)]$
24. $f(x) = e^{-x}\sin(2x+\pi)$
25. $g(x) = \ln(x^2-2x)$
26. $y = \ln\frac{x^2+1}{x^2-1}$
27. $y = e^{-x}\ln x$
28. $w = \ln(\sqrt{x+7} - \sqrt{x})$
29. $y = \frac{1}{12}\ln\frac{x}{\sqrt{144-x^2}}$
30. $y = \ln(\tan^2 x)$
31. $f(x) = \frac{e^x(x^2+4)}{x-2}$
32. $y = \sin^{-1}(x-1) + \sqrt{2x-x^2}$

33. $y = 2\sqrt{x} - 2\tan^{-1}\sqrt{x}$

34. $y = 4\tan^{-1}\frac{x}{2} + x$

35. $y = \sin^{-1}(2x - 1)$

36. $y = x^2 \tan^{-1}\frac{1}{x}$

37. $y = x\tan^{-1}x - \ln\sqrt{1+x^2}$

38. $y = \sqrt{1-x^2}(\sin^{-1}x)$

In Problems 39–44, find $y' = \frac{dy}{dx}$ using implicit differentiation.

39. $x = y^5 + y$

40. $x = \cos^5 y + \cos y$

41. $\ln x + \ln y = x\cos y$

42. $\tan(xy) = x$

43. $y = x + \sin(xy)$

44. $x = \ln(\csc y + \cot y)$

In Problems 45–48, find y' and y''.

45. $xy + 3y^2 = 10x$

46. $y^3 + y = x^2$

47. $xe^y = 4x^2$

48. $\ln(x+y) = 8x$

49. The function $f(x) = e^{2x}$ has an inverse function g. Find $g'(1)$.

50. The function $f(x) = \sin x$ defined on the restricted domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ has an inverse function g. Find $g'\left(\frac{1}{2}\right)$.

In Problems 51 and 52, express each limit in terms of the number e.

51. $\lim_{n\to\infty}\left(1+\frac{2}{5n}\right)^n$

52. $\lim_{h\to 0}(1+3h)^{2/h}$

53. If $f(x) = \sqrt{1-\sin^2 x}$, find the domain of f'.

54. If $f(x) = x^{1/2}(x-2)^{3/2}$ for all $x \geq 2$, find the domain of f'.

55. Let f be the function defined by $f(x) = \sqrt{1+6x}$.
(a) What are the domain and the range of f?
(b) Find the slope of the tangent line to the graph of f at $x = 4$.
(c) Find the y-intercept of the tangent line to the graph of f at $x = 4$.
(d) Give the coordinates of the point on the graph of f where the tangent line is parallel to the line $y = x + 12$.

56. **Tangent and Normal Lines** Find equations of the tangent and normal lines to the graph of $y = x\sqrt{x + (x-1)^2}$ at the point $(2, 2\sqrt{3})$.

57. If $f(x) = (x^2+1)^{(2-3x)}$, find $f'(1)$.

58. **Tangent Line** Find an equation of the tangent line to the graph of $4xy - y^2 = 3$ at the point $(1, 3)$.

59. Find y' at $x = \frac{\pi}{2}$ and $y = \pi$ if $x\sin y + y\cos x = 0$.

Preparing for the AP® Exam

AP® REVIEW PROBLEMS: CHAPTER 3

1. Find an equation of the tangent line to the graph of $y = 3^{\sin x} - 4$ when $x = 0$.
(A) $y = (\ln 3)x - 4$ (B) $y = (\ln 3)x - 3$
(C) $y = x - 3$ (D) $y = 3x - 3$

2. If $f(x) = \sin u$, $u = v - \frac{1}{v^2}$, and $v = \ln x$, find $f'(e)$.
(A) 0 (B) $\frac{3}{e}$ (C) $\frac{1}{e} + \frac{2}{e\ln 3}$ (D) $\frac{2}{e^2}$

3. If $e^{f(x)} = 2 + x^4$, then $f'(x) =$
(A) $\frac{4x^3}{e^x}$ (B) $4x^3e^x$
(C) $\frac{4x^3}{2+x^4}$ (D) $\frac{e^x}{2+x^4}$

4. Find y' if $y = \sin^{4/3}(4x - x^2)$.
(A) $y' = \frac{16-8x}{3}\sin^{1/3}(4x - x^2)$
(B) $y' = \frac{16-8x}{3}\sin^{1/3}(4x-x^2)\cdot\cos(4x-x^2)$
(C) $y' = \frac{4}{3}\sin^{1/3}(4x-x^2)\cdot\cos(4x-x^2)$
(D) $y' = -\frac{8x}{3}\sin^{1/3}(4x-x^2)\cdot\cos(4-2x)$

5. The slope of the normal line to the graph of $x^2 + (xy-2)^2 = 20$ at the point $(2, -1)$ is
(A) $\frac{3}{4}$ (B) -4 (C) $\frac{4}{3}$ (D) $-\frac{4}{3}$

6. If $y = \sin(3x + 2y)$, find the rate of change of y with respect to x at the origin.
(A) 0 (B) 3 (C) -1 (D) -3

7. The points $(1, 1)$, $(2, 3)$, and $(3, 13)$ are on the graph of the function $f(x) = x^3 - 2x^2 + x + 1$, $x \geq 1$. If the function g is the inverse function of f, then $g'(3) =$
(A) $\frac{1}{5}$ (B) $\frac{1}{16}$ (C) $\frac{1}{3}$ (D) $-\frac{1}{5}$

8. Find $\frac{d}{dx}\ln(\ln x)$.
(A) 1 (B) $\frac{x}{\ln x}$ (C) $\frac{1}{x\ln x}$ (D) $\frac{1}{(\ln x)^2}$

9. The derivative of $y = \tan^{-1}(xe^x)$ equals
(A) $\frac{xe^x + e^x}{\sqrt{1+x^2e^{2x}}}$ (B) $\frac{e^x+1}{1+xe^x}$
(C) $\frac{2x^2e^{2x} + 2xe^x}{1+x^2e^{2x}}$ (D) $\frac{xe^x+e^x}{1+x^2e^{2x}}$

Preparing for the AP® Exam

AP® CUMULATIVE REVIEW PROBLEMS: CHAPTERS 1–3

1. If $f(t)=t^2\sin t+\ln t$, then $f'(t)$ equals

   (A) $2t(\sin t+\ln t)+t^2\left(\cos t+\frac{1}{t}\right)$

   (B) $2t\sin t-t^2\cos t+\frac{1}{t}$

   (C) $2t\sin t+t^2\cos t+\frac{1}{t}\ln t$

   (D) $2t\sin t+t^2\cos t+\frac{1}{t}$

2. Find the point(s), if any, where the slope of the tangent line to the graph of $f(x)=\frac{x^3}{3}+x^2-3x+1$ is zero.

   (A) $(0,1)$ (B) $(0,-3)$

   (C) $(-3,0)$ and $(1,0)$ (D) $(-3,10)$ and $\left(1,-\frac{2}{3}\right)$

3. Given $f(x)=e^x(x+\cos x)$, find $f''(0)$.

   (A) 0 (B) 2 (C) 3 (D) 5

4. Suppose the function f is continuous on the closed interval $[-2,6]$, and $f(-2)=8$ and $f(6)=1$. By the Intermediate Value Theorem we know

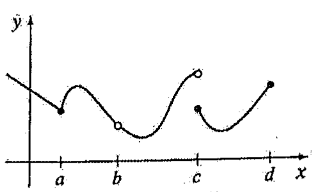
   (A) f has at least one zero on the interval $(-2,6)$.

   (B) $f(x)>0$ for all x in the interval $[-2,6]$.

   (C) $f(x)=3$ for at least one number in the interval $(-2,6)$.

   (D) $f'(x)$ is defined for all numbers in the interval $(-2,6)$.

5. The graph of f is shown in the figure below. Which statement is false?



Graph of f

   (A) $\lim_{x\to a}f(x)$ exists. (B) $\lim_{x\to b}f(x)$ exists.

   (C) $\lim_{x\to c}f(x)$ exists. (D) $\lim_{x\to d}f(x)$ does not exist.

6. $\lim_{h\to 0}\frac{\ln(e+h)^2-2}{h}=$

   (A) $f'(e^2)$ where $f(x)=\ln x$

   (B) $f'(e)$ where $f(x)=\ln x^2$

   (C) $f'(2)$ where $f(x)=\ln x$

   (D) $f'(e)$ where $f(x)=(\ln x)^2$

7. If $f(x)=\frac{4x+5}{5x-4}$, then $f'(x)$ equals

   (A) $\frac{40x+9}{(5x-4)^2}$ (B) $-\frac{41}{(5x-4)^2}$

   (C) $-\frac{9}{(5x-4)^2}$ (D) $\frac{9}{(5x-4)^2}$

8. Suppose the function f is defined by

$$f(x)=\begin{cases}5x+17 & \text{if } x<-3\\ 2 & \text{if } x=-3\\ 8-x^2 & \text{if } -3<x<3\\ x-4 & \text{if } x\geq 3\end{cases}$$

   For what numbers, if any, is f discontinuous?

   (A) none (B) -3 only (C) 3 only (D) -3 and 3

9. $\lim_{x\to 4}\frac{x^2-16}{\sqrt{x}-2}=$

   (A) 2 (B) 4 (C) 32 (D) Does not exist.

10. The graph of $y=\frac{2x^2+2x+3}{4x^2-4x}$ has

    (A) a horizontal asymptote at $y=\frac{1}{2}$, but no vertical asymptote.

    (B) no horizontal asymptote, but vertical asymptotes at $x=0$ and $x=1$.

    (C) a horizontal asymptote at $y=\frac{1}{2}$ and vertical asymptotes at $x=0$ and $x=1$.

    (D) a horizontal asymptote at $x=2$, but no vertical asymptote.

Ch. 3 Review AP Practice Problems (p.264) – Chain Rule, Implicit Diff. & Inverse Derivatives

1. Find an equation of the tangent line to the graph of $y = 3^{\sin x} - 4$ when $x = 0$.

   (A) $y = (\ln 3)x - 4$ (B) $y = (\ln 3)x - 3$

   (C) $y = x - 3$ (D) $y = 3x - 3$

2. If $f(x) = \sin u$, $u = v - \dfrac{1}{v^2}$, and $v = \ln x$, find $f'(e)$.

   (A) 0 (B) $\dfrac{3}{e}$ (C) $\dfrac{1}{e} + \dfrac{2}{e\ln 3}$ (D) $\dfrac{2}{e^2}$

3. If $e^{f(x)} = 2 + x^4$, then $f'(x) =$

   (A) $\dfrac{4x^3}{e^x}$ (B) $4x^3 e^x$

   (C) $\dfrac{4x^3}{2+x^4}$ (D) $\dfrac{e^x}{2+x^4}$

4. Find y' if $y = \sin^{4/3}(4x - x^2)$.

   (A) $y' = \dfrac{16 - 8x}{3}\sin^{1/3}(4x - x^2)$

   (B) $y' = \dfrac{16 - 8x}{3}\sin^{1/3}(4x - x^2) \cdot \cos(4x - x^2)$

   (C) $y' = \dfrac{4}{3}\sin^{1/3}(4x - x^2) \cdot \cos(4x - x^2)$

   (D) $y' = -\dfrac{8x}{3}\sin^{1/3}(4x - x^2) \cdot \cos(4 - 2x)$

5. The slope of the normal line to the graph of $x^2 + (xy - 2)^2 = 20$ at the point $(2, -1)$ is

   (A) $\dfrac{3}{4}$ (B) -4 (C) $\dfrac{4}{3}$ (D) $-\dfrac{4}{3}$

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6. If $y = \sin(3x + 2y)$, find the rate of change of y with respect to x at the origin.

(A) 0 (B) 3 (C) -1 (D) -3

7. The points (1, 1), (2, 3), and (3, 13) are on the graph of the function $f(x) = x^3 - 2x^2 + x + 1$, $x \geq 1$. If the function g is the inverse function of f, then $g'(3) =$

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8. Find $\frac{d}{dx} \ln(\ln x)$.

(A) 1 (B) $\frac{x}{\ln x}$ (C) $\frac{1}{x \ln x}$ (D) $\frac{1}{(\ln x)^2}$

9. The derivative of $y = \tan^{-1}(xe^x)$ equals

(A) $\frac{xe^x + e^x}{\sqrt{1 + x^2e^{2x}}}$ (B) $\frac{e^x + 1}{1 + xe^x}$

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(45)

AP Calculus Derivative Rules & Practice

0) Derivative Power Rule:

$$\frac{d}{dx}u^n = n * u^{n-1} * u'$$

A. Trig Derivatives

1) $\frac{d}{dx}\sin u = \cos u * u'$	2) $\frac{d}{dx}\cos u = -\sin u * u'$
3) $\frac{d}{dx}\tan u = sec^2\, u * u'$	4) $\frac{d}{dx}\cot\, u = -csc^2\, u * u'$
5) $\frac{d}{dx}\sec u = \sec u \tan u * u'$	6) $\frac{d}{dx}\csc u = -\csc u \cot u * u'$

B. Logs and Exponential Derivatives

7) $\frac{d}{dx}\ln u = \frac{u'}{u}$	8) $\frac{d}{dx}\, e^u = e^u * u'$
9) $\frac{d}{dx}log_a\, u = \left(\frac{1}{\ln a}\right)\frac{u'}{u}$	10) $\frac{d}{dx}\, a^u = (\ln a)\, a^u * u'$

C. ArcTrig Derivatives

11) $\frac{d}{dx}\arcsin u = \frac{u'}{\sqrt{1-u^2}}$	12) $\frac{d}{dx}\arccos u = -\frac{u'}{\sqrt{1-u^2}}$
13) $\frac{d}{dx}\arctan u = \frac{u'}{1+u^2}$	14) $\frac{d}{dx}\text{arccot}\, u = -\frac{u'}{1+u^2}$
15) $\frac{d}{dx}\text{arcsec}\, u = \frac{u'}{\lvert u\rvert\sqrt{u^2-1}}$	16) $\frac{d}{dx}\text{arccsc}\, u = -\frac{u'}{\lvert u\rvert\sqrt{u^2-1}}$

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AP Calculus Derivative & Integral Rules **(Blank Practice Sheet)**

Derivative Power Rule:

0) $\frac{d}{dx}u^n =$

A. Trig Derivatives

1) $\frac{d}{dx}\sin u =$	2) $\frac{d}{dx}\cos u =$
3) $\frac{d}{dx}\tan u =$	4) $\frac{d}{dx}\cot u =$
5) $\frac{d}{dx}\sec u =$	6) $\frac{d}{dx}\csc u =$

B. Logs and Exponential Derivatives

7) $\frac{d}{dx}\ln u =$	8) $\frac{d}{dx}\ e^u =$
9) $\frac{d}{dx}log_a u =$	10) $\frac{d}{dx}\ a^u =$

C. ArcTrig Derivatives

11) $\frac{d}{dx}\arcsin u =$	12) $\frac{d}{dx}\arccos u =$
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