BC Calculus - Unit 3 - Composite, Implicit, & Inverse Functions - Test Review Worksheet

Key

Show all appropriate work!

Find the derivative.

1. $h(x) = \cos^2(4x)$

$h(x) = [\cos(4x)]^2$

$h'(x) = 2[\cos(4x)] \cdot -\sin(4x) \cdot 4$

$\boxed{h'(x) = -8\cos(4x)\sin(4x)}$

2. $y = \ln\sqrt{x+3}$

$y = \ln(x+3)^{1/2}$

$y = \frac{1}{2}\ln(x+3)$

$y' = \frac{1}{2} \cdot \frac{1}{x+3} = \boxed{\frac{1}{2(x+3)}}$

3. $x^2 + 2y^5 = 10xy$

*implicit

*product Rule

$x^2 + 2y^5 = \underbrace{10x}_{f}\underbrace{y}_{g}$

$2x + 10y^4\left(\frac{dy}{dx}\right) = \underbrace{10}_{f'} \cdot \underbrace{y}_{g} + \underbrace{10x}_{f}\underbrace{\left(dy/dx\right)}_{g'}$

$10y^4\left(\frac{dy}{dx}\right) - 10x\left(\frac{dy}{dx}\right) = 10y - 2x$

$\frac{dy}{dx}\left(10y^4 - 10x\right) = 10y - 2x$

$\boxed{\frac{dy}{dx} = \frac{10y-2x}{10y^4-10x} \text{ or } \frac{5y-x}{5y^4-5x}}$

4. $y = \csc^{-1}(x^3)$

$* \frac{d}{dx}\operatorname{arccsc} u = \frac{-u'}{|u|\sqrt{u^2-1}}$

$y' = \frac{-3x^2}{|x^3|\sqrt{(x^3)^2-1}}$

$\boxed{y' = \frac{-3}{|x|\sqrt{x^6-1}}}$

For each problem, let f and g be differentiable functions where $g(x) = f^{-1}(x)$ for all x.

5. $f(6) = -1$, $f(4) = -2$, $f'(6) = 3$, and $f'(4) = 7$. What is the value of $g'(-1)$?

$f(\ ) = -1$	$g(-1) = $ ___
	$g'(-1) = $ ___

↓

$f(6) = -1$	$g(-1) = 6$
$f'(6) = 3$	$g'(-1) = \boxed{1/3}$

6. Let f be the function defined by $f(x) = x^3 + 3x + 1$. Let $g(x) = f^{-1}(x)$, where $g(-3) = -1$. What is the value of $g'(-3)$?

$f(-1) = -3$	$g(-3) = -1$
$f'(-1) = 6$	$g'(-3) = \boxed{1/6}$

$f'(x) = 3x^2 + 3$

$f'(-1) = 3(-1)^2 + 3 = 6$

Find $\frac{d^2y}{dx^2}$ based on the given information.

7. $y = x^5 - e^{4x}$

$$\frac{dy}{dx} = 5x^4 - e^{4x} \cdot 4$$

$$\frac{d^2y}{dx^2} = 20x^3 - 4e^{4x} \cdot 4$$

$$\boxed{\frac{d^2y}{dx^2} = 20x^3 - 16e^{4x}}$$

8. $y = y^2 + x$

$$1\left(\frac{dy}{dx}\right) = 2y\left(\frac{dy}{dx}\right) + 1$$

$$1\left(\frac{dy}{dx}\right) - 2y\left(\frac{dy}{dx}\right) = 1$$

$$\frac{dy}{dx}(1-2y) = 1$$

$$\frac{dy}{dx} = \frac{1}{(1-2y)} = (1-2y)^{-1}$$

$$\frac{d^2y}{dx^2} = -1(1-2y)^{-2}(-2)\left(\frac{dy}{dx}\right)$$

$$\frac{d^2y}{dx^2} = \frac{2}{(1-2y)^2} \cdot \frac{1}{(1-2y)}$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{2}{(1-2y)^3}}$$

9. Find the equation of the tangent line.
$x^2 + 7y^2 = 8y^3$ at $(-6, 2)$

$$2x + 14y\left(\frac{dy}{dx}\right) = 24y^2\left(\frac{dy}{dx}\right)$$

$$14y\left(\frac{dy}{dx}\right) - 24y^2\left(\frac{dy}{dx}\right) = -2x$$

$$\frac{dy}{dx}(14y - 24y^2) = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{14y - 24y^2}$$

$$\left.\frac{dy}{dx}\right|_{(-6,2)} = \frac{-2(-6)}{14(2) - 24(2)^2}$$

$$\left.\frac{dy}{dx}\right|_{(-6,2)} = \frac{12}{-68} = \frac{-3}{17}$$

point: $(-6, 2)$

slope: $m = -3/17$

$$\boxed{y - 2 = \frac{-3}{17}(x+6)}$$

10. If $x = y^2 - \cos x$ find $\frac{d^2y}{dx^2}$ at $(0, -1)$.

$$1 = 2y\left(\frac{dy}{dx}\right) - (-\sin x)$$

$$1 - \sin x = 2y\left(\frac{dy}{dx}\right)$$

$$\frac{1 - \sin x}{2y} = \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1 - \sin x}{2y} \rightarrow \left.\frac{dy}{dx}\right|_{(0,-1)} = \frac{1 - \sin 0}{2(-1)} = -\frac{1}{2}$$

$$\frac{d^2y}{dx^2} = \frac{\underbrace{-\cos x}_{f'} \cdot \underbrace{2y}_{g} - \underbrace{(1-\sin x)}_{f} \underbrace{2\left(\frac{dy}{dx}\right)}_{g'}}{\underbrace{(2y)^2}_{g^2}}$$

$$\left.\frac{d^2y}{dx^2}\right|_{(0,-1)} = \frac{-\cos 0 \cdot 2(-1) - (1 - \sin 0)(2)\left(\frac{-1}{2}\right)}{(-2)^2}$$

$$\left.\frac{d^2y}{dx^2}\right|_{(0,-1)} = \frac{(-1)(-2) - (1)(-1)}{4} = \frac{2+1}{4} = \boxed{\frac{3}{4}}$$

10. Find the equation of any horizontal tangent lines for the graph of $(y^3 + 1)^2 = x^2 + 4x + 4$.

$$2(y^3+1)(3y^2)\left(\frac{dy}{dx}\right) = 2x + 4$$

$$(6y^5 + 6y^2)\left(\frac{dy}{dx}\right) = 2x + 4$$

$$\frac{dy}{dx} = \frac{2x+4}{6y^5 + 6y^2}$$

* find horizontal tangents by setting numerator of $\frac{dy}{dx} = 0$

$2x + 4 = 0$

$x = -2$

$(y^3+1)^2 = (-2)^2 - 8 + 4$

$(y^3+1)^2 = 0$

$y^3 + 1 = 0$

$y^3 = -1$

$\boxed{y = -1}$

11. Slope of the tangent line of $g(x) = 4\sin^3 x$ at $x = \frac{\pi}{4}$.

$$g(x) = 4[\sin x]^3$$

$$g'(x) = 4 \cdot 3[\sin x]^2 \cdot \cos x$$

$$g'(\pi/4) = 12[\sin(\pi/4)]^2 \cos(\pi/4)$$

$$g'(\pi/4) = 12\left(\frac{\sqrt{2}}{2}\right)^2\left(\frac{\sqrt{2}}{2}\right) = 12 \cdot \frac{2}{4} \cdot \frac{\sqrt{2}}{2} = \boxed{3\sqrt{2}}$$

12. Let f and g be differentiable functions where $g(x) = f^{-1}(x)$ for all x. $f(6) = 8$, $f(8) = 2$, $f'(6) = -3$, and $f'(8) = 4$. What is the value of $g'(8)$?

$f(6) = 8$	$g(8) = 6$
$f'(6) = -3$	$g'(8) = \boxed{-1/3}$

Find $\frac{d^2y}{dx^2}$ based on the given information.

13. $y = e^{x^4}$

$$\frac{dy}{dx} = \underbrace{e^{x^4}}_{f} \cdot \underbrace{4x^3}_{g}$$

$$\frac{d^2y}{dx^2} = \underbrace{e^{x^4} 4x^3}_{f'} \cdot \underbrace{4x^3}_{g} + \underbrace{e^{x^4}}_{f} \cdot \underbrace{12x^2}_{g'}$$

$$\boxed{\frac{d^2y}{dx^2} = 16x^6 e^{x^4} + 12x^2 e^{x^4}}$$

14. $5y^2 + 3 = x^2$

$$10y\left(\frac{dy}{dx}\right) + 0 = 2x \rightarrow \frac{dy}{dx} = \frac{2x}{10y} = \frac{\overbrace{x}^{f}}{\underbrace{5y}_{g}}$$

$$\frac{d^2y}{dx^2} = \frac{\overbrace{1}^{f'} \cdot \overbrace{5y}^{g} - \overbrace{x}^{f} \cdot \overbrace{5(dy/dx)}^{g'}}{\underbrace{(5y)^2}_{g^2}} = \frac{5y - 5x\left(\frac{x}{5y}\right)}{25y^2}$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{5y - \frac{x^2}{y}}{25y^2}}$$

Evaluate the 2nd derivative at the given point.

15. If $f(x) = x^3 + \frac{5}{x}$, find $f''(-1)$.

$$y = x^3 + 5x^{-1}$$
$$y' = 3x^2 - 5x^{-2}$$
$$y'' = 6x + 10x^{-3}$$

$$y''(-1) = 6(-1) + 10(-1)^{-3}$$
$$= -6 + \frac{10}{(-1)^3}$$
$$= -6 - 10 = \boxed{-16}$$

16. If $x^2 + y^2 = 13$, find $\frac{d^2y}{dx^2}$ at $(2, 3)$.

$$2x + 2y\left(\frac{dy}{dx}\right) = 0$$
$$\frac{dy}{dx} = \frac{-2x}{2y} = \frac{-x}{y}$$
$$\frac{d^2y}{dx^2} = \frac{(-1)(y) - (-x)\left(\frac{dy}{dx}\right)}{y^2}$$

$$\frac{d^2y}{dx^2} = \frac{-y + x\left(\frac{-x}{y}\right)}{y^2}$$

$$\left.\frac{d^2y}{dx^2}\right|_{(2,3)} = \frac{-3 - \frac{2^2}{3}}{3^2}$$

$$= \frac{-3 - 4/3}{9} = \frac{-\frac{9}{3} - \frac{4}{3}}{9}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{-13}{3}}{9} = \frac{-13}{3} \cdot \frac{1}{9}$$

$$\boxed{\left.\frac{d^2y}{dx^2}\right|_{(2,3)} = \frac{-13}{27}}$$

The table below gives values of the differentiable functions g and h, as well as their derivatives, g' and h', at selected values of x.

x	$g(x)$	$g'(x)$	$h(x)$	$h'(x)$
−1	0	4	3	6
0	9	2	0	−4
3	−1	−2	9	4
9	3	1	16	9

18)

$h(3) = 9$	$(h^{-1})(9) = 3$
$h'(3) = 4$	$(h^{-1})'(9) = \boxed{1/4}$

17. If $f(x) = \frac{g(x)}{\sqrt{h(x)}}$, find $f'(3)$.

$$f(x) = \frac{g(x)}{(h(x))^{1/2}}$$

$$f'(x) = \frac{g'(x) \cdot \sqrt{h(x)} - g(x) \cdot \frac{1}{2}(h(x))^{-1/2}(h'(x))}{\left(\sqrt{h(x)}\right)^2}$$

18. Find $\frac{d}{dx} h^{-1}(9)$.

$$f'(3) = \frac{g'(3) \cdot \sqrt{h(3)} - g(3) \cdot \frac{1}{2}(h(3))^{-1/2} h'(3)}{h(3)}$$

$$= \frac{(-2)(\sqrt{9}) - (-1)(1/2)\left(\frac{1}{\sqrt{9}}\right)(4)}{9} = \frac{-2(3) + 2/3}{9}$$

$$\rightarrow -6 + \frac{2}{3} \rightarrow \frac{-\frac{18}{3} + \frac{2}{3}}{9} \rightarrow \boxed{\frac{-16}{27}}$$

19. Find the equation of the tangent line to $g^{-1}(x)$ at $x = 3$.

$g(9) = 3$	$(g^{-1})(3) = 9$
$g'(9) = 1$	$(g^{-1})'(3) = 1$

point: $(3, 9)$

slope: $m = 1$

$$\boxed{\begin{gathered} y - 9 = 1(x - 3) \\ \text{or} \\ y = x + 6 \end{gathered}}$$

Additional Practice Problems

Find $\frac{dy}{dx}$.

1. $y = \frac{e^{\tan 3x}}{3}$

$y = \frac{1}{3}e^{\tan(3x)}$

$y' = \frac{1}{3}e^{\tan(3x)} \cdot \sec^2(3x) \cdot 3$

$\boxed{y' = e^{\tan(3x)} \cdot \sec^2(3x)}$

2. $y = \ln(\sin 5x)$

$y' = \frac{\cos(5x) \cdot 5}{\sin(5x)}$

$y' = 5 \cdot \frac{\cos(5x)}{\sin(5x)}$

$\boxed{y' = 5\cot(5x)}$

3. $y = x\ln(4x)$

$y' = \underbrace{(1)}_{f'}\underbrace{(\ln(4x))}_{g} + \underbrace{x}_{f} \cdot \underbrace{\left(\frac{4}{4x}\right)}_{g'}$

$\boxed{y' = \ln(4x) + 1}$

4. $e^{y^2} = x^5 + 10$

$e^{y^2} \cdot 2y\left(\frac{dy}{dx}\right) = 5x^4 + 0$

$\boxed{\frac{dy}{dx} = \frac{5x^4}{2ye^{y^2}}}$

5. $y = \cos^{-1}(7x^3)$

$\frac{d}{dx}\cos^{-1}(u) = \frac{-u'}{\sqrt{1-u^2}}$

$y' = \frac{-21x^2}{\sqrt{1-(7x^3)^2}}$

$\boxed{y' = \frac{-21x^2}{\sqrt{1-49x^6}}}$

6. $2x^3 - xy = \ln(y)$

$6x^2 - (\underbrace{1}_{f'} \cdot \underbrace{y}_{g} + \underbrace{x}_{f} \cdot \underbrace{\frac{dy}{dx}}_{g'}) = \frac{1}{y}\left(\frac{dy}{dx}\right)$

$6x^2 - y - x\left(\frac{dy}{dx}\right) = \frac{1}{y}\left(\frac{dy}{dx}\right)$

$6x^2 - y = x\left(\frac{dy}{dx}\right) + \frac{1}{y}\left(\frac{dy}{dx}\right)$

$6x^2 - y = \frac{dy}{dx}\left(x + \frac{1}{y}\right)$

$\boxed{\frac{6x^2-y}{x+\frac{1}{y}} = \frac{dy}{dx}}$

Find the equation of the tangent line at the given point.

7. $4x^3 = -5xy + 4y$ at $(1, -4)$

$12x^2 = (-5)(y) + (-5x)\left(\frac{dy}{dx}\right) + 4\left(\frac{dy}{dx}\right)$

$12x^2 + 5y = \frac{dy}{dx}(-5x + 4)$

$\frac{12x^2+5y}{-5x+4} = \frac{dy}{dx}$

$\left.\frac{dy}{dx}\right|_{(1,-4)} = \frac{12(1)^2 + 5(-4)}{-5(1)+4}$

$\left.\frac{dy}{dx}\right|_{(1,-4)} = \frac{12-20}{-1} = \frac{-8}{-1} = 8$

point: $(1, -4)$

slope: $m = 8$

$\boxed{y + 4 = 8(x-1)}$

8. $y = \arccos(5x)$ at $x = -\frac{\sqrt{3}}{10}$

$y' = \frac{-5}{\sqrt{1-(5x)^2}} \rightarrow \frac{-5}{\sqrt{1-25x^2}}$

$y'\left(-\frac{\sqrt{3}}{10}\right) = \frac{-5}{\sqrt{1-25\left(-\frac{\sqrt{3}}{10}\right)^2}}$

$y'\left(-\frac{\sqrt{3}}{10}\right) = \frac{-5}{\sqrt{1-3/4}} = \frac{-5}{\sqrt{1/4}} = \boxed{-10}$

$y\left(-\frac{\sqrt{3}}{10}\right) = \arccos\left(-\frac{5\sqrt{3}}{10}\right)$

$= \arccos\left(-\frac{\sqrt{3}}{2}\right) \rightarrow \frac{5\pi}{6}$

point: $\left(-\frac{\sqrt{3}}{10}, \frac{5\pi}{6}\right)$ slope: $m = -10$

$\boxed{y - 5\pi/6 = -10\left(x + \frac{\sqrt{3}}{10}\right)}$

9. $h(x) = (2x-1)^3(x+2)$ at $x = 1$.

$h'(x) = 3(2x-1)^2(2)(x+2) + (2x-1)^3(1)$

$h'(1) = 3(1)(2)(3) + (1)(1) = 19$

$h(1) = (1)^3(3) = 3$

point: $(1, 3)$

slope: $m = 19$

$\boxed{y - 3 = 19(x-1)}$