Name: ______________________ **Period:** __________

BC Calculus

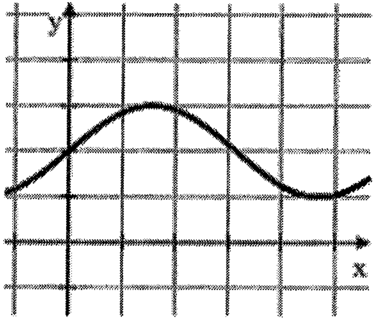
Unit 6

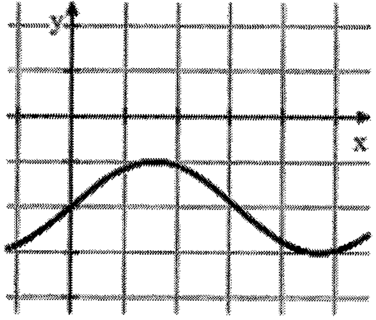
Integration
and
Accumulation of Change

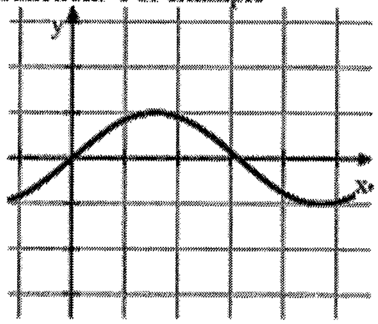
AP Calculus – 6.1 Notes – Exploring Accumulation of Change and Area

Area Under the Curve:

The region between a function and the x-axis is called the area under the curve. "Under" in this instance does not mean below. It means between the x-axis and the function. For example

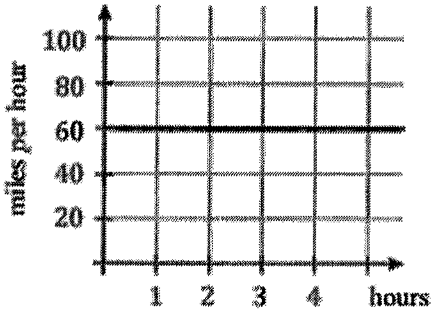






Let's take a rate of change function and examine its graph. The area under the curve gives us the **accumulation of change.**

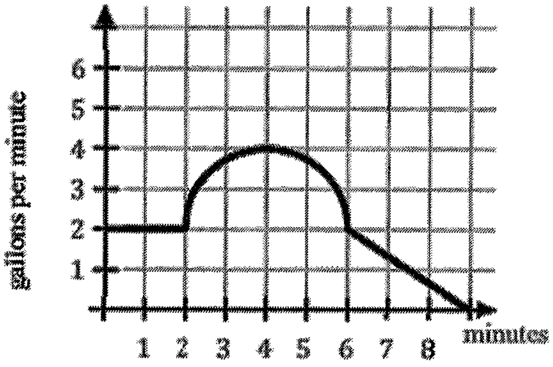
1. You are on a road trip and have your car on cruise control for 4 hours. You travel at 60 miles per hour. How far have you traveled?



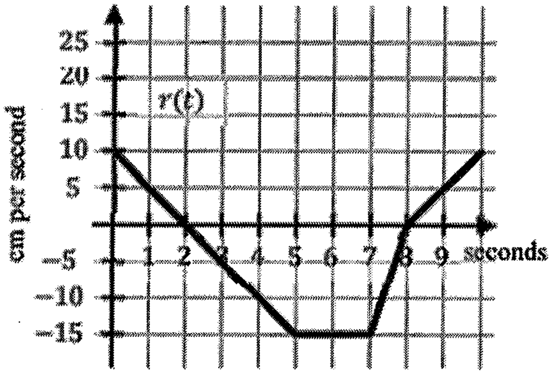
Units for Area Under the Curve:

The dependent unit multiplied by the independent unit. In other words, the unit for times the unit for .

2. The graph below represents the rate at which water is leaking out of a tank. The units are given in the graph. How much water has leaked out of the tank after 9 minutes?



3. A particle is moving along the x-axis at a rate modeled by $r(t)$ and shown in the graph below.

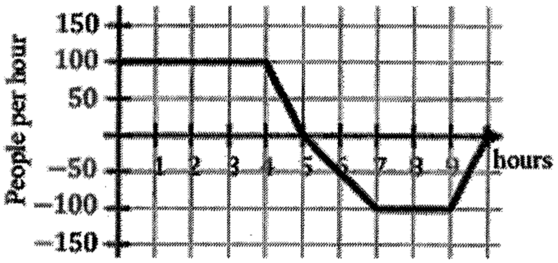


a. How far is the particle from its starting position after 10 seconds?

b. How far is the particle from its position at $t = 2$ after $t = 8$ seconds?

Practice:

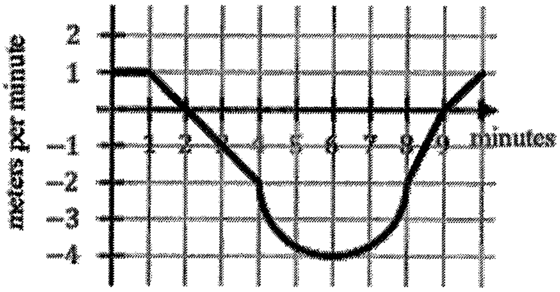
1. The graph below shows the rate of change for the number of people in a museum t hours after it opens.



a. How many people are in the museum after 5 hours?

b. How many people are in the museum after 10 hours?

2. The graph below shows the velocity of a particle moving along the x-axis, measured in meters per minute. At $t = 0$ minutes, the particle is at the origin.



a. Where is the particle after two minutes?

b. Where is the particle after six minutes?

c. Where is the particle after ten minutes?

3

6.1 Assess Your Understanding

Concepts and Vocabulary

1. Explain how rectangles can be used to approximate the area of the region bounded by the graph of a function $y = f(x) \geq 0$, the x-axis, and the lines $x = a$ and $x = b$.

2. *True or False* When a closed interval $[a, b]$ is partitioned into n subintervals each of the same width, the width of each subinterval is $\frac{a+b}{n}$.

3. If the closed interval $[-2, 4]$ is partitioned into 12 subintervals, each of the same width, then the width of each subinterval is ________.

4. *True or False* If the area A under the graph of a function f that is continuous and nonnegative on a closed interval $[a, b]$ is approximated using upper sums S_n, then $S_n \geq A$ and $A = \lim_{n\to\infty} S_n$.

Skill Building

PAGE 391 5. Approximate the area A of the region bounded by the graph of $f(x) = \frac{1}{2}x + 3$, the x-axis, and the lines $x = 2$ and $x = 4$ by dividing the closed interval $[2, 4]$ into four subintervals:

$$\left[2, \frac{5}{2}\right], \left[\frac{5}{2}, 3\right], \left[3, \frac{7}{2}\right], \left[\frac{7}{2}, 4\right]$$

(a) Using the left endpoint of each subinterval, draw four small rectangles that lie below the graph of f and sum the areas of the four rectangles.

(b) Using the right endpoint of each subinterval, draw four small rectangles that extend above the graph of f and sum the areas of the four rectangles.

(c) Compare the answers from parts (a) and (b) to the exact area $A = 9$ and to the estimates obtained in Example 1.

6. Approximate the area A of the region bounded by the graph of $f(x) = 6 - 2x$, the x-axis, and the lines $x = 1$ and $x = 3$ by dividing the closed interval $[1, 3]$ into four subintervals:

$$\left[1, \frac{3}{2}\right], \left[\frac{3}{2}, 2\right], \left[2, \frac{5}{2}\right], \left[\frac{5}{2}, 3\right]$$

(a) Using the right endpoint of each subinterval, draw four small rectangles that lie below the graph of f and sum the areas of the four rectangles.

(b) Using the left endpoint of each subinterval, draw four small rectangles that extend above the graph of f and sum the areas of the four rectangles.

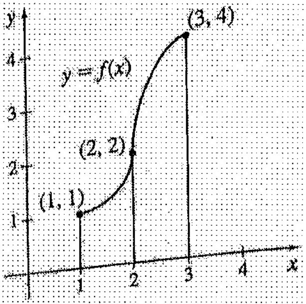
(c) Compare the answers from parts (a) and (b) to the exact area $A = 4$.

In Problems 7 and 8, refer to the graphs, (top, right). Approximate the shaded area under the graph of f:

(a) By constructing rectangles using the left endpoint of each subinterval.

(b) By constructing rectangles using the right endpoint of each subinterval.

7.



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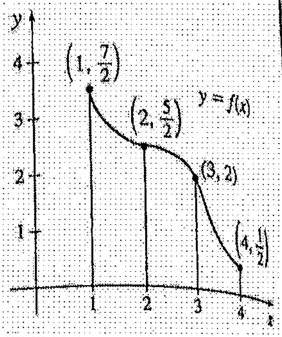


In Problems 9–12, partition each interval into n subintervals each of the same width.

9. $[1, 4]$ with $n = 3$

10. $[0, 9]$ with $n = 9$

11. $[-1, 4]$ with $n = 10$

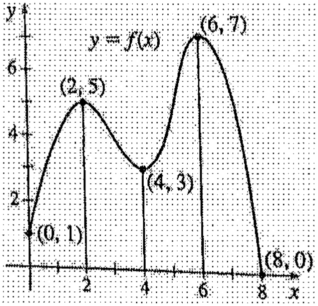
12. $[-4, 4]$ with $n = 16$

In Problems 13 and 14, refer to the graphs below. Using the indicated subintervals, approximate the shaded area:

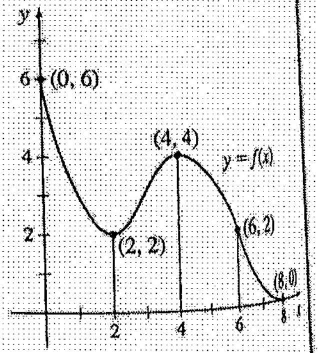
(a) By using lower sums s_n (rectangles that lie below the graph of f).

(b) By using upper sums S_n (rectangles that extend above the graph of f).

PAGE 392 13.



14.



15. **Area under a Graph**

(a) Graph $y = x$ and indicate the area under the graph from 0 to 3.

(b) Partition the interval $[0, 3]$ into n subintervals each of equal width.

(c) Show that $s_n = \sum_{i=1}^{n} (i-1)\left(\frac{3}{n}\right)^2$.

(d) Show that $S_n = \sum_{i=1}^{n} i\left(\frac{3}{n}\right)^2$.

(e) Show that $\lim_{n\to\infty} s_n = \lim_{n\to\infty} S_n = \frac{9}{2}$.

16. **Area under a Graph**

(a) Graph $y = 4x$ and indicate the area under the graph from 0 to 5.

(b) Partition the interval [0, 5] into n subintervals each of equal width.

(c) Show that $s_n = \sum_{i=1}^{n} (i-1)\frac{100}{n^2}$.

(d) Show that $S_n = \sum_{i=1}^{n} i\frac{100}{n^2}$.

(e) Show that $\lim_{n\to\infty} s_n = \lim_{n\to\infty} S_n = 50$.

In Problems 17–22, approximate the area A under the graph of each function f from a to b for $n = 4$ and $n = 8$ subintervals:

(a) By using lower sums s_n (rectangles that lie below the graph of f).

(b) By using upper sums S_n (rectangles that extend above the graph of f).

17. $f(x) = -x + 10$ on [0, 8]

18. $f(x) = 2x + 5$ on [2, 6]

19. $f(x) = 16 - x^2$ on [0, 4]

20. $f(x) = x^3$ on [0, 8]

21. $f(x) = \cos x$ on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

22. $f(x) = \sin x$ on $[0, \pi]$

23. Rework Example 3 (p. 393) by using lower sums s_n (rectangles that lie below the graph of f).

24. Rework Example 4 (p. 394) by using upper sums S_n (rectangles that extend above the graph of f).

In Problems 25–32, find the area A under the graph of f from a to b:

(a) By using lower sums s_n (rectangles that lie below the graph of f).

(b) By using upper sums S_n (rectangles that extend above the graph of f).

(c) Compare the work required in (a) and (b). Which is easier? Could you have predicted this?

25. $f(x) = 2x + 1$ from $a = 0$ to $b = 4$

26. $f(x) = 3x + 1$ from $a = 0$ to $b = 4$

27. $f(x) = 12 - 3x$ from $a = 0$ to $b = 4$

28. $f(x) = 5 - x$ from $a = 0$ to $b = 4$

29. $f(x) = 4x^2$ from $a = 0$ to $b = 2$

30. $f(x) = \frac{1}{2}x^2$ from $a = 0$ to $b = 3$

31. $f(x) = 4 - x^2$ from $a = 0$ to $b = 2$

32. $f(x) = 12 - x^2$ from $a = 0$ to $b = 3$

Applications and Extensions

In Problems 33–38, find the area under the graph of f from a to b. Partition the closed interval [a, b] into n subintervals

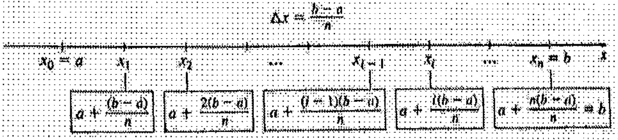
$$[x_0, x_1], [x_1, x_2], \ldots, [x_{i-1}, x_i], \ldots, [x_{n-1}, x_n],$$

where $a = x_0 < x_1 < \cdots < x_i < \cdots < x_{n-1} < x_n = b$, and each subinterval is of width $\Delta x = \frac{b-a}{n}$. As the figure below illustrates, the endpoints of each subinterval, written in terms of n, are

$$x_0 = a, \quad x_1 = a + \frac{b-a}{n}, \quad x_2 = a + 2\left(\frac{b-a}{n}\right), \ldots,$$

$$x_{i-1} = a + (i-1)\left(\frac{b-a}{n}\right), \quad x_i = a + i\left(\frac{b-a}{n}\right), \ldots,$$

$$x_n = a + n\left(\frac{b-a}{n}\right) = b$$



33. $f(x) = x + 3$ from $a = 1$ to $b = 3$

34. $f(x) = 3 - x$ from $a = 1$ to $b = 3$

35. $f(x) = 2x + 5$ from $a = -1$ to $b = 2$

36. $f(x) = 2 - 3x$ from $a = -2$ to $b = 0$

37. $f(x) = 2x^2 + 1$ from $a = 1$ to $b = 3$

38. $f(x) = 4 - x^2$ from $a = 1$ to $b = 2$

In Problems 39–42, approximate the area A under the graph of each function f by partitioning [a, b] into 20 subintervals of equal width and using an upper sum.

39. $f(x) = xe^x$ on [0, 8]

40. $f(x) = \ln x$ on [1, 3]

41. $f(x) = \frac{1}{x}$ on [1, 5]

42. $f(x) = \frac{1}{x^2}$ on [2, 6]

43. (a) Graph $y = \frac{4}{x}$ from $x = 1$ to $x = 4$ and shade the area under its graph.

(b) Partition the interval [1, 4] into n subintervals of equal width.

(c) Show that the lower sum s_n is

$$s_n = \sum_{i=1}^{n} \frac{4}{\left(1 + \frac{3i}{n}\right)}\left(\frac{3}{n}\right)$$

(d) Show that the upper sum S_n is

$$S_n = \sum_{i=1}^{n} \frac{4}{\left(1 + \frac{3(i-1)}{n}\right)}\left(\frac{3}{n}\right)$$

(e) Complete the following table:

n	5	10	50	100
s_n				
S_n				

(f) Use the table to give an upper and lower bound for the area.

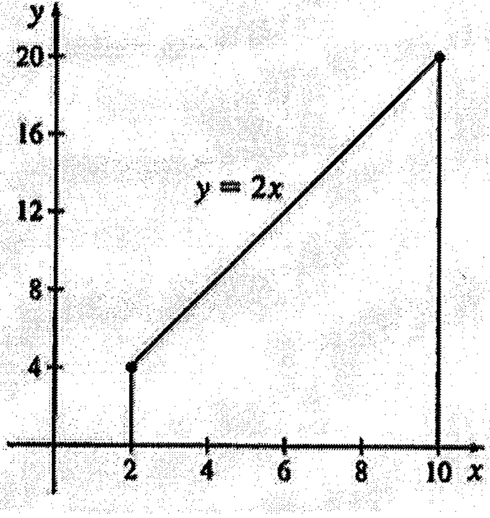
⑤

6.1 AP Practice Problems (p.398)

1. The approximate area under the graph of $f(x) = x^2 + 1$ from -1 to 3 found by partitioning the interval $[-1, 3]$ into 4 subintervals of equal width and using lower sum s_4 (rectangles that lie under the graph) is

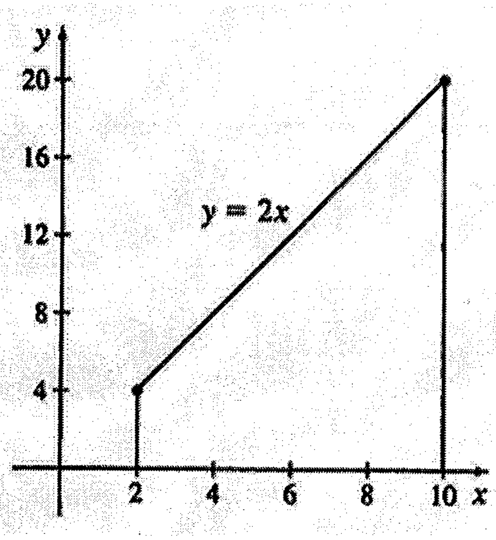
   (A) 8 (B) 9 (C) 10 (D) 18

2. The graph of the function $f(x) = 2x$ from 2 to 10 is shown below.



(a) Approximate the area under the graph of f by partitioning the interval [2, 10] into 4 subintervals of equal width and using lower sums s_4 (rectangles that lie under the graph of f).

2. The graph of the function $f(x) = 2x$ from 2 to 10 is shown below.



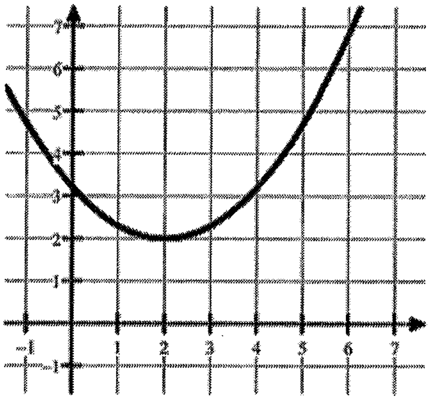
(b) Approximate the area under the graph of f by partitioning the interval [2, 10] into 4 subintervals of equal width and using upper sums S_4 (rectangles that extend above the graph of f).

(c) Find the exact area under the graph using geometry.

⑦

AP Calculus – 6.2a Notes – Approximating Area with Riemann Sums

The graph of the function $g(x)$ is shown to the right. Approximate the area under the curve on the interval $[2, 6]$ with n subintervals by using a left-rectangular approximation method.

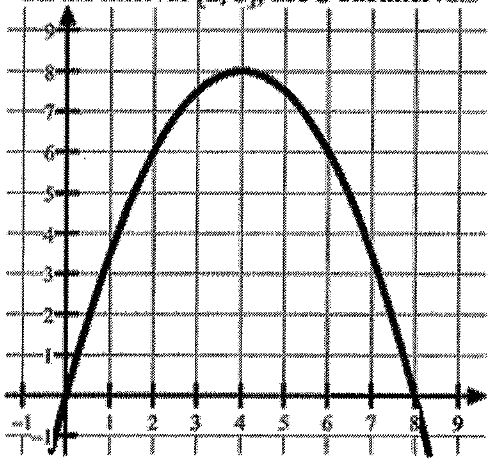


This approximation method is called a ________________. It was named after a German mathematician named Bernhard Riemann.

Below is the graph of $f(x) = 4x - \frac{1}{2}x^2$. Use Riemann Sums to find the approximation of the area under the curve.

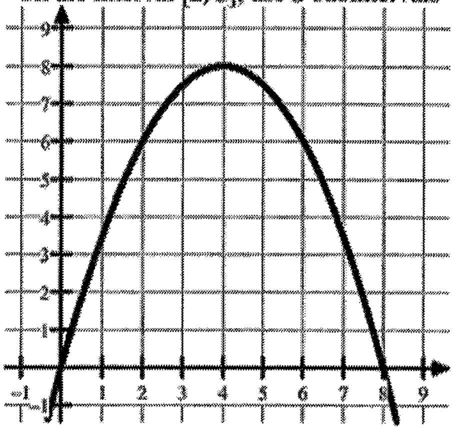
Left-Riemann Sum

On the interval $[2, 8]$, use 3 subintervals



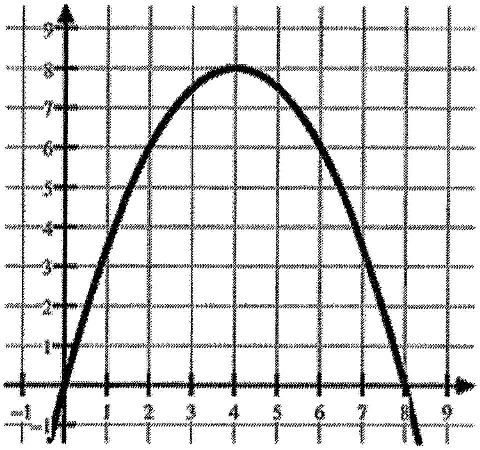
Right-Riemann Sum

On the interval $[2, 8]$, use 3 subintervals



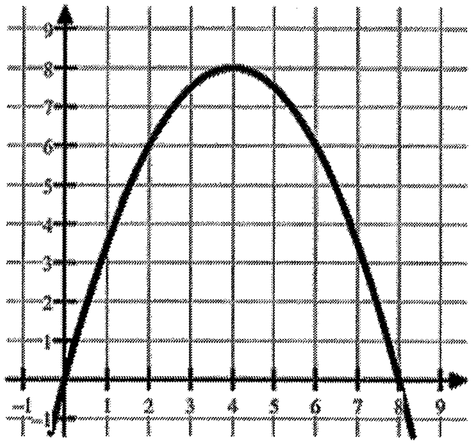
Midpoint-Riemann Sum

On the interval $[2, 8]$, use 3 subintervals



Trapezoidal Sum

On the interval $[2, 8]$, use 3 subintervals



⑧

Using Riemann Sums with a Table of Values

The rate at which water is being pumped into a tank is given by the continuous and increasing function $R(t)$. A table of selected values of $R(t)$, for the time interval $0 < t < 12$ minutes, is given below.

Time (minutes)	0	3	6	9	12
$R(t)$ (gallons/min)	7	13	18	23	27

Use the following Riemann sums (with the given intervals), to estimate the number of gallons of water pumped into the tank during the 12 minutes.

Right-Riemann sum with 4 subintervals

Is the approximation greater or less than the true value? Why?

Left-Riemann sum with 4 subintervals

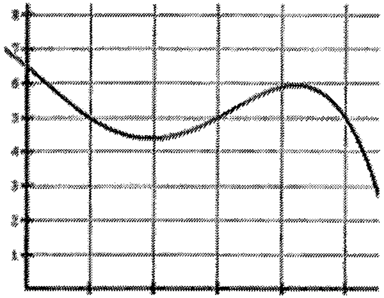
Is the approximation greater or less than the true value? Why?

Midpoint-Riemann sum with 2 subintervals

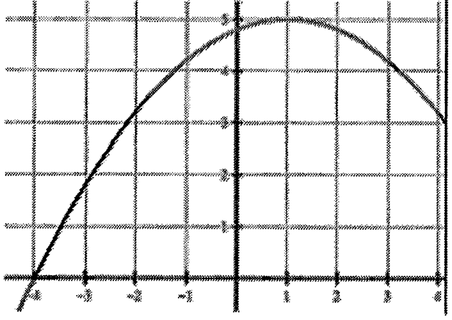
Trapezoidal sum with 4 subintervals

Sketch the following rectangular approximations. Find the width of each subinterval.

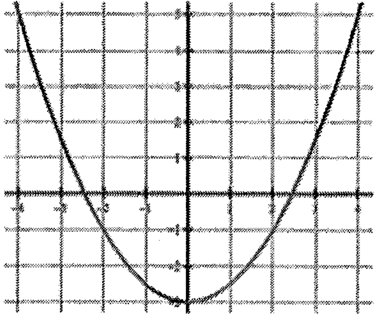
3. Midpoint on the interval [1,4]
with $n = 6$ subintervals
Width of each subinterval =



4. Right Endpoint on [-2,2]
with $n = 5$ subintervals
Width of each subinterval =



5. Left Endpoint on [-2,4]
with $n = 10$ subintervals
Width of each subinterval =



AP Calculus – 6.2b Notes –Summation Notation and Limit Definition of Area

I. Sigma Notation

____

$$\sum_{i=2}^{5} a_i = a_2 + a_3 + a_4 + a_5$$

____

________ ________

Ex. 1 $\sum_{i=2}^{4} i^2 + 1 =$

II. Summation Formulas:

1) $\sum_{i=1}^{n} 1 = n$

2) $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

3) $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

4) $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

5) $\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$

Example 2

$$\sum_{i=1}^{8}(3i^2 + 2) =$$

Example 3

$$\sum_{i=1}^{10}(i + 2)^2 =$$

Example 4

$$\sum_{k=1}^{n}\frac{1}{n}(k^2 - 1) =$$

III. Limits as *n* approaches infinity

*Think back about finding horizontal asymptotes

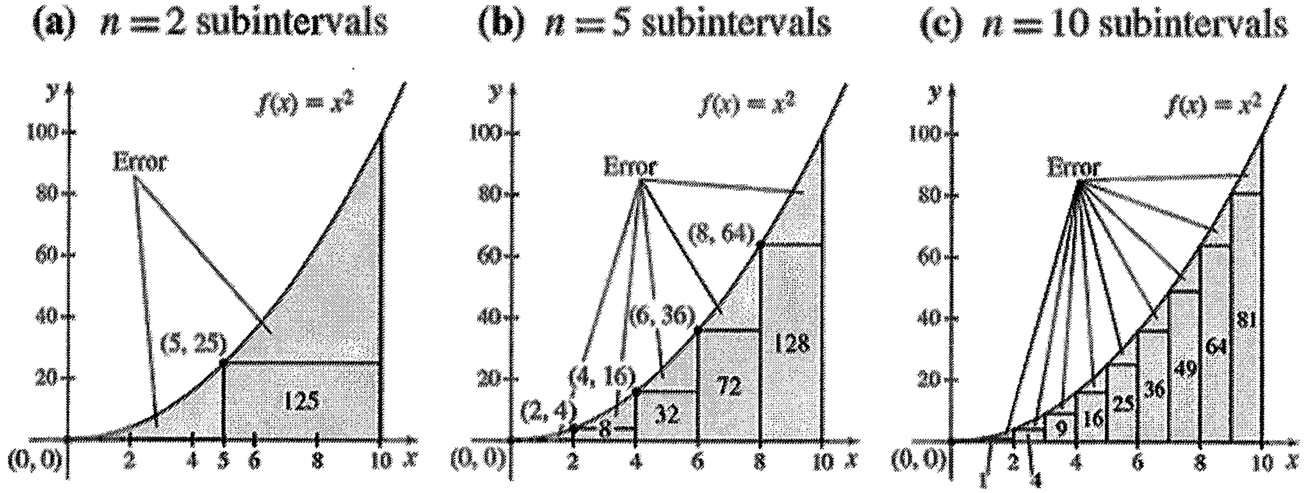
Example 5: $If\ S(n) = \frac{1}{n^2}\left[\frac{n(n+1)}{2}\right]$, $then\ find\ \lim_{n\to\infty} S(n)$

Example 6: *Find* $\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}\left(\frac{2i}{n}\right)\left(\frac{2}{n}\right)$

Example 7: *Find* $\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}\left(1+\frac{2i}{n}\right)^2\left(\frac{2}{n}\right)$

Moving from Area Approximation to Exact Area calculations:

We can continually improve the Area Approximation under the curve by increasing the number of rectangles (n = 2 to n = 5 to n = 10). If we let ***n*** go out to infinity (using limits), we will have something better than an approximation, we will achieve the actual area under the curve:



(a) $n = 2$ **subintervals** **(b)** $n = 5$ **subintervals** **(c)** $n = 10$ **subintervals**

(a) Two subintervals (b) Five subintervals (c) Ten subintervals

If $n \to \infty$ on the interval $[a, b]$, what does the width of each subinterval (rectangle) approach?

Finding Area Using Limit Definition:

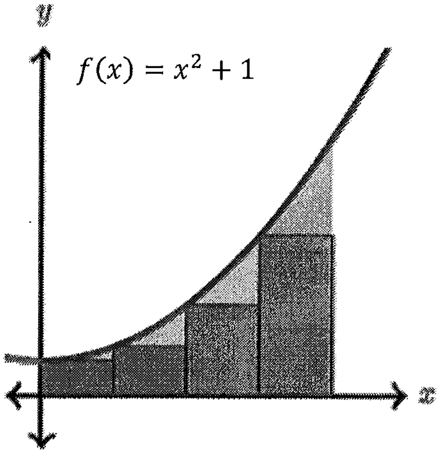
$$\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{b-a}{n}\right)*f\left(a+\frac{b-a}{n}*i\right)$$

$$\text{width} = \frac{b-a}{n}$$

Definite Integral Notation:

The area under the curve of $f(x)$ on the interval $[a,b]$ is represented by $\int_a^b f(x)\,dx$.

8a) Approximate the area under the curve $f(x) = x^2 + 1$ using 4 rectangles on interval [0,2]



b) Find the exact area under the curve $f(x) = x^2 + 1$ using limit definition of area on the interval [0,2]

Practice problems:

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{b-a}{n}\right)*f\left(a+\frac{b-a}{n}*i\right)$$

width $=\frac{b-a}{n}$

Examples:

1. Rewrite the definite integral using summation notation.

$$\int_{2}^{6}(x^2-3)\,dx=$$

2. Rewrite the summation notation expression as a definite integral.

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{6}{n}\right)\left(4+\frac{6k}{n}\right)^2=$$

a.

b.

c.

3. $\lim_{n\to\infty}\frac{2}{n}\left(\frac{1}{\frac{2}{n}+3}+\frac{1}{\frac{4}{n}+3}+\frac{1}{\frac{6}{n}+3}+\cdots+\frac{1}{\frac{2n}{n}+3}\right)$

Assuming the lower limit "a" is 0, write a definite integral that represents the above expression.

4. The expression $\frac{1}{10}\left(\cos\left(\frac{1}{10}\right)+\cos\left(\frac{2}{10}\right)+\cos\left(\frac{3}{10}\right)+\cdots+\cos\left(\frac{10}{10}\right)\right)$ is a Riemann sum approximation for what definite integral?

Where is the 10? Why isn't it written in the integral?

⑬

(a) Find a Riemann sum for v on the interval $[1, 8]$.

(b) What are the units of the Riemann sum?

(c) Interpret the Riemann sum in the context of the problem.

Solution

(a) Partition the interval $[1, 8]$ into four subintervals $[1, 2.5]$, $[2.5, 3]$, $[3, 5]$, and $[5, 8]$. On each subinterval choose u_i, $i = 1, 2, 3, 4$ as the left endpoint of the ith subinterval. (Alternatively the right endpoints or any combination of left and right endpoints could be chosen.) The Riemann sum of v over the interval $[1, 8]$ is

$$\begin{aligned}\sum_{i=1}^{4} v(u_i)\Delta t_i &= v(1)\Delta t_1 + v(2.5)\Delta t_2 + v(3)\Delta t_3 + v(5)\Delta t_4\\ &= 6(2.5-1) + 0(3-2.5) + (-4)(5-3) + 2(8-5)\\ &= (6)(1.5) + (0)(0.5) + (-4)(2) + (2)(3) = 7\end{aligned}$$

(b) Each term in the Riemann sum is the product of the velocity (in m/s) and the time (in s) so the units of the Riemann sum are meters.

(c) The Riemann sum approximates the **net displacement** of the object over the interval $[1, 8]$. ■

NOW WORK Problem 63 and AP® Practice Problem 5.

6.2 Assess Your Understanding

Concepts and Vocabulary

1. *True or False* In a Riemann sum $\sum_{i=1}^{n} f(u_i)\Delta x_i$, u_i is always the left endpoint, the right endpoint, or the midpoint of the ith subinterval, $i = 1, 2, \ldots, n$.

2. *Multiple Choice* In a regular partition of $[0, 40]$ into 20 subintervals, $\Delta x =$ [(a) 20 (b) 0.5 (c) 2 (d) 4].

3. *True or False* A function f defined on the closed interval $[a, b]$ has an infinite number of Riemann sums.

4. In the notation for a definite integral $\int_a^b f(x)\,dx$, a is called the ________; b is called the ________; $\int$ is called the ________; and $f(x)$ is called the ________.

5. If $f(a)$ is defined, $\int_a^a f(x)\,dx =$ ________

6. *True or False* If a function f is integrable over a closed interval $[a, b]$, then $\int_a^b f(x)\,dx = \int_b^a f(x)\,dx$.

7. *True or False* If a function f is continuous on a closed interval $[a, b]$, then the definite integral $\int_a^b f(x)\,dx$ exists.

8. *Multiple Choice* Since $\int_0^2 (3x-8)\,dx = -10$, then $\int_2^0 (3x-8)\,dx =$ [(a) -10 (b) 10 (c) 5 (d) 0].

Skill Building

In Problems 9 and 10, find the Riemann sum for $f(x) = x$, $0 \le x \le 2$, for the partition and the numbers u_i given below.

PAGE 400 9. Partition: $\left[0, \frac{1}{4}\right], \left[\frac{1}{4}, \frac{1}{2}\right], \left[\frac{1}{2}, \frac{3}{4}\right], \left[\frac{3}{4}, 1\right], [1, 2]$

$u_1 = \frac{1}{8}, u_2 = \frac{3}{8}, u_3 = \frac{5}{8}, u_4 = \frac{7}{8}, u_5 = \frac{9}{8}$

10. Partition: $\left[0, \frac{1}{2}\right], \left[\frac{1}{2}, 1\right], \left[1, \frac{3}{2}\right], \left[\frac{3}{2}, 2\right]$

$u_1 = \frac{1}{2}, u_2 = 1, u_3 = \frac{3}{2}, u_4 = 2$

In Problems 11 and 12, find the Riemann sum for $f(x) = 3x + 1$, $0 \le x \le 8$, by partitioning the interval $[0, 8]$ into four subintervals, each of the same width.

11. Choose u_i as the left endpoint of each subinterval.

12. Choose u_i as the right endpoint of each subinterval.

In Problems 13–18, find each definite integral.

PAGE 406 13. $\int_{-3}^{4} e\,dx$

14. $\int_0^3 (-\pi)\,dx$

15. $\int_3^0 (-\pi)\,dt$

16. $\int_7^2 2\,ds$

PAGE 402 17. $\int_4^4 2\theta\,d\theta$

18. $\int_{-1}^{-1} 8\,dr$

In Problems 19–26, write the limit of the Riemann sums as a definite integral. Here u_i is in the subinterval $[x_{i-1}, x_i]$, $i = 1, 2, \ldots n$ and $\Delta x_i = x_i - x_{i-1}$, $i = 1, 2, \ldots n$. Assume each limit exists.

PAGE 401 19. $\lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} (e^{u_i} + 2)\Delta x_i$ on $[0, 2]$

20. $\lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} \ln u_i \,\Delta x_i$ on $[1, 8]$

21. $\lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} \cos u_i \,\Delta x_i$ on $[0, 2\pi]$

22. $\lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} (\cos u_i + \sin u_i)\,\Delta x_i$ on $[0, \pi]$

23. $\lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} \frac{2}{u_i^2} \Delta x_i$ on $[1, 4]$

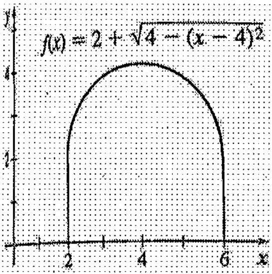
24. $\lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} u_i^{1/3} \Delta x_i$ on $[0, 8]$

25. $\lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} u_i \ln u_i \, \Delta x_i$ on $[1, e]$

26. $\lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} \ln(u_i + 1) \Delta x_i$ on $[0, e]$

In Problems 27–30, the graph of a function is shown. Express the shaded area as a definite integral.

27.



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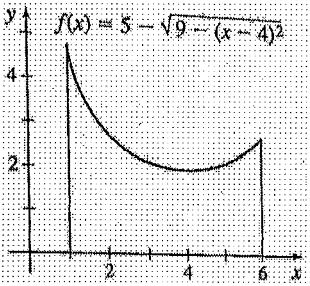

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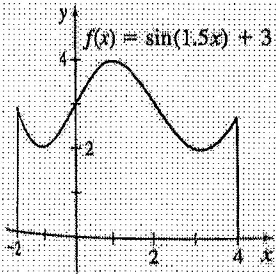

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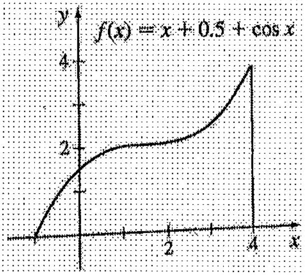


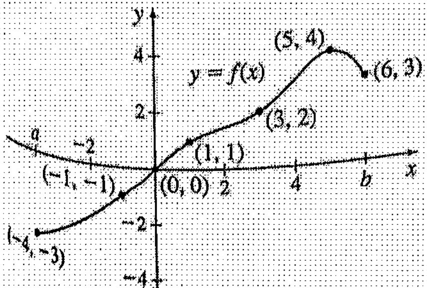
In Problems 31 and 32, the graph of a function f defined on an interval $[a, b]$ is given.

(a) Partition the interval $[a, b]$ into six subintervals (not necessarily of the same width) using the points shown on each graph.

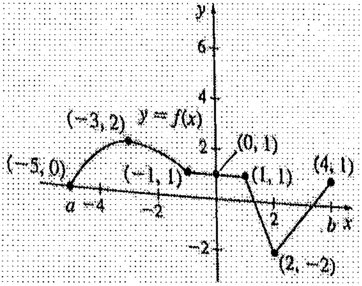
(b) Using Riemann sums, approximate $\int_a^b f(x)\,dx$ by choosing u_i as the left endpoint of each subinterval.

(c) Using Riemann sums, approximate $\int_a^b f(x)\,dx$ by choosing u_i as the right endpoint of each subinterval.

31.



32.



In Problems 33–38, express each definite integral as the limit of Riemann sums.

PAGE 402 **33.** $\int_0^{\pi} \sin x\,dx$

34. $\int_{-\pi/4}^{\pi/4} \tan x\,dx$

35. $\int_1^4 (x-2)^{1/3}dx$

36. $\int_1^4 (x+2)^{1/3}dx$

37. $\int_1^4 (|x|-2)\,dx$

38. $\int_{-2}^4 |x|\,dx$

In Problems 39–46, approximate each definite integral $\int_a^b f(x)\,dx$ by

(a) Partitioning $[a, b]$ into four subintervals and choosing u_i as the left endpoint of the ith subinterval.

(b) Partitioning $[a, b]$ into eight subintervals and choosing u_i as the right endpoint of the ith subinterval.

(c) Use technology to find each integral.

PAGE 403 **39.** $\int_0^8 (x^2-4)\,dx$

40. $\int_1^9 (9-x^2)\,dx$

41. $\int_{-4}^4 (x^2-x)\,dx$

42. $\int_{-2}^6 (x^2-4x)\,dx$

43. $\int_0^{2\pi} \sin^2 x\,dx$

44. $\int_0^{2\pi} \cos^2 x\,dx$

45. $\int_0^8 e^x\,dx$

46. $\int_1^9 \ln x\,dx$

In Problems 47–50,

(a) Approximate each definite integral by completing the table of Riemann sums using a regular partition of $[a, b]$.

n	10	50	100
Using left endpoints			
Using right endpoints			
Using the midpoint			

(b) Use technology to find each definite integral.

47. $\int_1^5 (2+\sqrt{x})\,dx$

48. $\int_{-1}^3 (e^x+e^{-x})\,dx$

49. $\int_{-1}^1 \frac{3}{1+x^2}\,dx$

50. $\int_0^2 \frac{1}{\sqrt{x^2+4}}\,dx$

In Problems 51–54,

(a) Find each definite integral using Riemann sums.

(b) Determine whether the integral represents an area.

51. $\int_0^1 (x-4)\,dx$

52. $\int_0^3 (3x-1)\,dx$

53. $\int_0^2 (2x+1)\,dx$

54. $\int_0^4 (1-x)\,dx$

55. The interval $[-3, 5]$ is partitioned into eight subintervals each of the same width.

(a) What is the largest Riemann sum of $f(x) = 2x + 6$ that can be found using this partition?

(b) What is the smallest Riemann sum?

(c) Compute the average of these sums.

(d) What integral has been approximated, and what is the integral's exact value?

56. The interval $[2, 7]$ is partitioned into five subintervals, each of the same width.

(a) What is the largest Riemann sum of $g(x) = 28 - 4x$ that can be formed using this partition?

(b) What is the smallest Riemann sum?

(c) Find the average of these sums.

(d) What integral has been approximated, and what is the integral's exact value?

Applications and Extensions

In Problems 57–60, for the given function f:

(a) Graph f.

(b) Express the area under the graph of f as an integral.

(c) Evaluate the integral.

(d) Confirm the answer to (c) using geometry.

57. $f(x) = \sqrt{9 - x^2}, 0 \le x \le 3$

58. $f(x) = \sqrt{25 - x^2}, -5 \le x \le 5$

59. $f(x) = 3 - \sqrt{6x - x^2}, 0 \le x \le 6$

60. $f(x) = \sqrt{4x - x^2} + 2, 0 \le x \le 4$

61. Find an approximate value of $\int_1^2 \frac{1}{x}\,dx$ by finding Riemann sums corresponding to a partition of $[1, 2]$ into four subintervals, each of the same width, and evaluating the integrand at the midpoint of each subinterval. Compare your answer with the actual value, $\ln 2 = 0.6931\ldots$.

62. (a) Find the approximate value of $\int_0^2 \sqrt{4-x^2}\,dx$ by finding Riemann sums corresponding to a partition of $[0, 2]$ into 16 subintervals, each of the same width, and evaluating the integrand at the left endpoint of each subinterval.

(b) Can $\int_0^2 \sqrt{4-x^2}\,dx$ be interpreted as area? If it can, describe the area; if it cannot, explain why.

(c) Find the actual value of $\int_0^2 \sqrt{4-x^2}\,dx$ by graphing $y = \sqrt{4-x^2}$ and using a familiar formula from geometry.

63. An object in rectilinear motion has acceleration a. The table below shows a (in meters per second squared) of the object for select times t, in seconds.

t	1	2	3.5	5	7
$a(t)$	6	4	4	2	3

(a) Find a Left Riemann sum for a on the interval $[1, 7]$.

(b) What are the units of the Riemann sum?

(c) Interpret the Riemann sum in the context of the problem.

64. An object in rectilinear motion moves with velocity v. The table below shows v (in meters per second) for select times t, in seconds.

t	0	1	3.5	6	9
$v(t)$	4	5	9	6	5

(a) Find a Right Riemann sum for v on the interval $[0, 9]$.

(b) What are the units of the Riemann sum?

(c) Interpret the Riemann sum in the context of the problem.

65. The table below gives the rate of change of revenue R (in millions of dollars per year) of a manufacturing company for selected times t, in years.

t	1	2	3	4	5
$\frac{dR}{dt}$	6	10	12	10	14

(a) Find a Right Riemann sum for $\frac{dR}{dt}$ on the interval $[1, 5]$.

(b) What are the units of the Riemann sum?

(c) Interpret the Riemann sum in the context of the problem.

66. The table below shows the rate of change of volume V with respect to time t (in liters per minute) of a balloon for selected times t, in minutes.

t	1	2.5	3	4	5
$\frac{dV}{dt}$	6	5	4	3	1

(a) Find a Left Riemann sum for $\frac{dV}{dt}$ on the interval $[1, 5]$.

(b) What are the units of the Riemann sum?

(c) Interpret the Riemann sum in the context of the problem.

67. Units of an Integral In the definite integral $\int_0^5 F(x)\,dx$, F represents a force measured in newtons and x, $0 \le x \le 5$, is measured in meters. What are the units of $\int_0^5 F(x)\,dx$?

68. Units of an Integral In the definite integral $\int_0^{50} C(x)\,dx$, C represents the concentration of a drug in grams per liter and x, $0 \le x \le 50$, is measured in liters of alcohol. What are the units of $\int_0^{50} C(x)\,dx$?

69. Units of an Integral In the definite integral $\int_a^b v(t)\,dt$, v represents velocity measured in meters per second and time t is measured in seconds. What are the units of $\int_a^b v(t)\,dt$?

6.2 AP Practice Problems (p.411-412)

1. For the function $f(x) = 2 - 3x, 0 \le x \le 4$, the interval $[0, 4]$ is partitioned into four subintervals $[0, 1], [1, 2], [2, 3], [3, 4]$. The Right Riemann sum equals

   (A) -88 (B) -24 (C) -22 (D) -10

2. The Riemann sums for a function f on the interval $[1, 5]$ are given as $\sum_{i=1}^{n} (2 - u_i^2) \Delta x_i$ where $[1, 5]$ is partitioned into n subintervals $[x_{i-1}, x_i]$ of width Δx_i and u_i is some number in $[x_{i-1}, x_i]$. If $\lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} (2 - u_i^2) \Delta x_i$ exists, it equals

   (A) $\int_1^5 (2 - u_i^2) \Delta x_i$ (B) $\sum_{i=1}^{n} (2 - u_i^2) \Delta x_i$

   (C) $\int_1^5 (2 - x^2) dx$ (D) $\int_5^1 (2 - x^2) dx$

3. Approximate $\int_0^6 (2x - 5) dx$ by partitioning the interval $[0, 6]$ into three subintervals each of width 2 and using a Right Riemann sum.

   (A) -10 (B) 9 (C) 18 (D) 22

4. $\int_{-3}^{5}(3-x^2)dx =$

(A) $\lim\limits_{\Delta x_i \to 0}\sum\limits_{i=1}^{n}\left(3-u_i^2\right)\Delta x_i$

(B) $\lim\limits_{n \to \infty}\sum\limits_{i=1}^{n}(3-x^2)\frac{2}{n}$

(C) $\lim\limits_{n \to \infty}\sum\limits_{i=1}^{n}\left(3-u_i^2\right)\frac{8}{n}$

(D) $\lim\limits_{n \to \infty}\sum\limits_{i=1}^{n}\left(3-u_i^2\right)\frac{2}{n}$

5. A function f is continuous on the closed interval [0, 10] and has values

x	0	1	4	8	10
$f(x)$	4	5	10	12	8

Find an approximation to $\int_0^{10} f(x)\,dx$ using a Right Riemann sum with the four subintervals [0, 1], [1, 4], [4, 8], [8, 10].

(A) 70 (B) 99 (C) 83 (D) 62

6. $\lim\limits_{n \to \infty}\frac{1}{n}\left[2\left(\frac{1}{n}\right)^{2/3}+2\left(\frac{2}{n}\right)^{2/3}+2\left(\frac{3}{n}\right)^{2/3}+\cdots+2\left(\frac{n}{n}\right)^{2/3}\right] =$

(A) $\int_0^1 2x^{2/3}dx$ (B) $\int_0^2 x^{2/3}dx$

(C) $2\int_0^1\left(\frac{1}{x}\right)^{2/3}dx$ (D) $\frac{2}{n^{2/3}}\int_0^1 dx$

7. The expression

$$\frac{2}{25}\left[\sqrt{\frac{2}{25}}+\sqrt{\frac{4}{25}}+\sqrt{\frac{6}{25}}+\cdots+\sqrt{\frac{48}{25}}+\sqrt{\frac{50}{25}}\right]$$

is a Riemann sum approximation for

(A) $\frac{1}{25}\int_0^2\sqrt{x}dx$ (B) $\int_0^2\sqrt{x}dx$

(C) $\int_0^{50}\sqrt{x}dx$ (D) $\int_0^1\sqrt{2x}dx$

8. The integral $\int_{-2}^{4}(x^3-4)dx$ is approximated by partitioning the closed interval $[-2, 4]$ into three subintervals of equal width and using a Right Riemann sum. Which of the following is true?

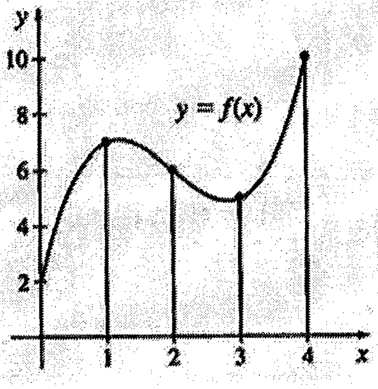
(A) The Right Riemann sum $= -24$;
it underestimates $\int_{-2}^{4}(x^3-4)dx$.

(B) The Right Riemann sum $= 60$;
it overestimates $\int_{-2}^{4}(x^3-4)dx$.

(C) The Right Riemann sum $= 120$;
it overestimates $\int_{-2}^{4}(x^3-4)dx$.

(D) The Right Riemann sum $= 120$; there is not enough information to determine whether it overestimates or underestimates $\int_{-2}^{4}(x^3-4)dx$.

9. The graph of $f(x) = (x-2)^3 - 2x + 10$ is shown below.



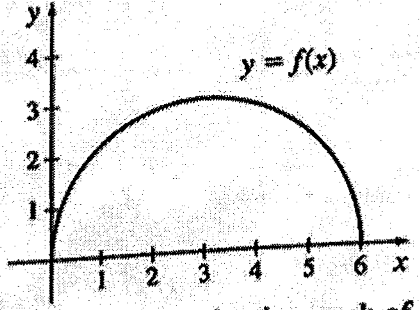
(a) Approximate the area under the graph of f using a Left Riemann sum with $n = 4$ subintervals of width 1.

(b) Express the area under the graph of f as a definite integral.

(c) Use technology to find the area under the graph of f.

⑲

10. The graph of $f(x) = \sqrt{6x - x^2}$ is shown below.



(a) Approximate the area under the graph of $f(x) = \sqrt{6x - x^2}$ using a Midpoint Riemann sum with three subintervals of equal width.

(b) Express the area under the graph of f as a definite integral.

(c) Evaluate the integral.

(d) Confirm the answer to (c) using geometry.

11. Express $\int_0^5 e^x\,dx$ as the limit of Riemann sums.

AP Calculus – 6.3 Notes –Antiderivative Power Rule, Theorems of Calculus, & Definite Integral

If $f(x) = x^2$*, what is* $f'(x)$*?*

Using Power Rule, $\frac{d}{dx}u^n = n * u^{n-1}$*, we know that* $f'(x) = 2x$

To put the steps for Derivatives Power rule into words:

1) Bring exponent down in front of variable and ______________

2) ______________ exponent by 1

If $f'(x) = 2x$*, what steps can we take to find* $f(x)$*?*

We can "undo" the previous derivative steps:

1) ______________ 1 to the exponent

2) ______________ by the new exponent

__Power Rule for Integration:__

$$\int u^n du = \frac{u^{n+1}}{n+1} + C$$

__Antidifferentiation Notation:__

NOTATION: $\int 2x\,dx = x^2 + C$

Integral integrand identifies the independent variable constant of integration

Consider the below functions:

$f(x) = x^2 + 5$
$f(x) = x^2 - 13$
$f(x) = x^2 + 126$

Since we can add a constant to any of these functions and still result in the same derivative, the **antiderivative** of a function will be in the form of $f(x) + C$ to show the family of functions that share the same derivative.

The process of integration is called **antidifferentiation** or taking the indefinite integral.

The indefinite integral results in a function.

The definite integral results in a number.

Recall Derivative Power Rule Conditions:

1) Rewrite expression as rational exponent **2)** All variables in numerator **3)** Expand terms (remove parentheses)

(21)

Integration Formulas

1. $\int u^n du = \frac{u^{n+1}}{n+1} + C$
2. $\int a\, dx = ax + C$
3. $\int \frac{1}{u}\, du = \ln|u| + C$

Important: The derivative and integral are <u>Inverse</u> operations of each other.

4) $\int f'(x)dx = f(x) + C$

5) $\frac{d}{dx}[\int f(x)\, dx] = f(x)$

Recall Derivative Power Rule Conditions:

1) Rewrite expression as rational exponent **2)** All variables in numerator **3)** Expand terms (remove parentheses)

Class Examples:

1. $\int 7x\, dx =$

2. $\int 7x^3 dx =$

3. $\int 2x + 3x^2 - 5x^4\, dx =$

4. $\int (3x-1)^2 dx =$

5. $\int \frac{x+1}{\sqrt{x}} dx =$

6. $\int \frac{3}{y\sqrt{y}} dy =$

7. $\int \frac{3\sqrt{x}(1-x)^2}{\sqrt[3]{x}} dx =$

22

AP Calculus – 6.3b Notes – Trig Integrals

Review Derivative Trig Rules:

1) $\frac{d}{dx}\sin u =$ 3) $\frac{d}{dx}\cos u =$

2) $\frac{d}{dx}\tan u =$ 4) $\frac{d}{dx}\cot u =$

5) $\frac{d}{dx}\sec u =$ 6) $\frac{d}{dx}\csc u =$

Integral Trig Rules:

1) $\int \sin u\, du =$ 2) $\int \cos u\, du =$

3) $\int \sec^2 u\, du =$ 4) $\int \csc^2 u\, du =$

5) $\int \sec u \tan u\, du =$ 6) $\int \csc u \cot u\, du =$

Example:

$\int (3\sec^2 x)\, dx$

$\int (x - 2\csc^2 x)\, dx$

$\int (3\csc x \cot x - 1)\, dx$

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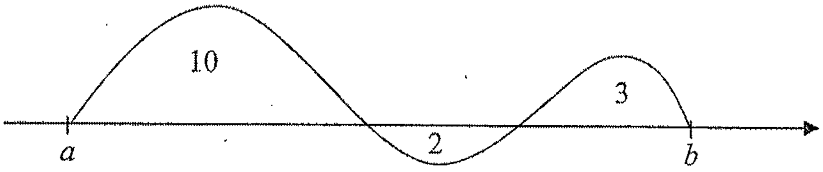
AP CALCULUS AB Integral Unit Additional Notes

Displacement is how far you are from where you started. displacement = $\int_a^b v(t)dt$ where $v(t)$ is the velocity function. (*displacement = integral of velocity*)

Distance is the total amount a particle has traveled. distance = $\int_a^b |v(t)|dt$.
(distance = integral of absolute value of velocity)

Ex) Suppose the graph below represents an object's velocity function. The numbers inside of each region represent the area of that region.



Based on this graph the object's displacement from *a* to *b* would be the integral of this function from *a* to *b* :

The object's distance traveled would be the integral of the absolute value of this function :

Position, Velocity, Acceleration

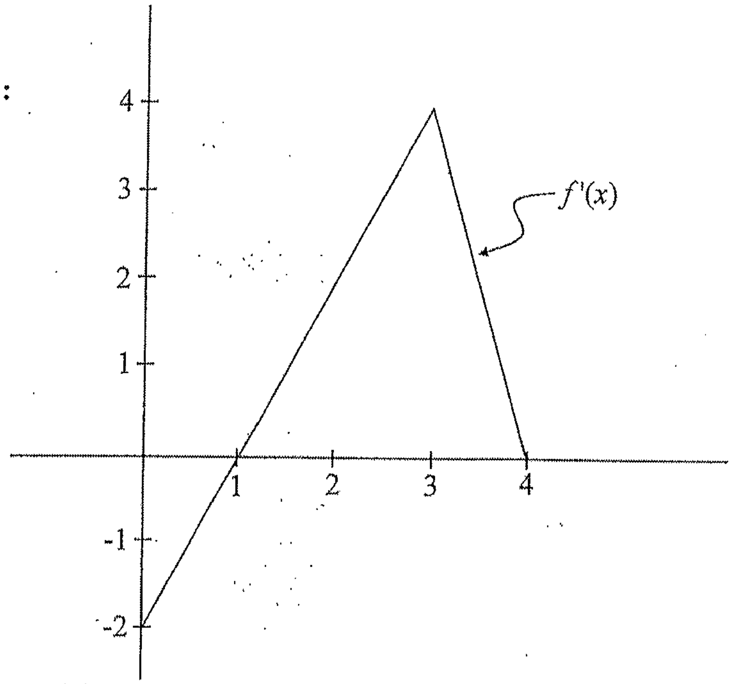
the derivative of position is velocity, the derivative of velocity is acceleration

So, the antiderivative of acceleration is velocity and the antiderivative of velocity is position

Acceleration s''(t)	+	+	-	-
Velocity s'(t)	+	-	+	-
Speed	increasing	decreasing	decreasing	increasing

Remember: Speed is increasing when s''(t) and s'(t) have the same signs
Speed is decreasing when s''(t) and s'(t) have opposite signs

Example 2:



The derivative of $f(x)$ is graphed above. Use this graph to answer the following:

a) Find the x-coordinates of all relative extrema. Justify your answer.	b) Find the x-coordinates of all points of inflection. Justify your answer.
c) If $f(0) = 1$, find $f(1), f(2), f(3)$, and $f(4)$.	d) State the absolute extrema and the x-values where they occur.
e) Find total distance traveled from $t = 0$ to $t = 4$	f) Find total displacement from $t = 0$ to $t = 4$
g) On the interval $1 < t < 3$, is speed increasing or decreasing? State reason:	h) On the interval $0 < t < 1$, is speed increasing or decreasing? State reason

i) Using the data from parts a, b, and c, sketch a graph of $f(x)$ below

(b) Now use $F(x) = -\cos x + 1$ to determine F at the required numbers.

$$F\left(\frac{\pi}{2}\right) = -\cos\frac{\pi}{2} + 1 = 1$$

$$F(\pi) = -\cos\pi + 1 = 1 + 1 = 2$$

$$F\left(\frac{3\pi}{2}\right) = -\cos\frac{3\pi}{2} + 1 = 0 + 1 = 1$$

$$F(2\pi) = -\cos(2\pi) + 1 = -1 + 1 = 0$$

(c) The sine function is positive over the interval $(0, \pi)$ and is negative over the interval $(\pi, 2\pi)$. The accumulation function F of $f(t) = \sin t$ increases from 0 to 1 to 2 as x increases from 0 to π and then decreases from 2 to 1 to 0 as x goes from π to 2π. The net accumulation over the interval $[0, 2\pi]$ equals 0. ■

NOW WORK Problem 45 and AP® Practice Problem 10.

6.3 Assess Your Understanding

Concepts and Vocabulary

1. According to Part 1 of the Fundamental Theorem of Calculus, if a function f is continuous on a closed interval $[a, b]$, then $\dfrac{d}{dx}\displaystyle\int_a^x f(t)\,dt =$ ________ for all numbers x in (a, b).
2. *True or False* By Part 2 of the Fundamental Theorem of Calculus, $\int_a^b x\,dx = b - a$.
3. *True or False* By Part 2 of the Fundamental Theorem of Calculus, $\int_a^b f(x)\,dx = f(b) - f(a)$.
4. *True or False* $\int_a^b F'(x)\,dx$ can be interpreted as the rate of change in F from a to b.

Skill Building

In Problems 5–18, find each derivative using Part 1 of the Fundamental Theorem of Calculus.

PAGE 413 **5.** $\dfrac{d}{dx}\displaystyle\int_1^x \sqrt{t^2+1}\,dt$

6. $\dfrac{d}{dx}\displaystyle\int_3^x \frac{t+1}{t}\,dt$

7. $\dfrac{d}{dt}\displaystyle\int_0^t (3+x^2)^{3/2}dx$

8. $\dfrac{d}{dx}\displaystyle\int_{-4}^x \left(t^3+8\right)^{1/3} dt$

9. $\dfrac{d}{dx}\displaystyle\int_1^x \ln u\,du$

10. $\dfrac{d}{dt}\displaystyle\int_4^t e^x dx$

PAGE 414 **11.** $\dfrac{d}{dx}\displaystyle\int_1^{2x^3} \sqrt{t^2+1}\,dt$

12. $\dfrac{d}{dx}\displaystyle\int_1^{\sqrt{x}} \sqrt{t^4+5}\,dt$

13. $\dfrac{d}{dx}\displaystyle\int_2^{x^5} \sec t\,dt$

14. $\dfrac{d}{dx}\displaystyle\int_3^{1/x} \sin^5 t\,dt$

PAGE 414 **15.** $\dfrac{d}{dx}\displaystyle\int_x^5 \sin(t^2)\,dt$

16. $\dfrac{d}{dx}\displaystyle\int_x^3 (t^2-5)^{10}\,dt$

17. $\dfrac{d}{dx}\displaystyle\int_{5x^2}^5 (6t)^{2/3}\,dt$

18. $\dfrac{d}{dx}\displaystyle\int_{x^2}^0 e^{10t}\,dt$

In Problems 19–36, use Part 2 of the Fundamental Theorem of Calculus to find each definite integral.

19. $\displaystyle\int_{-2}^3 dx$

20. $\displaystyle\int_{-2}^3 2\,dx$

21. $\displaystyle\int_{-1}^2 x^3 dx$

22. $\displaystyle\int_1^3 \frac{1}{x^3}dx$

23. $\displaystyle\int_0^1 \sqrt{u}\,du$

24. $\displaystyle\int_1^8 \sqrt[3]{y}\,dy$

PAGE 416 **25.** $\displaystyle\int_{\pi/6}^{\pi/2} \csc^2 x\,dx$

26. $\displaystyle\int_0^{\pi/2} \cos x\,dx$

27. $\displaystyle\int_0^{\pi/4} \sec x \tan x\,dx$

28. $\displaystyle\int_{\pi/6}^{\pi/2} \csc x \cot x\,dx$

29. $\displaystyle\int_{-1}^0 e^x dx$

30. $\displaystyle\int_{-1}^0 e^{-x} dx$

PAGE 416 **31.** $\displaystyle\int_1^e \frac{1}{x}\,dx$

32. $\displaystyle\int_e^1 \frac{1}{x}\,dx$

33. $\displaystyle\int_0^1 \frac{1}{1+x^2}\,dx$

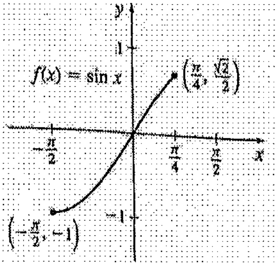
34. $\displaystyle\int_0^{\sqrt{2}/2} \frac{1}{\sqrt{1-x^2}}\,dx$

35. $\displaystyle\int_{-1}^8 x^{2/3}\,dx$

36. $\displaystyle\int_0^4 x^{3/2}\,dx$

In Problems 37–42, find $\int_a^b f(x)\,dx$ over the domain of f indicated in the graph.

37.



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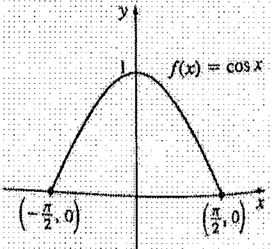

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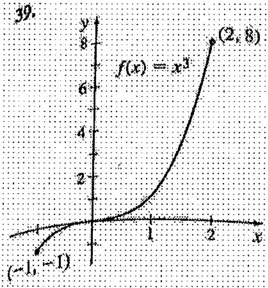

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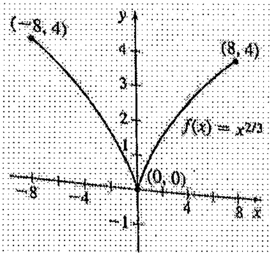
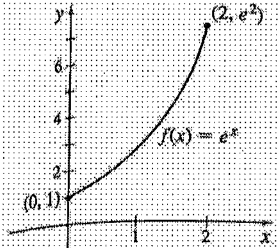
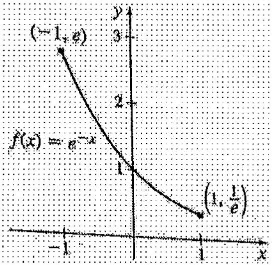


41.



42.



In Problems 43–50,

(a) Write the accumulation function F associated with f over the indicated interval as an integral.

(b) Find F at the given numbers.

(c) Graph F.

43. $f(t) = \cos t,\ 0 \le t \le \frac{4\pi}{3};\ F\left(\frac{\pi}{6}\right), F\left(\frac{\pi}{2}\right), F\left(\frac{4\pi}{3}\right)$

44. $f(t) = \sin(2t),\ 0 \le t \le \frac{3\pi}{2};\ F\left(\frac{\pi}{6}\right), F\left(\frac{\pi}{2}\right), F\left(\frac{3\pi}{2}\right)$

45. $f(t) = t^3,\ -2 \le t \le 3;\ F(-1), F(0), F(3)$

46. $f(t) = \sqrt[3]{t},\ -8 \le t \le 8;\ F(-1), F(0), F(8)$

47. $f(t) = 4 - t^2,\ -1 \le t \le 3;\ F(0), F(2), F(3)$

48. $f(t) = 8 - t^3,\ 0 \le t \le 4;\ F(1), F(2), F(4)$

49. $f(t) = \sqrt{t} - 2,\ 0 \le t \le 9;\ F(1), F(4), F(9)$

50. $f(t) = t^2 - 2t - 3,\ 0 \le t \le 5;\ F(1), F(3), F(5)$

Applications and Extensions

51. Given that $f(x) = (2x^3 - 3)^2$ and $f'(x) = 12x^2(2x^3 - 3)$, find $\int_0^2 [12x^2(2x^3 - 3)]\,dx$.

52. Given that $f(x) = (x^2 + 5)^3$ and $f'(x) = 6x(x^2 + 5)^2$, find $\int_{-1}^2 6x(x^2 + 5)^2\,dx$.

53. Area Find the area under the graph of $f(x) = \frac{1}{\sqrt{1 - x^2}}$ from 0 to $\frac{1}{2}$.

54. Area Find the area under the graph of $f(x) = \sec x \tan x$ from 0 to $\frac{\pi}{4}$.

55. Area Find the area under the graph of $f(x) = \frac{1}{x^2 + 1}$ from 0 to $\sqrt{3}$.

56. Area Find the area under the graph of $f(x) = \frac{1}{1 + x^2}$ from 0 to r, where $r > 0$. What happens as $r \to \infty$?

57. Area Find the area under the graph of $y = \frac{1}{\sqrt{x}}$ from $x = 1$ to $x = r$, where $r > 1$. Then examine the behavior of this area as $r \to \infty$.

58. Area Find the area under the graph of $y = \frac{1}{x^2}$ from $x = 1$ to $x = r$, where $r > 1$. Then examine the behavior of this area as $r \to \infty$.

59. Interpreting an Integral The function $R = R(t)$ models the rate of sales of a corporation measured in millions of dollars per year as a function of the time t in years. Interpret the integral $\int_0^2 R(t)\,dt = 23$ in the context of the problem.

60. Interpreting an Integral The function $v = v(t)$ models the speed v in meters per second of an object at time t in seconds. Interpret the integral $\int_0^{10} v(t)\,dt = 4.8$ in the context of the problem.

61. Interpreting an Integral Helium is leaking from a large advertising balloon at a rate of $H(t)$ cubic centimeters per minute, where t is measured in minutes.

(a) Write an integral that models the change in the amount of helium in the balloon over the interval $a \le t \le b$.

(b) What are the units of the integral from (a)?

(c) Interpret $\int_0^{300} H(t)\,dt = -100$ in the context of the problem.

62. Interpreting an Integral Water is being added to a reservoir at a rate of $w(t)$ kiloliters per hour, where t is measured in hours.

(a) Write an integral that models the change in the amount of water in the reservoir over the interval $a \le t \le b$.

(b) What are the units of the integral from (a)?

(c) Interpret $\int_0^{36} w(t)\,dt = 100$ in the context of the problem.

63. Population Growth The growth rate of a colony of bacteria is $B'(t) = 3.455(1.259)^t$ grams per hour, where t is the number of hours since time $t = 0$.

(a) Find $\int_0^6 B'(t)\,dt$.

(b) Interpret $\int_0^6 B'(t)\,dt$ in the context of the problem.

(c) If 15 grams of bacteria are present at time $t = 0$, how many grams of bacteria are present after 6 h?

64. Return on Investment An investment in a hedge fund is growing at a continuous rate of $A'(t) = 1105.17(1.105)^t$ dollars per year.

(a) Find $\int_0^{10} A'(t)\,dt$.

(b) Interpret $\int_0^{10} A'(t)\,dt$ in the context of the problem.

(c) If initially $1000 is invested, what is the value of the investment after 10 years?

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PAGE 419 **65. Increase in Revenue** The marginal revenue function $R'(x)$ for selling x units of a product is $R'(x) = \sqrt[3]{x}$ (in hundreds of dollars per unit).

(a) Interpret $\int_a^b R'(x)\,dx$ in the context of the problem.

(b) How much additional revenue is attained if sales increase from 40 to 50 units?

66. Increase in Cost The marginal cost function $C'(x)$ of producing x thousand units of a product is $C'(x) = 2x + 6$ thousand dollars.

(a) Interpret $\int_a^b C'(x)\,dx$ in the context of the problem.

(b) What does it cost the company to increase production from 10,000 units to 13,000 units?

67. Free Fall The speed v of an object dropped from rest is given by $v(t) = 9.8t$, where v is in meters per second and time t is in seconds.

(a) Express the distance traveled in the first 5.2 s as an integral.

(b) Find the distance traveled in 5.2 s.

68. Area Find h so that the area under the graph of $y^2 = x^3$, $0 \le x \le 4$, $y \ge 0$, is equal to the area of a rectangle of base 4 and height h.

69. Area If P is a polynomial that is positive for $x > 0$, and for each $k > 0$, the area under the graph of P from $x = 0$ to $x = k$ is $k^3 + 3k^2 + 6k$, find P.

70. Put It Together If $f(x) = \int_0^x \frac{1}{\sqrt{t^3+2}}\,dt$, which of the following is *false*?

(a) f is continuous at x for all $x \ge 0$.

(b) $f(1) > 0$

(c) $f(0) = \frac{1}{\sqrt{2}}$

(d) $f'(1) = \frac{1}{\sqrt{3}}$

In Problems 71–74:

(a) Use Part 2 of the Fundamental Theorem of Calculus to find each definite integral.

(b) Determine whether the integrand is an even function, an odd function, or neither.

(c) Can you make a conjecture about the definite integrals in (a) based on the analysis from (b)?

71. $\int_0^4 x^2 dx$ and $\int_{-4}^4 x^2 dx$

72. $\int_0^4 x^3 dx$ and $\int_{-4}^4 x^3 dx$

73. $\int_0^{\pi/4} \sec^2 x\,dx$ and $\int_{-\pi/4}^{\pi/4} \sec^2 x\,dx$

74. $\int_0^{\pi/4} \sin x\,dx$ and $\int_{-\pi/4}^{\pi/4} \sin x\,dx$

75. Area Find c, $0 < c < 1$, so that the area under the graph of $y = x^2$ from 0 to c equals the area under the same graph from c to 1.

76. Area Let A be the area under the graph of $y = \frac{1}{x}$ from $x = m$ to $x = 2m$, $m > 0$. Which of the following is true about the area A?

(a) A is independent of m.

(b) A increases as m increases.

(c) A decreases as m increases.

(d) A decreases as m increases when $m < \frac{1}{2}$ and increases as m increases when $m > \frac{1}{2}$.

(e) A increases as m increases when $m < \frac{1}{2}$ and decreases as m increases when $m > \frac{1}{2}$.

77. Put It Together If F is a function whose derivative is continuous for all real x, find

$$\lim_{h\to 0} \frac{1}{h}\int_c^{c+h} F'(x)\,dx$$

78. Suppose the closed interval $\left[0, \frac{\pi}{2}\right]$ is partitioned into n subintervals, each of length Δx, and u_i is an arbitrary number in the subinterval $[x_{i-1}, x_i]$, $i = 1, 2, \ldots, n$. Explain why

$$\lim_{n\to\infty} \sum_{i=1}^{n} [(\cos u_i)\,\Delta x] = 1$$

79. The interval $[0, 4]$ is partitioned into n subintervals, each of width Δx, and a number u_i is chosen in the subinterval $[x_{i-1}, x_i]$, $i = 1, 2, \ldots, n$. Find

$$\lim_{n\to\infty} \sum_{i=1}^{n} (e^{u_i}\,\Delta x)$$

80. If u and v are differentiable functions and f is a continuous function, find a formula for

$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)\,dt$$

81. Suppose that the graph of $y = f(x)$ contains the points $(0, 1)$ and $(2, 5)$. Find $\int_0^2 f'(x)\,dx$. (Assume that f' is continuous.)

82. If f' is continuous on the interval $[a, b]$, show that

$$\int_a^b f(x)f'(x)\,dx = \frac{1}{2}\left\{[f(b)]^2 - [f(a)]^2\right\}$$

Hint: Look at the derivative of $F(x) = \frac{[f(x)]^2}{2}$.

83. If f'' is continuous on the interval $[a, b]$, show that

$$\int_a^b xf''(x)\,dx = bf'(b) - af'(a) - f(b) + f(a)$$

Hint: Look at the derivative of $F(x) = xf'(x) - f(x)$.

Challenge Problems

84. What conditions on f and f' guarantee that $f(x) = \int_0^x f'(t)\,dt$?

6.3 AP Practice Problems (p.423-424)

1. $\int_0^{\pi/4} \sec^2 x\, dx =$

(A) $-\frac{1}{2}$ (B) $\sqrt{2}$ (C) 1 (D) $\frac{1}{2}$

2. If $F(x) = \int_0^x \sin t\, dt$ then $F'\left(\frac{\pi}{2}\right)$ is

(A) 0 (B) $\frac{1}{2}$ (C) 1 (D) -1

3. $\int_0^x e^t dt =$

(A) e^x (B) $e^x - e$ (C) $e^x - 1$ (D) $\frac{e^x - 1}{\ln|x|}$

4. $\int_1^e \frac{1}{x^3} dx =$

(A) $\frac{e-1}{2e}$ (B) $\frac{e^2-1}{2}$ (C) $\frac{1-e^2}{2}$ (D) $\frac{e^2-1}{2e^2}$

5. If P is a polynomial function of degree n, what is the degree of the function $Q(x) = \int_0^x P(t)\, dt$?

(A) $n-1$ (B) n (C) $n+1$ (D) $2n$

6. If $F(x) = \int_1^{x^3+1} \sqrt{t^2-1}\, dt$, then $F'(x)$ equals

(A) $\sqrt{(x^3+1)^2-1}$ (B) $3x^2\sqrt{x^3}$

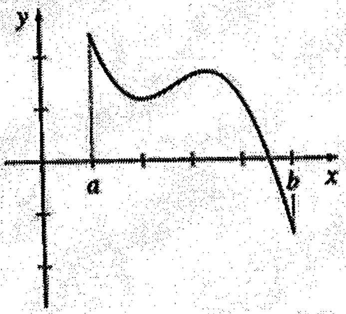
(C) x^3 (D) $3x^2\sqrt{(x^3+1)^2-1}$

29

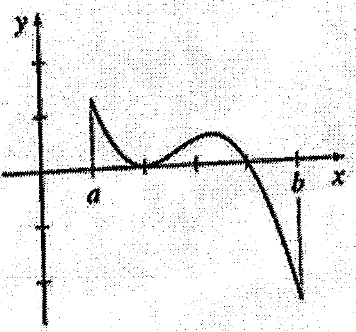
7. If $x > 1$, then $\dfrac{d}{dx}\displaystyle\int_{e}^{x} \frac{1}{t}\,dt =$

(A) $\dfrac{1}{x}$ (B) $\ln x$ (C) $\ln x - 1$ (D) $\dfrac{1}{x} - \dfrac{1}{e}$

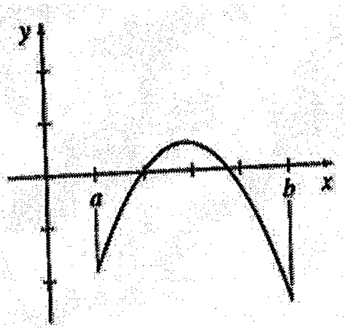
8. The graph of a function $y = g(x)$, where $a \leq x \leq b$, is shown below.



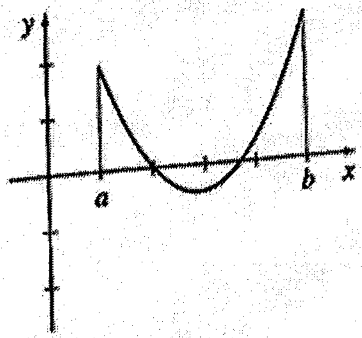
If $g(x) = \int_{a}^{x} f(t)\,dt$, then which of the following could be the graph of f on $[a, b]$?



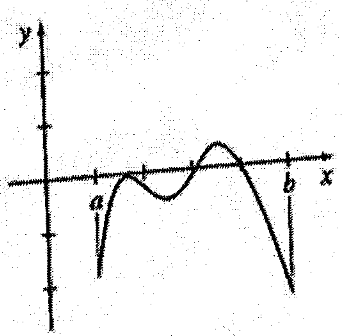
(A)



(B)



(C)



(D)

9. If $f(x) = \int_1^{3x^2} \sqrt{t^2+1}\,dt$, then $f'(-2)$ equals

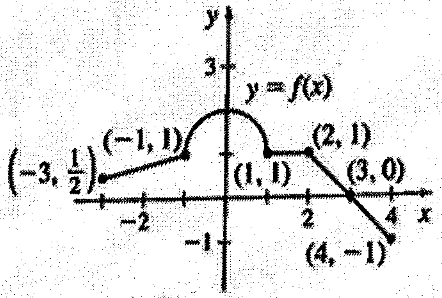
(A) $-12\sqrt{145}$ (B) $-12\sqrt{37}$

(C) $12\sqrt{37}$ (D) $\sqrt{145}$

10. The graph of a function f is shown below. If

$$g(t) = \int_{-3}^{t} f(x)\,dx$$

for what number t is $g(t)$ the greatest?



(A) 0 (B) 2 (C) 3 (D) 4

31

11. The area under the graph of $f(x) = \sin x$ from 0 to k, $0 \leq k \leq \frac{\pi}{2}$, is 0.2. Then k equals

(A) 1.2 (B) 0.644 (C) 0.356 (D) 0.314

12. Suppose $g(x) = \int_0^x \sin\left(t - \frac{\pi}{2}\right) dt$ for $0 \leq x \leq \frac{3\pi}{2}$.

On which interval is g increasing?

(A) $\left[0, \frac{\pi}{2}\right]$ (B) $[0, \pi]$

(C) $\left[\frac{\pi}{2}, \pi\right]$ (D) $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$

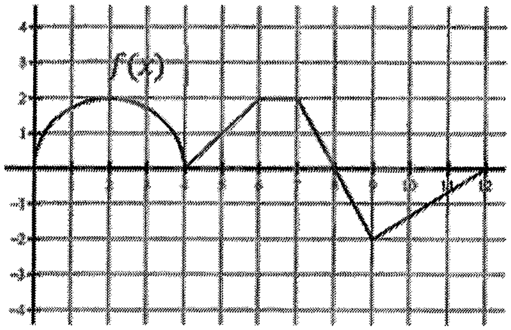
13. A biologist is growing bacteria for a lab experiment. There are 10 mg of bacteria in a controlled environment when she changes the temperature. The amount P of bacteria then grows at a rate of $20(0.95)^t$ per hour, where t is the number of hours since the temperature changed.

(a) Write an integral that models the amount of bacteria at time $t \geq 0$.

(b) Find the amount after 24 hours.

AP Calculus – 6.4 Notes Properties of Definite Integral

The area under the curve of $f(x)$ on the interval $[a, b]$ is represented by $\int_a^b f(x)\,dx$.



1. $\int_0^{12} f(x)\,dx =$

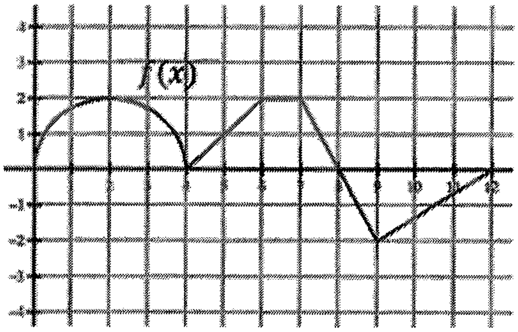
2. $\int_{12}^{0} f(x)\,dx =$

Properties of Definite Integrals

Equivalent Limits	Reversal of Limits
$\int_a^a f(x)\,dx =$	$\int_b^a f(x)\,dx =$

Multiply by constant (k = constant)	Adjacent Intervals ($a < c < b$)
$\int_a^b kf(x)\,dx =$	$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx =$

Addition	Subtraction
$\int_a^b [f(x) + g(x)]\,dx =$	$\int_a^b [f(x) - g(x)]\,dx =$



3. $\int_7^6 f(x)\,dx =$

4. $\int_{12}^{8} 3f(x)\,dx =$

5. Given that $\int_{-2}^{1} f(x)\,dx = 4$, $\int_1^5 f(x)\,dx = -3$, and $\int_{-2}^{1} g(x)\,dx = 8$, find the following.

a. $\int_5^1 f(x)\,dx$

b. $\int_{-2}^{5} f(x)\,dx$

c. $\int_{-2}^{1} [f(x) + 2g(x)]\,dx$

d. $\int_0^1 f(x)\,dx$

e. $\int_1^{-2} 3f(x)\,dx$

f. $\int_5^5 [f(x) - g(x)]\,dx$

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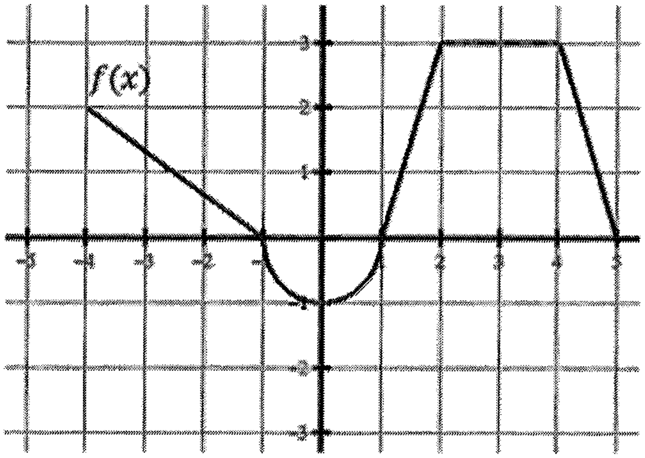
Practice:

The graph of f consists of line segments and a semicircle. Evaluate each definite integral.

1.



a. $\int_{-4}^{-1} f(x)\,dx =$

b. $\int_{2}^{1} f(x)\,dx =$

c. $\int_{1}^{5} 2f(x)\,dx =$

d. $\int_{-4}^{5} f(x)\,dx =$

e. $\int_{4}^{2} f(x)\,dx =$

f. $\int_{-4}^{1} |f(x)|\,dx =$

Let f and g be continuous functions that produce the following definite integral values.

$$\int_{-3}^{2} f(x)\,dx = 2 \qquad \int_{2}^{7} f(x)\,dx = -5 \qquad \int_{-3}^{2} g(x)\,dx = 6$$

Find the following.

5. $\int_{2}^{7} 2f(x)\,dx$

6. $4\int_{-3}^{2} f(x)\,dx$

7. $\int_{-3}^{7} f(x)\,dx$

8. $\int_{2}^{-3} g(x)\,dx$

9. $\int_{-3}^{2} [g(x) - f(x)]\,dx$

10. $\left|\int_{2}^{7} f(x)\,dx\right|$

11. $-\int_{7}^{2} f(x)\,dx$

34

Since $v(t)=3t^2-18t+15=3(t-1)(t-5)$ we find that $v(t)\geq 0$ if $0\leq t\leq 1$ or if $5\leq t\leq 6$ and $v(t)\leq 0$ if $1\leq t\leq 5$. Now express $|v(t)|$ as the piecewise-defined function

$$|v(t)|=\begin{cases} v(t)=3t^2-18t+15 & \text{if } 0\leq t\leq 1 \\ -v(t)=-(3t^2-18t+15) & \text{if } 1<t<5 \\ v(t)=3t^2-18t+15 & \text{if } 5\leq t\leq 6 \end{cases}$$

Then

$$\begin{aligned}\int_0^6 |v(t)|dt &= \int_0^1 v(t)\,dt+\int_1^5[-v(t)]\,dt+\int_5^6 v(t)\,dt\\ &= [t^3-9t^2+15t]_0^1-[t^3-9t^2+15t]_1^5+[t^3-9t^2+15t]_5^6\\ &= (7-0)-(-25-7)+[-18-(-25)]=7+32+7=46\end{aligned}$$

The object traveled a total distance of 46 m from $t=0$ to $t=6$ min. ■

Figure 34 shows the distinction between the net displacement and the total distance traveled by the object assuming that the object is at the origin when the motion starts.

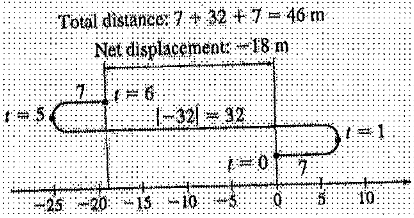


Figure 34

NOW WORK Problem 85 and and AP® Practice Problems 13 and 17.

6.4 Assess Your Understanding

Concepts and Vocabulary

1. *True or False* $\int_2^3 (x^2+x)\,dx=\int_2^3 x^2dx+\int_2^3 x\,dx$

2. *True or False* $\int_0^3 5e^{x^2}dx=\int_0^3 5\,dx\cdot\int_0^3 e^{x^2}dx$

3. *True or False*

$$\int_0^5 (x^3+1)\,dx=\int_0^{-3}(x^3+1)\,dx+\int_{-3}^5 (x^3+1)\,dx$$

4. If f is continuous on an interval containing the numbers a, b, and c, and if $\int_a^c f(x)\,dx=3$ and $\int_c^b f(x)\,dx=-5$, then $\int_a^b f(x)\,dx=$ ______.

5. If a function f is continuous on the closed interval $[a, b]$, then $\bar{y}=\dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$ is the ______ of f over $[a, b]$.

6. *True or False* If a function f is continuous on a closed interval $[a, b]$ and if m and M denote the absolute minimum value and the absolute maximum value, respectively, of f on $[a, b]$, then

$$m\leq\int_a^b f(x)\,dx\leq M$$

Skill Building

In Problems 7–16, find each definite integral given that $\int_1^3 f(x)\,dx=5$, $\int_1^3 g(x)\,dx=-2$, $\int_3^5 f(x)\,dx=2$, $\int_3^5 g(x)\,dx=1$.

7. $\int_1^3 [f(x)-g(x)]\,dx$ **8.** $\int_1^3 [f(x)+g(x)]\,dx$

9. $\int_1^3 [5f(x)-3g(x)]\,dx$ **10.** $\int_1^3 [3f(x)+4g(x)]\,dx$

11. $\int_1^5 [2f(x)-3g(x)]\,dx$ **12.** $\int_1^5 [f(x)-g(x)]\,dx$

13. $\int_1^3 [5f(x)-3g(x)+7]\,dx$ **14.** $\int_1^3 [f(x)+4g(x)+2x]\,dx$

15. $\int_1^5 [2f(x)-g(x)+3x^2]\,dx$ **16.** $\int_1^5 [5f(x)-3g(x)-4]\,dx$

In Problems 17–36, find each definite integral using the Fundamental Theorem of Calculus and properties of the definite integral.

PAGE 425 **17.** $\int_0^1 (t^2-t^{3/2})\,dt$ **18.** $\int_{-2}^0 (x+x^2)\,dx$

PAGE 425 **19.** $\int_{\pi/2}^{\pi} 4\sin x\,dx$ **20.** $\int_0^{\pi/2} 3\cos x\,dx$

21. $\int_{-\pi/4}^{\pi/4} (1+2\sec x\,\tan x)\,dx$ **22.** $\int_0^{\pi/4} (1+\sec^2 x)\,dx$

23. $\int_1^4 (\sqrt{x}-4x)\,dx$ **24.** $\int_0^1 (\sqrt[5]{t^2}+1)\,dt$

25. $\int_{-2}^3 [(x-1)\,(x+3)]\,dx$ **26.** $\int_0^1 (z^2+1)^2 dz$

27. $\int_1^2 \dfrac{x^2-12}{x^4}\,dx$ **28.** $\int_1^e \dfrac{5s^2+s}{s^2}\,ds$

PAGE 426 **29.** $\int_0^1 \dfrac{e^{2x}-1}{e^x}\,dx$ **30.** $\int_0^1 \dfrac{e^{2x}+4}{e^{-x}}\,dx$

35

31. $\displaystyle\int_{\pi/3}^{\pi/2} \frac{x\sin x + 2}{x}\,dx$

32. $\displaystyle\int_{\pi/6}^{\pi/2} \frac{x\cos x - 4}{x}\,dx$

33. $\displaystyle\int_{1}^{2} \frac{x^2+2}{x^2}\,dx$

34. $\displaystyle\int_{1}^{4} \frac{x-\sqrt{x}}{x}\,dx$

35. $\displaystyle\int_{0}^{1/2} \left(5 + \frac{1}{\sqrt{1-x^2}}\right) dx$

36. $\displaystyle\int_{0}^{1} \left(1 + \frac{5}{1+x^2}\right) dx$

In Problems 37–48, use properties of integrals and the Fundamental Theorem of Calculus to find each integral.

37. $\displaystyle\int_{-2}^{1} f(x)\,dx$, where $f(x) = \begin{cases} 1 & \text{if } x < 0 \\ x^2+1 & \text{if } x \geq 0 \end{cases}$

38. $\displaystyle\int_{-1}^{2} f(x)\,dx$, where $f(x) = \begin{cases} x+1 & \text{if } x < 0 \\ x^2+1 & \text{if } x \geq 0 \end{cases}$

39. $\displaystyle\int_{-2}^{2} f(x)\,dx$, where $f(x) = \begin{cases} 3x & \text{if } -2 \leq x < 0 \\ 2x^2 & \text{if } 0 \leq x \leq 2 \end{cases}$

40. $\displaystyle\int_{0}^{4} h(x)\,dx$, where $h(x) = \begin{cases} x-2 & \text{if } 0 \leq x \leq 2 \\ 2-x & \text{if } 2 < x \leq 4 \end{cases}$

41. $\displaystyle\int_{-2}^{1} H(x)\,dx$, where $H(x) = \begin{cases} 1+x^2 & \text{if } -2 \leq x < 0 \\ 1+3x & \text{if } 0 \leq x \leq 1 \end{cases}$

42. $\displaystyle\int_{-\pi/2}^{\pi/2} f(x)\,dx$, where $f(x) = \begin{cases} x^2+x & \text{if } -\dfrac{\pi}{2} \leq x \leq 0 \\ \sin x & \text{if } 0 < x < \dfrac{\pi}{4} \\ \dfrac{\sqrt{2}}{2} & \text{if } \dfrac{\pi}{4} \leq x \leq \dfrac{\pi}{2} \end{cases}$

43. $\displaystyle\int_{-4}^{2} f(x)\,dx$, where

$$f(x) = \begin{cases} 8-x & \text{if } -4 \leq x \leq -2 \\ \dfrac{x^2+7x+10}{x+2} & \text{if } -2 < x \leq 2 \end{cases}$$

44. $\displaystyle\int_{0}^{5} f(x)\,dx$, where $f(x) = \begin{cases} \dfrac{x^3-8}{x-2} & \text{if } 0 \leq x < 1 \\ 4x-1 & \text{if } 1 \leq x \leq 5 \end{cases}$

45. $\displaystyle\int_{0}^{2\pi} f(x)\,dx$, where $f(x) = \begin{cases} \sin x & \text{if } 0 \leq x < \pi \\ \cos x & \text{if } \pi \leq x \leq 2\pi \end{cases}$

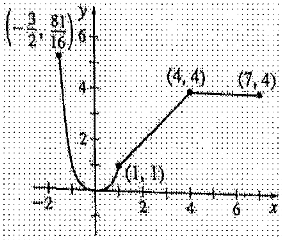
46. $\displaystyle\int_{-1}^{e} f(x)\,dx$, where $f(x) = \begin{cases} x^2+4 & \text{if } -1 \leq x < 1 \\ \dfrac{3}{x} & \text{if } 1 \leq x \leq e \end{cases}$

47. $\displaystyle\int_{0}^{5} f(x)\,dx$, where $f(x) = \begin{cases} \dfrac{x^2-1}{x+1} & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1 \\ x & \text{if } 1 < x \leq 5 \end{cases}$

48. $\displaystyle\int_{-1}^{6} f(x)\,dx$, where $f(x) = \begin{cases} x^2 & \text{if } -1 < x < 3 \\ 0 & \text{if } x = 3 \\ \dfrac{1}{x} & \text{if } 3 < x \leq 6 \end{cases}$

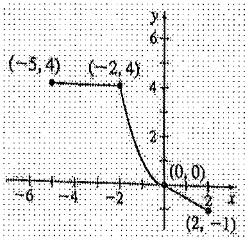
In Problems 49–52, the domain of f is a closed interval [a, b]. Find $\int_a^b f(x)\,dx$.

49.



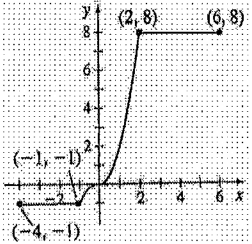
$f(x) = \begin{cases} x^4 \text{ if } -\dfrac{3}{2} \leq x < 1 \\ x \text{ if } 1 \leq x < 4 \\ 4 \text{ if } 4 \leq x \leq 7 \end{cases}$

50.



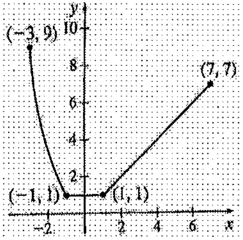
$f(x) = \begin{cases} 4 \text{ if } -5 \leq x < -2 \\ x^2 \text{ if } -2 \leq x \leq 0 \\ -\dfrac{x}{2} \text{ if } 0 < x \leq 2 \end{cases}$

51.



$f(x) = \begin{cases} -1 \text{ if } -4 \leq x \leq -1 \\ x^3 \text{ if } -1 < x < 2 \\ 8 \text{ if } 2 \leq x \leq 6 \end{cases}$

52.



$f(x) = \begin{cases} x^2 \text{ if } -3 \leq x \leq -1 \\ 1 \text{ if } -1 < x \leq 1 \\ x \text{ if } 1 < x \leq 7 \end{cases}$

In Problems 53–56, use properties of definite integrals to verify each statement. Assume that all integrals involved exist.

53. $\displaystyle\int_{3}^{11} f(x)\,dx - \int_{7}^{11} f(x)\,dx = \int_{3}^{7} f(x)\,dx$

54. $\displaystyle\int_{-2}^{6} f(x)\,dx - \int_{3}^{6} f(x)\,dx = \int_{-2}^{3} f(x)\,dx$

55. $\displaystyle\int_{0}^{4} f(x)\,dx - \int_{6}^{4} f(x)\,dx = \int_{0}^{6} f(x)\,dx$

56. $\displaystyle\int_{-1}^{3} f(x)\,dx - \int_{5}^{3} f(x)\,dx = \int_{-1}^{5} f(x)\,dx$

In Problems 57–64, use the Bounds on an Integral Theorem to obtain a lower estimate and an upper estimate for each integral.

57. $\displaystyle\int_{1}^{3} (5x+1)\,dx$

58. $\displaystyle\int_{0}^{1} (1-x)\,dx$

59. $\displaystyle\int_{\pi/4}^{\pi/2} \sin x\,dx$

60. $\displaystyle\int_{\pi/6}^{\pi/3} \cos x\,dx$

61. $\displaystyle\int_{0}^{1} \sqrt{1+x^2}\,dx$

62. $\displaystyle\int_{-1}^{1} \sqrt{1+x^4}\,dx$

63. $\displaystyle\int_{0}^{1} e^x\,dx$

64. $\displaystyle\int_{1}^{10} \frac{1}{x}\,dx$

In Problems 65–70, for each integral find the number(s) u guaranteed by the Mean Value Theorem for Integrals.

PAGE 430 **65.** $\int_0^3 (2x^2+1)\,dx$ **66.** $\int_0^2 (2-x^3)\,dx$

67. $\int_0^4 x^2\,dx$ **68.** $\int_0^4 (-x)\,dx$

69. $\int_0^{2\pi} \cos x\,dx$ **70.** $\int_{-\pi/4}^{\pi/4} \sec x \tan x\,dx$

In Problems 71–80, find the average value of each function f over the given interval.

PAGE 431 **71.** $f(x)=e^x$ over $[0, 1]$ **72.** $f(x)=\frac{1}{x}$ over $[1, e]$

73. $f(x)=x^{2/3}$ over $[-1, 1]$ **74.** $f(x)=\sqrt{x}$ over $[0, 4]$

75. $f(x)=\sin x$ over $\left[0, \frac{\pi}{2}\right]$ **76.** $f(x)=\cos x$ over $\left[0, \frac{\pi}{2}\right]$

77. $f(x)=1-x^2$ over $[-1, 1]$

78. $f(x)=16-x^2$ over $[-4, 4]$

79. $f(x)=e^x-\sin x$ over $\left[0, \frac{\pi}{2}\right]$

80. $f(x)=x+\cos x$ over $\left[0, \frac{\pi}{2}\right]$

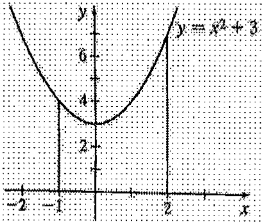
In Problems 81–84, find:

(a) The area under the graph of the function over the indicated interval.

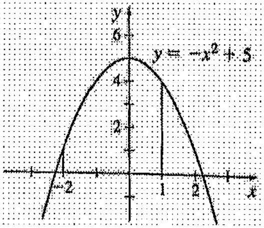
(b) The average value of each function over the indicated interval.

(c) Interpret the results geometrically.

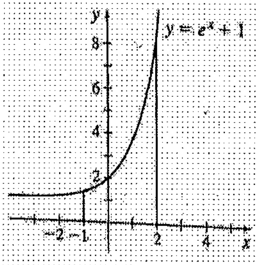
81. $[-1, 2]$



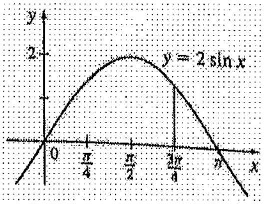
82. $[-2, 1]$



83. $[-1, 2]$



84. $\left[0, \frac{3\pi}{4}\right]$



In Problems 85–88, an object in rectilinear motion is moving along a horizontal line with velocity $v=v(t)$ over the indicated interval.

(a) Find the net displacement of the object. Interpret the result in the context of the problem.

(b) Find the total distance traveled by the object and interpret the result.

PAGE 432 **85.** $v(t)=6t^2+12t-18$ m/min; $0 \le t \le 3$

86. $v(t)=t^2-2t-8$ m/min; $0 \le t \le 6$

87. $v(t)=t^3+2t^2-8t$ m/s; $-2 \le t \le 3$

88. $v(t)=t^3+5t^2-6t$ m/s; $-1 \le t \le 3$

Applications and Extensions

In Problems 89–92, find the indicated derivative.

89. $\frac{d}{dx}\int_x^{4x} \sqrt{t^2+4}\,dt$ **90.** $\frac{d}{dx}\int_x^{x^2} (t^2+4)^{3/2}\,dt$

91. $\frac{d}{dx}\int_{x^2}^{x^3} \ln t\,dt$ **92.** $\frac{d}{dx}\int_{-x}^{x^2} e^{t^2}\,dt$

In Problems 93–96, find each definite integral using the Fundamental Theorem of Calculus and properties of definite integrals.

93. $\int_{-2}^{3} (x+|x|)\,dx$ **94.** $\int_0^3 |x-1|\,dx$

95. $\int_0^2 |3x-1|\,dx$ **96.** $\int_0^5 |2-x|\,dx$

97. Average Temperature A rod 3 m long is heated to $25x$ °C, where x is the distance in meters from one end of the rod. Find the average temperature of the rod.

98. Average Daily Rainfall The rainfall per day, x days after the beginning of the year, is modeled by the function $r(x)=0.00002(6511+366x-x^2)$, measured in centimeters. Find the average daily rainfall for the first 180 days of the year.

99. Structural Engineering A structural engineer designing a member of a structure must consider the forces that will act on that member. Most often, natural forces like snow, wind, or rain distribute force over the entire member. For practical purposes, however, an engineer determines the distributed force as a single resultant force acting at one point on the member. If the distributed force is given by the function $W=W(x)$, in newtons per meter (N/m), then the magnitude F_R of the resultant force is

$$F_R=\int_a^b W(x)\,dx$$

The position $\bar{x}$ of the resultant force measured in meters from the origin is given by

$$\bar{x}=\frac{\int_a^b xW(x)\,dx}{\int_a^b W(x)\,dx}$$

If the distributed force is $W(x)=0.75x^3$, $0 \le x \le 5$, find:

(a) The magnitude of the resultant force.

(b) The position from the origin of the resultant force.

Source: Problem contributed by the students at Trine University, Avalon, IN.

6.4 AP Practice Problems (p. 437)

1. If $\int_0^2 f(x)\,dx = -3$ and $\int_0^5 f(x)\,dx = 7$, then $\int_2^5 [4f(x) - 1]\,dx$ equals

(A) 13 (B) 15 (C) 37 (D) 39

2. An object in rectilinear motion moves along the x-axis with velocity $v = v(t)$ meters per second. If $v(t) = 3t^2 + t - 2$, what is the average velocity of the object during the interval $0 \le t \le 6$?

(A) 37 m/s (B) 38 m/s (C) 41 m/s (D) 222 m/s

3. If $g(x) = 2f(x) - 4$ on the interval $[-2, 8]$, then $\int_{-2}^{8} [f(x) + g(x)]\,dx$ equals

(A) $3\int_{-2}^{8} f(x)\,dx - 24$ (B) $3\int_{-2}^{8} f(x)\,dx - 40$

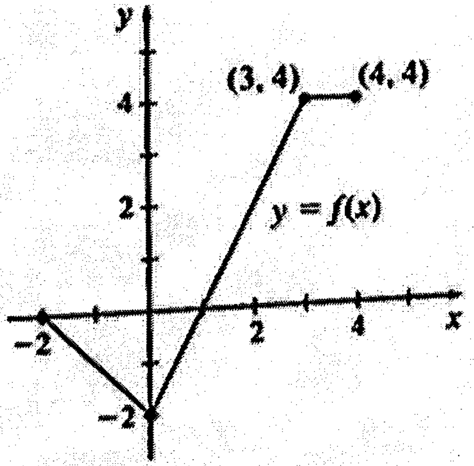
(C) $3\int_{-2}^{8} f(x)\,dx - 4$ (D) $\int_{-2}^{8} f(x)\,dx - 40$

4. $\displaystyle\int_1^e \frac{3x^2+1}{x}\,dx =$

(A) $\dfrac{3e^2-1}{2}$ (B) $\dfrac{3e^2+1}{2}$

(C) $e^3 + 1$ (D) $3e^2 - 1$

5. The graph of the piecewise function f is below. What is $\int_{-2}^{4} f(x)\,dx$?



(A) 2 (B) 5 (C) $\frac{17}{2}$ (D) 9

6. $\int_{1}^{4} \sqrt{x}\left(x - \frac{1}{x}\right) dx =$

(A) $\frac{32}{5}$ (B) $\frac{44}{5}$ (C) $\frac{52}{5}$ (D) $\frac{56}{15}$

7. The average value of $f(x) = \sin x$ on the interval $\left[-\frac{\pi}{3}, \frac{\pi}{2}\right]$ is

(A) $-\frac{3}{5\pi}$ (B) $\frac{3}{5\pi}$ (C) $\frac{1}{2}$ (D) $\frac{5\pi}{12}$

39

8. If $f(x) = \begin{cases} x^3 \text{ if } x \leq 1 \\ \frac{1}{x} \text{ if } x > 1 \end{cases}$ then $\int_0^e f(x)\,dx$ equals

(A) 0 (B) $\frac{5}{4}$ (C) $\frac{1}{4} + e$ (D) $\frac{e^4}{4}$

9. What is the average value of $f(x) = \frac{1}{x}$ on the closed interval $[1, 4]$?

(A) $\ln \frac{4}{3}$ (B) $\frac{\ln 3}{3}$ (C) $\frac{\ln 4}{3}$ (D) $\ln 4$

10. The area under the graph of $f(x) = x^2(3 - x)$ from 0 to 3 is

(A) 6 (B) $\frac{27}{4}$ (C) 7 (D) 7.5

11. If $\int_1^8 f(x)\,dx = 5$ and $\int_8^4 f(x)\,dx = 9$, then $\int_1^4 f(x)\,dx$ equals

(A) -4 (B) 4 (C) 8 (D) 14

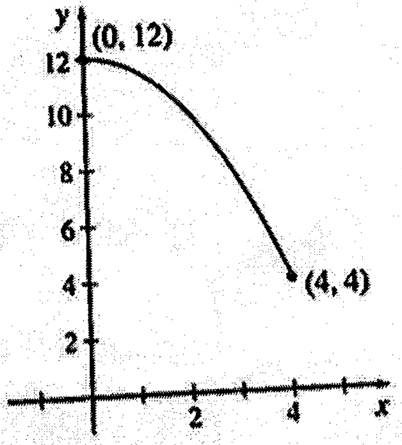
12. What is the average value of the part of the graph of $f(x) = x^3(2 - x)$ that lies in the first quadrant?

(A) $\frac{4}{5}$ (B) 1 (C) $\frac{8}{5}$ (D) $\frac{12}{5}$

13. $\displaystyle\int_0^6 |x - 4|\,dx =$

(A) 6 (B) 10 (C) 22 (D) 42

14. The graph of f is shown below.



Then $\int_0^4 f(x)\,dx$ must be between

(A) 4 and 12 (B) 20 and 32 (C) 16 and 48 (D) 40 and 60

15. Find $\int_{-2}^{10} f(x)\,dx$ where $f(x) = \begin{cases} x+3 & \text{if } -2 \le x < 2 \\ 3x & \text{if } \;\; 2 \le x \le 10 \end{cases}$

(A) 144 (B) 150 (C) 156 (D) 306

16. A car's velocity $v = v(t)$ (in ft/s) is measured each second t for $t = 0$ to $t = 8$ and posted in the table. Use a Right Riemann sum with four subintervals of equal length to approximate the car's average velocity over the interval from 0 to 8 seconds.

t	0	1	2	3	4	5	6	7	8
$v(t)$	0	2	4	6	7	7	8	6	2

(A) 42 ft/s (B) 7 ft/s (C) 5 ft/s (D) 5.25 ft/s

17. An object in rectilinear motion is moving along a horizontal line with velocity $v(t) = 3t^2 - 6t$, $1 \leq t \leq 4$ (in meters per second).
    (a) Find the total distance the object moves from $t = 1$ to $t = 4$.
    (b) For what time(s) is the object at rest?
    (c) If at time $t = 1$, the object is 2 m from the origin, what is its position at $t = 4$?
    (d) Find the average velocity of the object from $t = 1$ to $t = 4$.

Calculus Chapter 4.5a U-Substitution Method for Indefinite Integrals

U-Substitution is the reverse of the derivative chain rule. We want to look for a function and its derivative to be in the integral.

Example 1:

Suppose $f(x) = \sin(3x)$

$f'(x) = \cos(3x) \cdot 3$
$f'(x) = 3\cos(3x)$

This means that:

$$\int 3\cos(3x)dx = \sin(3x) + C$$

U-Substitution Steps:

1. Assign the 'u' value to the expression inside the parentheses.
2. Find the derivative of u: $\frac{du}{dx}$
3. Solve for dx.
4. Rewrite the integral in terms of u and du. (Check to make sure no 'x' or 'dx' remains)
5. Evaluate the Integral
6. Write the answer in terms of x.
7. **Not all Integral problems require U-Substitution. Check first to see if expansion/rewriting problem will allow problem to only need the Power Rule**

Ex. 2: $\int x(x^2+1)^{15}dx$

Ex. 3: $\int x^2 \sec^2(2x^3)dx$

Ex. 4: $\int x^3\sqrt{5-x^4}\,dx$

Ex. 5: $\int \tan^5 x \sec^2 x \, dx$

Ex. 6: $\int (3-y)\left(\frac{1}{\sqrt{y}}\right)dy$

Change of Variable U-Substitution Method:

Ex. 7: $\int x\sqrt{x+3}\,dx$

Ex. 8: $\int x^2\sqrt{2-x}\,dx$

45

Calculus **Chapter 4.5b** **U-Substitution Method for Definite Integrals**

U-Substitution with definite integrals: Be sure the bounds matches the variable of the problem

Ex. 1: $\int_{1}^{2} 2x(x^2-2)^3\,dx$

Ex. 2: $\int_{1}^{5} \frac{x}{\sqrt{2x-1}}\,dx$

Integrals of Odd and Even Functions

Review: Suppose $\int_{10}^{3} f(x)dx = 9$ and $\int_{-1}^{3} f(x)dx = 5$, find $\int_{-1}^{10} f(x)dx$

Even/Odd Rules:

Even: $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$

Odd: $\int_{-a}^{a} f(x)dx = 0$

Ex. 3: Suppose g(x) is an even function where $\int_{0}^{3} g(x)dx = 2$ and $\int_{-4}^{-3} g(x)dx = 4$. Find $\int_{-4}^{3} g(x)dx$.

(Sketch a possible graph using the above given information)

Ex. 4: Same as Example 3, but g(x) is an odd function: $\int_{0}^{3} g(x)dx = 2$ and $\int_{-4}^{-3} g(x)dx = 4$. Find $\int_{-4}^{3} g(x)dx$.

Ex. 5: If f(x) is even and $\int_{3}^{6} f(x)dx = 7$ and $\int_{-6}^{3} f(x)dx = 12$, find $\int_{0}^{6} f(x)dx$

AP Calculus – 6.5c Notes – Integrating using Long Division & Completing the square

What would happen if we attempted to apply power rule for this problem?

$$\int \frac{1}{x}\,dx$$

Recall Derivative Rule:

$$\frac{d}{dx}\ln u = \frac{u'}{u}$$

Natural Log Integral Rule

$$\int \frac{1}{u}\,du = \ln|u| + C$$

Example 1: $\int \frac{2x}{x^2+1}\,dx$

Example 2: $\int \frac{1}{x\ln x}\,dx$

Completing the Square and Long Division are two skills that help us manipulate the integrand until it becomes something we can work with. We are practicing those two skills in this lesson.

Using Long Division to Rewrite the Integrand:

1. $\int \frac{3x^3-x^2-5x+1}{x-2}\,dx$

2. $\int \frac{6x^4-7x^3+x^2+2x}{3x-5}\,dx$

$\frac{d}{dx}\arcsin u = \frac{u'}{\sqrt{1-u^2}}$ $\frac{d}{dx}\arctan u = \frac{u'}{1+u^2}$ $\frac{d}{dx}arc\sec u = \frac{u'}{\lvert u\rvert\sqrt{u^2-1}}$	$\frac{d}{dx}\arccos u = -\frac{u'}{\sqrt{1-u^2}}$ $\frac{d}{dx}arc\cot u = -\frac{u'}{1+u^2}$ $\frac{d}{dx}arc\csc u = -\frac{u'}{\lvert u\rvert\sqrt{u^2-1}}$	**Inverse Trig Integral Rules:** ***a is a constant*** 1. $\int\frac{du}{\sqrt{a^2-u^2}} = \arcsin\left(\frac{u}{a}\right)+C$ 2. $\int\frac{du}{a^2+u^2} = \frac{1}{a}\arctan\left(\frac{u}{a}\right)+C$ 3. $\int\frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}arc\sec\left(\frac{\lvert u\rvert}{a}\right)+C$

Ex. 1: $\int\frac{5}{x\sqrt{x^2-9}}dx$

Ex. 2: $\int\frac{1}{4+(x-1)^2}dx$

Completing the Square Steps:

1. Write in standard form: $x^2 + bx + c$
2. Add spaces $x^2 + bx +$ ___ $+ c -$ ___
3. Put $\left(\frac{b}{2}\right)^2$ into the spaces
4. Factor expression

Using Completing the Square to Rewrite the Integrand:

3. $\int\frac{1}{x^2+6x+10}dx$

4. $\int\frac{1}{\sqrt{-x^2+8x-15}}dx$

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EXAMPLE 12 **Using Properties of Integrals**

If f is an even function and $\int_0^2 f(x)\,dx = -6$ and $\int_{-5}^0 f(x)\,dx = 8$, find $\int_2^5 f(x)\,dx$.

Solution

$\int_2^5 f(x)\,dx = \int_2^0 f(x)\,dx + \int_0^5 f(x)\,dx$

Now $\int_2^0 f(x)\,dx = -\int_0^2 f(x)\,dx = 6$

Since f is even, $\int_0^5 f(x)\,dx = \int_{-5}^0 f(x)\,dx = 8$. Then

$$\int_2^5 f(x)\,dx = \int_2^0 f(x)\,dx + \int_0^5 f(x)\,dx = 6 + 8 = 14 \qquad \blacksquare$$

NOW WORK Problem 87.

6.5 Assess Your Understanding

Concepts and Vocabulary

1. $\dfrac{d}{dx}\displaystyle\int f(x)\,dx =$ ________

2. *True or False* If k is a constant, then $\int kf(x)\,dx = \int k\,dx \cdot \int f(x)\,dx$

3. If a is a real number, $a \neq -1$, then $\int x^a\,dx =$ ________

4. *True or False* When finding the indefinite integral of a function f, a constant of integration C is added to the result because $\int f(x)\,dx$ denotes all the antiderivatives of f.

5. If the substitution $u = 2x + 3$ is used with $\int \sin(2x+3)\,dx$, the result is $\int$ ________ du.

6. *True or False* If the substitution $u = x^2 + 3$ is used with $\displaystyle\int_0^1 x(x^2+3)^3 dx$, then $\displaystyle\int_0^1 x(x^2+3)^3 dx = \frac{1}{2}\int_0^1 u^3 du$.

7. *Multiple Choice* $\displaystyle\int_{-4}^4 x^3 dx =$ [(a) 128 (b) 4 (c) 0 (d) 64]

8. *True or False* $\displaystyle\int_0^5 x^2\,dx = \frac{1}{2}\int_{-5}^5 x^2\,dx$

Skill Building

In Problems 9–20, find each indefinite integral.

PAGE 439 9. $\displaystyle\int x^{2/3}\,dx$

10. $\displaystyle\int t^{-4}\,dt$

PAGE 439 11. $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx$

12. $\displaystyle\int \frac{1}{1+x^2}\,dx$

PAGE 441 13. $\displaystyle\int \frac{5x^2+2xe^x-1}{x}\,dx$

14. $\displaystyle\int \frac{xe^x+1}{x}\,dx$

15. $\displaystyle\int \frac{\tan x}{\cos x}\,dx$

16. $\displaystyle\int \frac{1}{\sin^2 x}\,dx$

PAGE 440 17. $\displaystyle\int \frac{2}{5\sqrt{1-x^2}}\,dx$

18. $\displaystyle\int -\frac{4}{x\sqrt{x^2-1}}\,dx$

19. $\displaystyle\int \frac{e^t+e^{-t}}{2}\,dt$

20. $\displaystyle\int \frac{e^{2t}-e^{-2t}}{2}\,dt$

In Problems 21–26, find each indefinite integral using the given substitution.

PAGE 442 21. $\displaystyle\int e^{3x+1}dx$; let $u = 3x+1$

22. $\displaystyle\int \frac{dx}{x\ln x}$; let $u = \ln x$

23. $\displaystyle\int (1-t^2)^6 t\,dt$; let $u = (1-t^2)$

24. $\displaystyle\int \sin^5 x\cos x\,dx$; let $u = \sin x$

25. $\displaystyle\int \frac{x^2\,dx}{\sqrt{1-x^6}}$; let $u = x^3$

26. $\displaystyle\int \frac{e^{-x}}{6+e^{-x}}\,dx$; let $u = 6+e^{-x}$

In Problems 27–60, find each indefinite integral.

PAGE 442 27. $\displaystyle\int \sin(3x)\,dx$

28. $\displaystyle\int x\sin x^2\,dx$

29. $\displaystyle\int \sin x\cos^2 x\,dx$

30. $\displaystyle\int \tan^2 x\sec^2 x\,dx$

31. $\displaystyle\int \frac{e^{1/x}}{x^2}\,dx$

32. $\displaystyle\int \frac{e^{\sqrt[3]{x}}}{\sqrt[3]{x^2}}\,dx$

PAGE 443 33. $\displaystyle\int \frac{x\,dx}{x^2-1}$

34. $\displaystyle\int \frac{5x\,dx}{1-x^2}$

35. $\displaystyle\int \frac{e^x}{\sqrt{1+e^x}}\,dx$

36. $\displaystyle\int \frac{dx}{x(\ln x)^7}$

37. $\displaystyle\int \frac{1}{\sqrt{x}(1+\sqrt{x})^4}\,dx$

38. $\displaystyle\int \frac{dx}{\sqrt{x}(1+\sqrt{x})}$

39. $\displaystyle\int \frac{3e^x}{\sqrt[4]{e^x-1}}\,dx$

40. $\displaystyle\int \frac{[\ln(5x)]^3}{x}\,dx$

41. $\displaystyle\int \frac{\cos x\,dx}{2\sin x-1}$

42. $\displaystyle\int \frac{\cos(2x)\,dx}{\sin(2x)}$

PAGE 444 43. $\displaystyle\int \sec(5x)\,dx$

44. $\displaystyle\int \tan(2x)\,dx$

45. $\int \sqrt{\tan x}\sec^2 x\,dx$

46. $\int (2+3\cot x)^{3/2}\csc^2 x\,dx$

47. $\int \frac{\sin x}{\cos^2 x}\,dx$

48. $\int \frac{\cos x}{\sin^2 x}\,dx$

49. $\int \sin x \cdot e^{\cos x}\,dx$

50. $\int \sec^2 x \cdot e^{\tan x}\,dx$

PAGE 444 **51.** $\int x\sqrt{x+3}\,dx$

52. $\int x\sqrt{4-x}\,dx$

53. $\int [\sin x + \cos(3x)]\,dx$

54. $\int [x^2 + \sqrt{3x+2}]\,dx$

PAGE 445 **55.** $\int \frac{dx}{x^2+25}$

56. $\int \frac{\cos x}{1+\sin^2 x}\,dx$

57. $\int \frac{dx}{\sqrt{9-x^2}}$

58. $\int \frac{dx}{\sqrt{16-9x^2}}$

59. $\int \sin x \cos x\,dx$

60. $\int \sec^2 x \tan x\,dx$

In Problems 61–68, find each definite integral two ways:

(a) By finding the related indefinite integral and then using the Fundamental Theorem of Calculus.

(b) By making a substitution in the integrand and using the substitution to change the limits of integration.

(c) Which method did you prefer? Why?

PAGE 446 **61.** $\int_{-2}^{0} \frac{x}{(x^2+3)^2}\,dx$

62. $\int_{-1}^{1} (x^2-1)^5 x\,dx$

63. $\int_{0}^{1} x^2 e^{x^3+1}\,dx$

64. $\int_{0}^{1} x e^{x^2-2}\,dx$

65. $\int_{1}^{6} x\sqrt{x+3}\,dx$

66. $\int_{2}^{6} x^2\sqrt{x-2}\,dx$

67. $\int_{0}^{2} x \cdot 3^{2x^2}\,dx$

68. $\int_{0}^{1} x \cdot 10^{-x^2}\,dx$

In Problems 69–78, find each definite integral.

PAGE 447 **69.** $\int_{1}^{3} \frac{1}{x^2}\sqrt{1-\frac{1}{x}}\,dx$

70. $\int_{0}^{\pi/4} \frac{\sin(2x)}{\sqrt{5-2\cos(2x)}}\,dx$

71. $\int_{0}^{2} \frac{e^{2x}}{e^{2x}+1}\,dx$

72. $\int_{1}^{3} \frac{e^{3x}}{e^{3x}-1}\,dx$

73. $\int_{2}^{3} \frac{dx}{x\ln x}$

74. $\int_{2}^{3} \frac{dx}{x(\ln x)^2}$

75. $\int_{0}^{\pi} e^x \cos(e^x)\,dx$

76. $\int_{0}^{\pi} e^{-x}\cos(e^{-x})\,dx$

77. $\int_{0}^{1} \frac{x\,dx}{1+x^4}$

78. $\int_{0}^{1} \frac{e^x}{1+e^{2x}}\,dx$

In Problems 79–86, use properties of even and odd functions to find each integral.

PAGE 448 **79.** $\int_{-2}^{2} (x^2-4)\,dx$

80. $\int_{-1}^{1} (x^3-2x)\,dx$

PAGE 448 **81.** $\int_{-\pi/2}^{\pi/2} \frac{1}{3}\sin\theta\,d\theta$

82. $\int_{-\pi/4}^{\pi/4} \sec^2 x\,dx$

83. $\int_{-1}^{1} \frac{3}{1+x^2}\,dx$

84. $\int_{-5}^{5} (x^{1/3}+x)\,dx$

85. $\int_{-5}^{5} |2x|\,dx$

86. $\int_{-1}^{1} [|x|-3]\,dx$

PAGE 449 **87.** If f is an odd function and $\int_0^3 f(x)\,dx = 6$ and $\int_3^5 f(x)\,dx = 2$, find $\int_{-3}^{5} f(x)\,dx$.

88. If f is an odd function and $\int_{-5}^{10} f(x)\,dx = 8$, find $\int_5^{10} f(x)\,dx$.

89. If f is an even function and $\int_0^4 f(x)\,dx = -2$ and $\int_0^6 f(x)\,dx = 6$, find $\int_{-4}^{6} f(x)\,dx$.

90. If f is an even function and $\int_{-3}^{0} f(x)\,dx = 4$ and $\int_{-5}^{0} f(x)\,dx = 1$, find $\int_3^5 f(x)\,dx$.

Applications and Extensions

In Problems 91 and 92, find each indefinite integral by

(a) Using substitution.

(b) Expanding the integrand.

91. $\int (x+1)^2\,dx$

92. $\int (x^2+1)^2 x\,dx$

In Problems 93 and 94, find each integral three ways:

(a) By using substitution.

(b) By using even-odd properties of the definite integral.

(c) By using trigonometry to simplify the integrand before integrating.

(d) Verify the results are equivalent.

93. $\int_{-\pi/2}^{\pi/2} \cos(2x+\pi)\,dx$

94. $\int_{-\pi/4}^{\pi/4} \sin(7\theta-\pi)\,d\theta$

95. Area Find the area under the graph of $f(x) = \frac{x^2}{\sqrt{2x+1}}$ from 0 to 4.

96. Area Find the area under the graph of $f(x) = \frac{x}{(x^2+1)^2}$ from 0 to 2.

97. Area Find the area under the graph of $y = \frac{1}{3x^2+1}$ from $x = 0$ to $x = 1$.

98. Area Find the area under the graph of $y = \frac{1}{x\sqrt{x^2-4}}$ from $x = 3$ to $x = 4$.

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6.5 AP Practice Problems (p.453-454)

1. $\int \frac{1}{3}\left(e^x - \frac{2}{x}\right) dx =$

(A) $\frac{1}{3}e^x - 2\ln|x| + C$ (B) $\frac{1}{3}\left(e^x - \frac{2}{x^2}\right) + C$

(C) $\frac{1}{3}e^x - \frac{2}{3}\ln|x| + C$ (D) $\frac{1}{3}(e^x - \ln|2x|) + C$

2. The velocity $v = v(t)$ of an object in rectilinear motion is given by $v(t) = \int t \sin\frac{t}{2}\, dt$. What function gives the acceleration of the object?

(A) $a(t) = \frac{1}{2}t\cos\frac{t}{2} + \sin\frac{t}{2}$

(B) $a(t) = \frac{1}{2}t\cos\frac{t}{2}$

(C) $a(t) = t^2\cos\frac{t}{2}$

(D) $a(t) = t\sin\frac{t}{2}$

3. $\int \frac{1 + \sin x}{\cos^2 x}\, dx =$

(A) $\tan x + \sec x + C$ (B) $\csc x + \sec^2 x + C$

(C) $x + \sec x + C$ (D) $\sec x + C$

4. Using the substitution $u = \sin x$, $\int_{\pi/6}^{\pi/2} \sin^4 x \cos x \, dx$ becomes

(A) $-\int_{1/2}^{1} u^4 du$ (B) $\int_{\pi/6}^{\pi/2} u^4 du$

(C) $\int_{\sqrt{3}/2}^{0} u^4 du$ (D) $\int_{1/2}^{1} u^4 du$

5. If the function f is continuous for all real numbers, and if $F'(x) = f(x)$, then $\int_{1}^{4} f(3x) \, dx$ equals

(A) $F(12) - F(3)$ (B) $\frac{1}{3}[F(4) - F(1)]$

(C) $\frac{1}{3}[F(12) - F(3)]$ (D) $3[F(4) - F(1)]$

6. $\int \frac{e^{-3x}}{2 + e^{-3x}} \, dx =$

(A) $-3\ln(2 + e^{-3x}) + C$ (B) $-\frac{1}{3}\ln(2 + e^{-3x}) + C$

(C) $2x - \frac{e^{-3x}}{3} + C$ (D) $\frac{1}{3}\ln|-3e^{-3x}| + C$

7. $\int \frac{x}{1+x^2}\,dx =$

(A) $\arctan x + C$ (B) $\frac{1}{2}\ln|1+x^2| + C$

(C) $\ln|2(1+x^2)| + C$ (D) $2\ln|1+x^2| + C$

8. $\int_2^6 \frac{dx}{\sqrt{2x-3}} =$

(A) 1 (B) 2 (C) 4 (D) 78

9. What is the average value of $f(x) = \sin x \cos x$ on the closed interval $\left[0, \frac{\pi}{2}\right]$?

(A) $\frac{1}{\pi}$ (B) $\frac{2}{\pi}$ (C) $\frac{1}{2}$ (D) $\frac{\pi}{4}$

10. $\int x^3\sqrt{x^2+3}\,dx =$

(A) $\frac{1}{5}(x^2+3)^{5/2} + (3+x^2)^{3/2} + C$

(B) $\frac{1}{5}(x^2+3)^{5/2} - (3+x^2)^{3/2} + C$

(C) $\frac{2}{5}(x^2+3)^{5/2} - 2(3+x^2)^{3/2} + C$

(D) $\frac{4}{5}(x^2+3)^{5/2} + 4(3+x^2)^{3/2} + C$

11. An object in rectilinear motion has acceleration $a(t) = 4t - 5$ at time $t \geq 0$. If the object's initial velocity is 3, at what time t does the object first change direction?

(A) 0 (B) 1 (C) $\frac{3}{2}$ (D) 3

12. $\int_{e}^{e^4} \frac{1}{x\sqrt{\ln x}}\,dx =$

(A) $\frac{1}{2}$ (B) 1 (C) 2 (D) $2(e^2 - \sqrt{e})$

13. An object moving along the y-axis has velocity $v(t) = 2\cos\left(2t + \frac{\pi}{2}\right)$, $t \geq 0$. If the object is at $y = 5$ when $t = \pi$, find its position when $t = \frac{\pi}{2}$.

(A) 1 (B) 3 (C) 4 (D) 5

14. $\int \frac{4}{100 + x^2}\,dx =$

(A) $\frac{1}{25}\tan^{-1}\frac{x}{10} + C$ (B) $\frac{1}{25}\tan^{-1}(10x) + C$

(C) $\frac{2}{5}\tan^{-1}x + C$ (D) $\frac{2}{5}\tan^{-1}\frac{x}{10} + C$

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AP Calculus AB Integration Technique Checklist

1) **Power Rule** (Can you rearrange problem to rely on just power rule?)

*Some examples include: $\int (3-x)^2\left(\frac{2}{\sqrt{x}}\right)dx$ and $\int \frac{2x(5-3x+x^4)}{3(\sqrt{x^7})}dx$

a) convert radicals to rational exponential form (example: $\sqrt{x^5} = x^{\frac{5}{2}}$)

b) move denominator variable to numerator

c) resolve parentheses and separate the terms.

*typically, if there are multiple terms in denominator separated by addition or subtraction, power rule alone will not be enough to make progress. Proceed to Option #2

2) If unable to rely on just power rule, then explore **U-Substitution** options.

a) Big picture: We want to choose a u-value that will lead to an exact match with a **<u>known Integral rule.</u>** (Needs to be a perfect match outside of coefficient terms, and with no x-variables remaining)

b) If expression can be rewritten using parentheses, the u-value is usually the expression inside the set of parentheses.

c) u-value is more than just replacing an "x", and may involve replacing a significant portion of the expression.

d) For fractional expressions, the u-value usually comes from the denominator.

(potential notable exceptions are log functions like $\ln x$ and radical expressions like $\sqrt{x}$)

e) u-value are typically higher degree expressions when choosing between 2 expressions with different degrees.

2b) U-Substitution (using **change of variable**)

a) If the initial round of u-substitution is not enough to remove the remaining x's in the integrand, then explore option of rearranging the expression assigned to u, and solving for x.

b) Once we make that second set of substitutions, the problem is now purely in terms of u, and with all x's removed and replaced.

3) Rewrite <u>rational expression</u> using **Long Division** (synthetic division)

a) Condition needed to apply **long division** is the **numerator degree** ≥ **denominator degree.** (example: $\int \frac{2x^3-4x+1}{x^2+3}dx$)

b) For long division problems, we can apply **synthetic division** <u>only if denominator degree is = 1</u> (linear degree) (example: $\int \frac{4x^3-7x+2}{x-5}dx$)

c) Once our rewrite is complete, we can typically find the antiderivative by using a combination of power rule and u-substitution across the different terms.

Integration Technique Checklist (continued)

4) **ArcTrig Integral Rule** (From Ch. 5.7)

a) If the **denominator degree > numerator degree by 2 or more degrees**, consider the ArcTrig Integral rules as potential match for the problem.

b) If the "a" and "u" values of the ArcTrig Integral rule are not clearly visible, then apply the **completing the square method** in the denominator. This process will create a different (but equivalent) form that allows the "a" and "u" values in the denominator to become more visible.

*Completing the square steps:

i) Write expression in the form of $x^2 + bx + c$
ii) Add spaces: $x^2 + bx + \_\_ + c - \_\_$
iii) Insert $\left(\frac{b}{2}\right)^2$ into both the above ____ spaces
iv) Factor, then identify the a-value(constant) and u-value(variable expression)
v) Apply the ArcTrig Integral Rule

AP Calculus AB Visual Comparison between Integral Rules (mostly Rational expressions)

Compare Numerator and Denominator to help determine Integral Rule(s) to apply	**Example #1**	**Example #2**
1) **Only 1 Term in the Denominator** (regardless of degree differences between numerator and denominator) Solution: Consider expanding and splitting up the terms into individual fractions and applying integral rule for each term separately.	$\int \frac{x^4 - 5x^3 + 1}{2x^4} dx$	$\int \frac{4e^{4x} - e^{2x}}{6e^{3x}} dx$
2) Multiple terms in the denominator and the **Denominator has variable exponent degree that is 1 higher than the Numerator** Solution: Consider U-Substitution	$\int \frac{5x}{7x^2 - 4} dx$	$\int \frac{2x^2}{\sqrt[5]{3x^3 - 4}} dx$
3) Multiple terms in the denominator and the **Numerator has variable exponent that is Same degree OR Higher than the Denominator.** Solution: Consider Long Division and/or Synthetic Division	$\int \frac{4x - 3}{x - 5} dx$ Apply long division or synthetic division	$\int \frac{x^4 + x - 4}{x^2 + 2} dx$ Apply long division (synthetic division does **not** apply)
4) Multiple terms in the denominator and the **Denominator has variable exponent that is higher than the Numerator by 2 or more degrees**: Solution: Consider ArcTrig Integral Rules	$\int \frac{1}{x^2 - 8x + 4} dx$ Apply Arctan Integral Rule	$\int \frac{5x}{\sqrt{1 - x^4}} dx$ Apply Arcsin Integral Rule

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AP Calculus AB **Visual Comparison between Integral Rules (mostly Rational expressions)** Key

Compare Numerator and Denominator to help determine Integral Rule(s) to apply	**Example #1**	**Example #2**
1) **Only 1 Term in the Denominator** (regardless of degree differences between numerator and denominator) Solution: Consider expanding and splitting up the terms into individual fractions and applying integral rule for each term separately.	$\int \frac{x^4-5x^3+1}{2x^4}dx$ $\int \frac{x^4}{2x^4}-\frac{5x^3}{2x^4}+\int\frac{1}{2x^4}dx$ $\int \frac{1}{2}-\frac{5}{2}\left(\frac{1}{x}\right)+\frac{1}{2}x^{-4}dx$ $\boxed{\frac{1}{2}x-\frac{5}{2}\ln\lvert x\rvert+\frac{1}{2}\left(\frac{x^{-3}}{-3}\right)+C}$	$\int \frac{4e^{4x}-e^{2x}}{6e^{3x}}dx$ $\int \frac{4e^{4x}}{6e^{3x}}-\frac{e^{2x}}{6e^{3x}}dx \rightarrow \int \frac{2}{3}e^{x}-\frac{1}{6}e^{-x}dx$ $\boxed{\frac{2}{3}e^{x}+\frac{1}{6}e^{-x}+C}$
2) Multiple terms in the denominator and the **Denominator has variable exponent degree that is 1 higher than the Numerator** Solution: Consider U-Substitution	$\int \frac{5x}{7x^2-4}dx$ $u=7x^2-4$ $\frac{du}{dx}=14x$ $14x\,dx=du$ $dx=\frac{du}{14x}$ $\int \frac{5x}{u}\cdot\frac{du}{14x}$ $\frac{5}{14}\int\frac{1}{u}du$ $\boxed{\frac{5}{14}\ln\lvert 7x^2-4\rvert+C}$	$\int \frac{2x^2}{\sqrt[5]{3x^3-4}}dx \rightarrow \int\frac{2x^2}{(3x^3-4)^{1/5}}dx$ $u=3x^3-4$ $\frac{du}{dx}=9x^2$ $dx=\frac{du}{9x^2}$ $\int\frac{2x^2}{u^{1/5}}\cdot\frac{du}{9x^2}$ $\frac{2}{9}\int u^{-1/5}du$ $\frac{2}{9}\left(\frac{u^{4/5}}{4/5}\right)+C$ $\frac{10}{36}u^{4/5}+C$ $\boxed{\frac{5}{18}(3x^3-4)^{4/5}+C}$
3) Multiple terms in the denominator and the **Numerator has variable exponent that is Same degree OR Higher than the Denominator.** Solution: Consider Long Division and/or Synthetic Division $\begin{array}{r\|rr} 5 & 4 & -3 \\ & \downarrow & 20 \\ \hline & 4 & 17 \end{array}$ $\int 4+\frac{17}{x-5}dx$ $\boxed{4x+17\ln\lvert x-5\rvert+C}$	$\int \frac{4x-3}{x-5}dx$ Apply long division or synthetic division	$\int \frac{x^4+x-4}{x^2+2}dx$ u-sub: $u=x^2+2$ $\frac{du}{dx}=2x$ $dx=\frac{du}{2x}$ Apply long division (synthetic division does **not** apply) $x^2-2+\frac{x}{x^2+2}$ $x^2+2\,)\overline{x^4+x-4}$ $\ominus x^4 \ominus{+2x^2}$ $-2x^2+x-4$ $\oplus -2x^2 \quad \oplus 4$ x ← Remainder $\int x^2-2+\frac{x}{x^2+2}dx$ $\int\frac{x}{u}\cdot\frac{du}{2x}$ $\frac{1}{2}\int\frac{1}{u}du$ $\boxed{\frac{x^3}{3}-2x+\frac{1}{2}\ln\lvert x^2+2\rvert+C}$
4) Multiple terms in the denominator and the **Denominator has variable exponent that is higher than the Numerator by 2 or more degrees:** Solution: Consider ArcTrig Integral Rules x^2-8x+4 $\left(\frac{b}{2}\right)^2 \rightarrow \left(\frac{8}{2}\right)^2=16$ $x^2-8x+\underline{16}+4+\underline{16}$ $(x-4)^2+20$	$\int \frac{1}{x^2-8x+4}dx$ Apply Arctan Integral Rule $\int\frac{dx}{(x-4)^2+(\sqrt{20})^2}$ $u=x-4$ $a=\sqrt{20}$ $\int\frac{du}{a^2+u^2}=\frac{1}{a}\arctan\left(\frac{u}{a}\right)+C$ $\frac{1}{\sqrt{20}}\arctan\left(\frac{x-4}{\sqrt{20}}\right)+C$	$\int \frac{5x}{\sqrt{1-x^4}}dx$ $\frac{5}{2}\arcsin\left(\frac{x^2}{1}\right)+C$ $\boxed{\frac{5}{2}\arcsin(x^2)+C}$ Apply Arcsin Integral Rule $\int\frac{5x}{\sqrt{(\ )^2-(\ )^2}}dx$ $\int\frac{5x}{\sqrt{(1)^2-(x^2)^2}}dx$ $a=1$ $u=x^2$ $\frac{du}{dx}=2x$ $dx=\frac{du}{2x}$ $\int\frac{5x}{\sqrt{a^2-u^2}}\cdot\frac{du}{2x}$ $\frac{5}{2}\int\frac{du}{\sqrt{a^2-u^2}}$

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AP Calculus AB Derivative & Integral Rules Formula Patterns **(Blank Practice Sheet)**

I. Derivative Rules

A. Trig Derivatives

1) $\frac{d}{dx}\sin u =$

2) $\frac{d}{dx}\cos u =$

3) $\frac{d}{dx}\tan u =$

4) $\frac{d}{dx}\cot u =$

5) $\frac{d}{dx}\sec u =$

6) $\frac{d}{dx}\csc u =$

B. Logs and Exponential Derivatives

7) $\frac{d}{dx}\ln u =$

8) $\frac{d}{dx} e^{u} =$

9) $\frac{d}{dx} log_a u =$

10) $\frac{d}{dx} a^{u} =$

C. ArcTrig Derivatives

11) $\frac{d}{dx}\arcsin u =$

12) $\frac{d}{dx}\arccos u =$

13) $\frac{d}{dx}\arctan u =$

14) $\frac{d}{dx}\mathrm{arccot}\, u =$

15) $\frac{d}{dx}\mathrm{arcsec}\, u =$

16) $\frac{d}{dx}\mathrm{arccsc}\, u =$

II. Integral Rules

A. Basic Trig Integrals

1) $\int \sin u \, du =$

2) $\int \cos u \, du =$

3) $\int sec^2 u \, du =$

4) $\int \sec u \tan u \;\, du =$

5) $\int csc^2 u \, du =$

6) $\int \csc u \cot u \, du =$

B. Trig Integrals involving Natural Logs

7) $\int \tan u \, du =$

8) $\int \cot u \, du =$

9) $\int \sec u \, du =$

10) $\int \csc u \, du =$

C. Power Rule, Logs and Exponentials

11) $\int u^n du =$

12) $\int \frac{1}{u} du =$

13) $\int \; e^u du =$

14) $\int \; a^u du =$

D. Inverse Trig Integrals

15) $\frac{1}{\sqrt{a^2-u^2}} du =$

16) $\frac{1}{a^2+u^2} du =$

17) $\frac{1}{u\sqrt{u^2-a^2}} du =$

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Key ⑦

AP Calculus AB Derivative & Integral Rules Formula Patterns

I. Derivative Rules

A. Trig Derivatives

1) $\frac{d}{dx}\sin u = \cos u * u'$

2) $\frac{d}{dx}\cos u = -\sin u * u'$

3) $\frac{d}{dx}\tan u = sec^2\, u * u'$

4) $\frac{d}{dx}\cot\, u = -\,csc^2\, u * u'$

5) $\frac{d}{dx}\sec u = \sec u \tan u * u'$

6) $\frac{d}{dx}\csc u = -\csc u \cot u * u'$

B. Logs and Exponential Derivatives

7) $\frac{d}{dx}\ln u = \frac{u'}{u}$

8) $\frac{d}{dx}\, e^u = e^u * u'$

9) $\frac{d}{dx}\, log_a\, u = \left(\frac{1}{\ln a}\right)\frac{u'}{u}$

10) $\frac{d}{dx}\, a^u = (\ln a)\, a^u * u'$

C. ArcTrig Derivatives

11) $\frac{d}{dx}\arcsin u = \frac{u'}{\sqrt{1-u^2}}$

12) $\frac{d}{dx}\arccos u = -\frac{u'}{\sqrt{1-u^2}}$

13) $\frac{d}{dx}\arctan u = \frac{u'}{1+u^2}$

14) $\frac{d}{dx}\operatorname{arccot} u = -\frac{u'}{1+u^2}$

15) $\frac{d}{dx}\operatorname{arcsec} u = \frac{u'}{|u|\sqrt{u^2-1}}$

16) $\frac{d}{dx}\operatorname{arccsc} u = -\frac{u'}{|u|\sqrt{u^2-1}}$

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II. Integral Rules

A. Basic Trig Integrals

1) $\int \sin u \, du = -\cos u + C$

2) $\int \cos u \, du = \sin u + C$

3) $\int sec^2 u \, du = \tan u + C$

4) $\int \sec u \tan u \, du = \sec u + C$

5) $\int csc^2 u \, du = -\cot u + C$

6) $\int \csc u \cot u \, du = -\csc u + C$

B. Trig Integrals involving Natural Logs

7) $\int \tan u \, du = -\ln|\cos u| + C$

8) $\int \cot u \, du = \ln|\sin u| + C$

9) $\int \sec u \, du = \ln|\sec u + \tan u| + C$

10) $\int \csc u \, du = -\ln|\csc u + \cot u| + C$

C. Power Rule, Logs and Exponentials

11) $\int u^n du = \frac{u^{n+1}}{n+1} + C$

12) $\int \frac{1}{u} du = ln|u| + C$

13) $\int e^u du = e^u + C$

14) $\int a^u du = \left(\frac{1}{\ln a}\right) a^u + C$

~~D. Inverse Trig Integrals~~

15) $\frac{1}{\sqrt{a^2-u^2}} du = \arcsin\left(\frac{u}{a}\right) + C$

16) $\frac{1}{a^2+u^2} du = \frac{1}{a} \arctan\left(\frac{u}{a}\right) + C$

17) $\frac{1}{u\sqrt{u^2-a^2}} du = \frac{1}{a} \operatorname{arcsec}\left(\frac{|u|}{a}\right) + C$