Name______________________________ Period________

BC Calculus

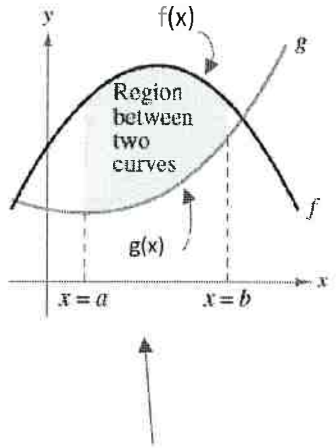
Unit 8

Area & Volume

BC Calculus November 2023 Class Calendar

Monday	Tuesday	Wednesday	Thursday	Friday
Oct 30 Ch. 6 Test FRQ Calculator Review Day 1	Oct 31 7.1 Solving Differential Equations Day 1 HW: 7.1 AP Practice #1-4 all	Nov 1 7.2 - Solving Differential Equations (Day 2) HW: 7.2 AP Practice #1-10 all	2 **Unit 6 Test (Calculator Portion)** **3 FRQs** 1.Derivative Graph 2.Riemann Sums 3. Particle Motion	3 7.3 – Slope Fields HW: **7.3 AP Practice** #17,18, 1-3 all **Unit 7 Cumulative AP Practice** #1,2,5
6 7.1-7.3 Quiz Review	7 **Election Day (No School)**	8 7.1-7.3 Quiz Review	9 8.1 Notes – Area between Graphs HW: pg. 579-582 *3,7,15,21,23,25,27* HW: pg. 579-582 8.1 AP Problems 1-10 all	10 **Chapter 7 Quiz Differential Equations and Slope Fields**
13 8.2a – Volume by Disc Method and Washer Method HW: pg. 592-596 Disc: # 5,7,8 Washer:# 10,47, 49	14 8.2b – Volume by Disc Method and Washer Method HW: 8.2 AP Practice Problems #1-10 all	15 Senior Service Day 8.4 – Volumes with Cross Sections of various Shapes HW: pg. 610-612 #1,5,7,14, AP Practice #1-8 all	16 8.1-8.4 Quiz Review HW: Ch. 8 Exercise Problems Pg. 633 *1-7 all*	17 8.1-8.4 Quiz Review
20 Thanksgiving Break	21 Thanksgiving Break	22 Thanksgiving Break	23 Thanksgiving Break	24 Thanksgiving Break

AP Calculus Ch. 8.1 – Area Between Two Curves



Vertical Orientation: (vertical rectangles between graphs)

Right bound → x_2

$$\boldsymbol{Area} = \int_{x_1}^{x_2} (Top\ graph - Bottom\ graph)\, dx$$

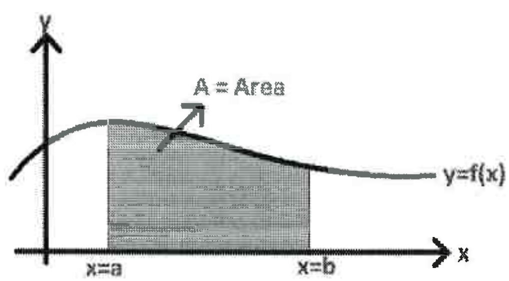
Left bound → x_1

Expressions in terms of **x**

(Equations in the form of " y = ___ ")

Example 1: Area = ______________________________

Example 2:



Area = ______________________________

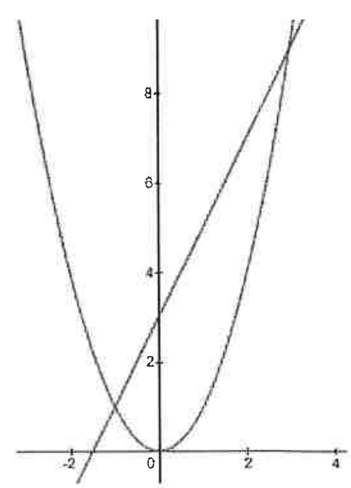
Example 3: Find the area of the region bounded by $y = x^2$ and $y = 2x + 3$

Steps:

i) **Find bounds**: Find the point of intersection between the 2 graphs (by setting equations equal, & solving for x).

ii) Identify the **top and bottom** function

iii) Apply the Integral **Area Formula.**



Horizontal Orientation: (horizontal rectangles between graphs)

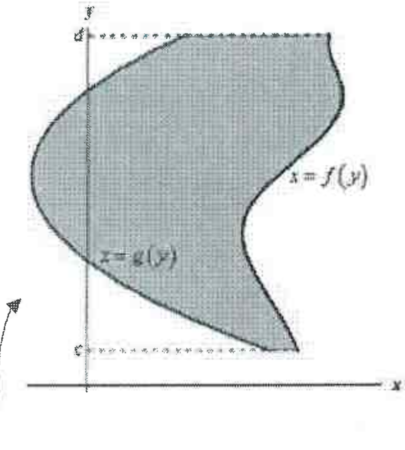
Upper bound

$$\boldsymbol{Area} = \int_{y_1}^{y_2} (Right\ graph - Left\ graph)\ dy$$

Lower bound

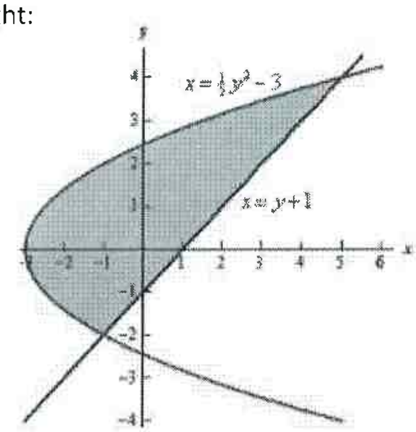
Expressions in terms of **y**

(Equations in the form of " x = ___ ")



Example 3: Area = ______________________

Example 4: Find area of the region bounded by the equations on right:



Steps:

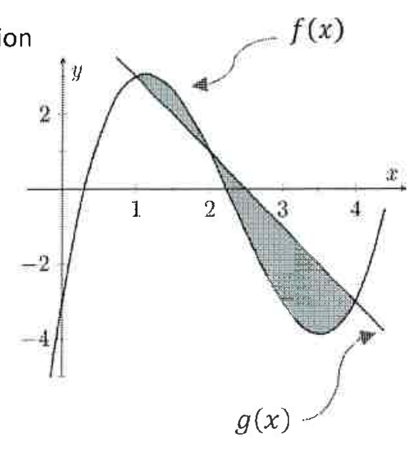
i) **Find bounds**: Find the point of intersection between the 2 graphs

(by setting equations equal, & solving for y).

ii) Identify the **right and left** function

iii) Apply the Integral **Area Formula**

Example 5: Represent the area of shaded region to the right using integral notation



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Ch. 8.1b Area between Curves Area FRQ Graphing Calculator Practice Problems

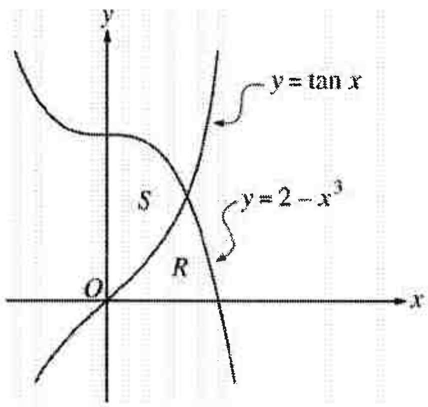
1. Let R and S be the regions in the first quadrant shown in the figure above. The region R is bounded by the x-axis and the graphs of $y = 2 - x^3$ and $y = \tan x$. The region S is bounded by the y-axis and the graphs of $y = 2 - x^3$ and $y = \tan x$.

$$Area = \int_{x_1}^{x_2} (Top\ graph - Bottom\ graph)dx \qquad \int_{y_1}^{y_2} (Right\ graph - Left\ graph)dy$$

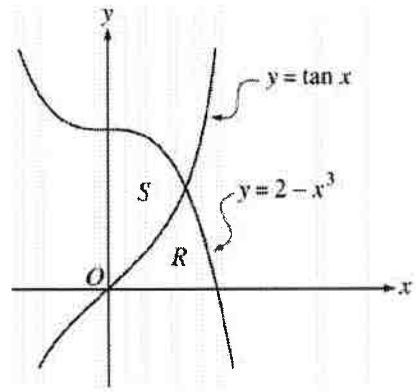
(in the forms of " y = __ ") (in the form of " x = ___ ")

a) Find the area of **S**

i) (Top – Bottom Method)

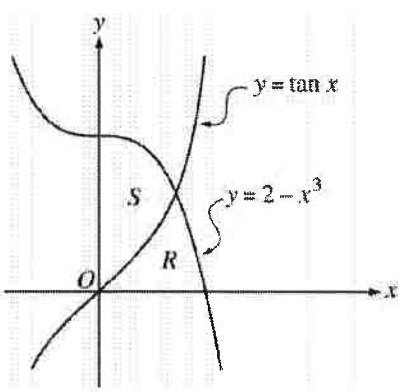


ii) (Right – Left Method)

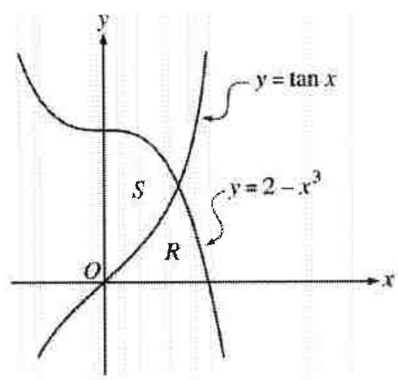


b) Find the area of **R**

i) (Top – Bottom Method)



ii) (Right – Left Method)

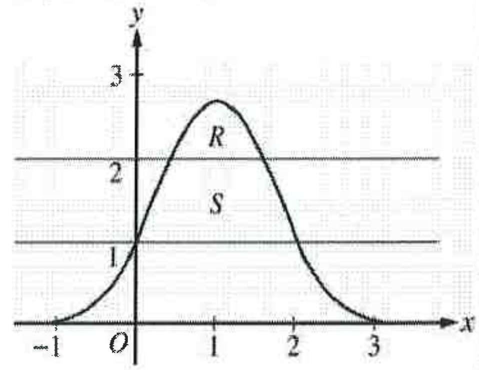


④

2) Let R be the region bounded by the graph of $y = e^{2x-x^2}$ and the horizontal line $y = 2$, and let S be the region bounded by the graph of $y = e^{2x-x^2}$ and the horizontal lines $y = 1$ and $y = 2$, as shown above.

(a) Find the area of R.

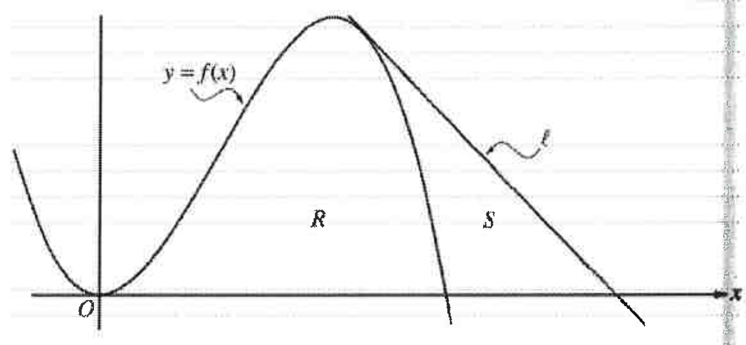
(b) Find the area of S.



3) Let f be the function given by $f(x) = 4x^2 - x^3$, and let ℓ be the line $y = 18 - 3x$, where ℓ is tangent to the graph of f. Let R be the region bounded by the graph of f and the x-axis, and let S be the region bounded by the graph of f, the line ℓ, and the x-axis, as shown above.

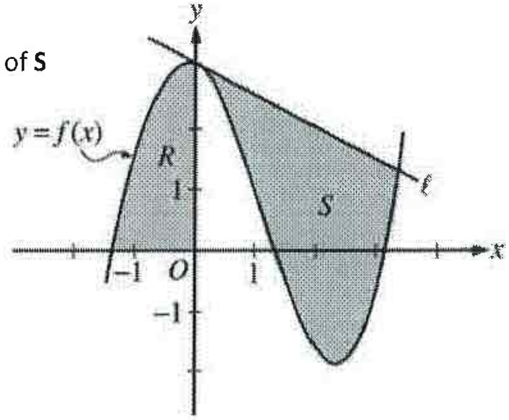
(a) Show that ℓ is tangent to the graph of $y = f(x)$ at the point $x = 3$.

(b) Find the area of S.



4) Let f be the function given by $f(x) = \frac{x^3}{4} - \frac{x^2}{3} - \frac{x}{2} + 3\cos x$. Let R be the shaded region in the second quadrant bounded by the graph of f, and let S be the shaded region bounded by the graph of f and line ℓ, the line tangent to the graph of f at $x = 0$, as shown above.

(a) Find the area of R. **(b)** Write an <u>integral expression</u> for Area of **S**



(5) Key

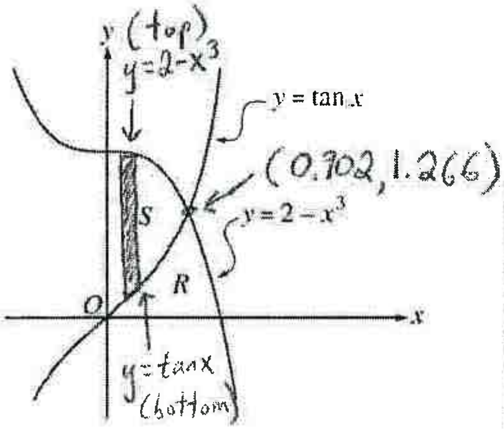
1. Let R and S be the regions in the first quadrant shown in the figure above. The region R is bounded by the x-axis and the graphs of $y = 2 - x^3$ and $y = \tan x$. The region S is bounded by the y-axis and the graphs of $y = 2 - x^3$ and $y = \tan x$.

(a) Find the area of S

$$Area = \int_{x_1}^{x_2} (Top\ \ graph - Bottom\ graph)dx \qquad \int_{y_1}^{y_2} (Right\ graph - Left\ graph)dy$$

(in the forms of "y = ___") (in the form of "x = ___")

i) (Top – Bottom Method)

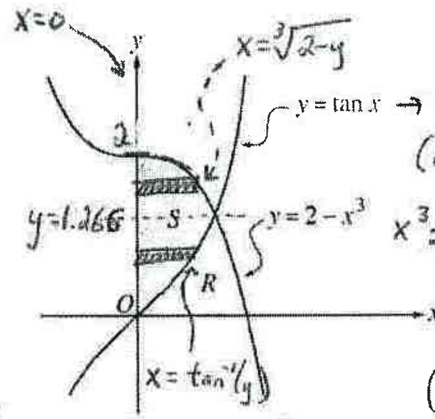


$$Area = \int_0^{0.902} 2 - x^3 - (\tan x)\,dx$$

Top − Bottom

Area = 1.161

ii) (Right – Left Method)



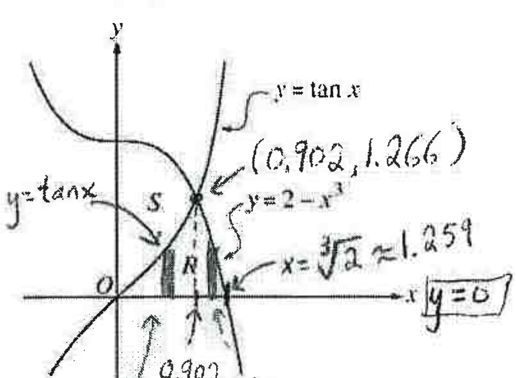
$$\int_0^{1.266} \tan^{-1}(y) - 0\,dy + \int_{1.266}^{2} \sqrt[3]{2-y} - 0\,dy$$

(Right) − (Left)   (Right) − (Left)

Area = 0.664 + 0.4965 = **1.161**

(b) Find the area of R

i) (Top – Bottom Method)

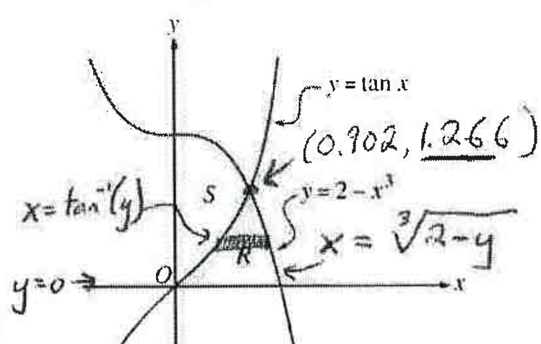


$$Area = \int_0^{0.902} \tan x - 0\,dx + \int_{0.902}^{1.259} 2 - x^3 - 0\,dx$$

(top) − (bottom)   (top) − (bottom)

Area = 0.478 + 0.251 = **0.729**

ii) (Right – Left Method)



$$Area = \int_0^{1.266} \sqrt[3]{2-y} - \tan^{-1}(y)\,dy$$

(Right) − (Left)

Area = 0.729

(6)

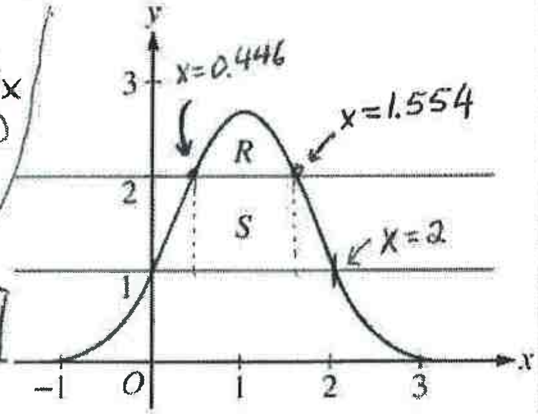
2) Let R be the region bounded by the graph of $y = e^{2x-x^2}$ and the horizontal line $y = 2$, and let S be the region bounded by the graph of $y = e^{2x-x^2}$ and the horizontal lines $y = 1$ and $y = 2$, as shown above.

(a) Find the area of R.

(b) Find the area of S.

a) $\text{Area}_{(R)} = \int_{0.446}^{1.554} e^{2x-x^2} - 2\,dx$ (top) − (bottom)

$\text{Area}_{(R)} = 0.514$



b) option 1:

$\int_0^{0.446} e^{2x-x^2} - 1\,dx + \int_{0.446}^{1.554} 2 - 1\,dx + \int_{1.554}^{2} e^{2x-x^2} - 1\,dx = 1.546$

option 2: $\int(R+S) - \int R$

$\int_0^2 e^{2x-x^2} - 1\,dx - \int_{0.446}^{1.554} e^{2x-x^2} - 2\,dx = 1.546$

(top) − (bottom)

3) Let f be the function given by $f(x) = 4x^2 - x^3$, and let ℓ be the line $y = 18 - 3x$, where ℓ is tangent to the graph of f. Let R be the region bounded by the graph of f and the x-axis, and let S be the region bounded by the graph of f, the line ℓ, and the x-axis, as shown above.

(a) Show that ℓ is tangent to the graph of $y = f(x)$ at the point $x = 3$.

(b) Find the area of S.

$f(x) = 4x^2 - x^3$

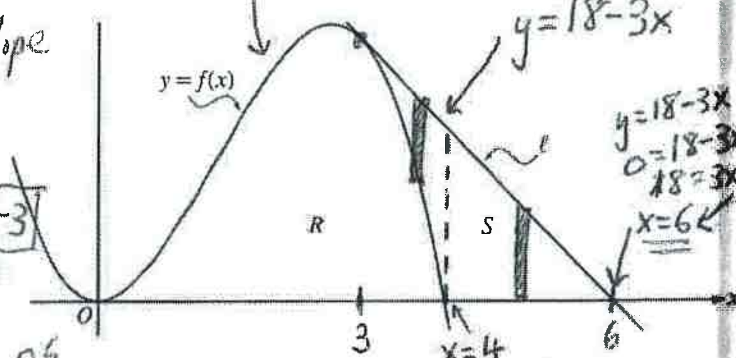
a) * Show that the graph f(x) has same slope as line $y = 18 - 3x$ at $x = 3$

* slope of graph: $f'(x) = 8x - 3x^2$

$f'(3) = 8(3) - 3(3)^2 = -3$

slope of line:

$y = -3x + 18 \rightarrow m = -3$ ← same slope



$y = 18 - 3x$

$0 = 18 - 3x$

$18 = 3x$

$x = 6$

b) $\text{Area}_{(S)} = \int_3^4 18 - 3x - (4x^2 - x^3)\,dx + \int_4^6 18 - 3x - 0\,dx$

(Top) − (Bottom) (Top) − (Bottom)

$0 = 4x^2 - x^3$

$0 = x^2(4 - x)$

$x = 4$

$\text{Area}(S) = 1.917 + 6 = 7.917$

4) Let f be the function given by $f(x) = \frac{x^3}{4} - \frac{x^2}{3} - \frac{x}{2} + 3\cos x$. Let R be the shaded region in the second quadrant bounded by the graph of f, and let S be the shaded region bounded by the graph of f and line ℓ, the line tangent to the graph of f at $x = 0$, as shown above.

(a) Find the area of R.

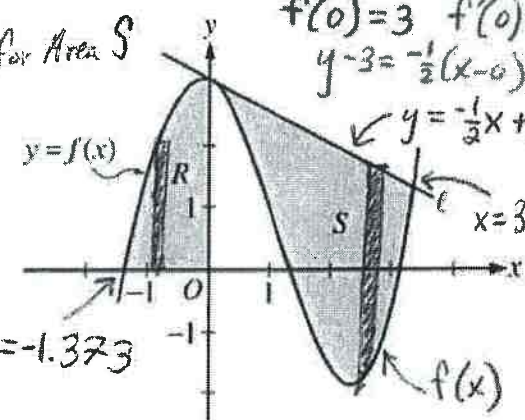
b) Write integral expression for Area S

$f(0) = 3$ $f'(0) = -\frac{1}{2}$

$y - 3 = -\frac{1}{2}(x - 0)$

$y = -\frac{1}{2}x + 3$

a) $\int_{-1.373}^{0} f(x) - 0\,dx = 2.903$

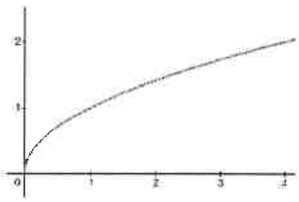


b) Area of S: $\int_0^{3.389} -\frac{1}{2}x + 3 - f(x)\,dx$

(top) − (bottom)

Calculus Ch.3.2a: Volume by Disc Method

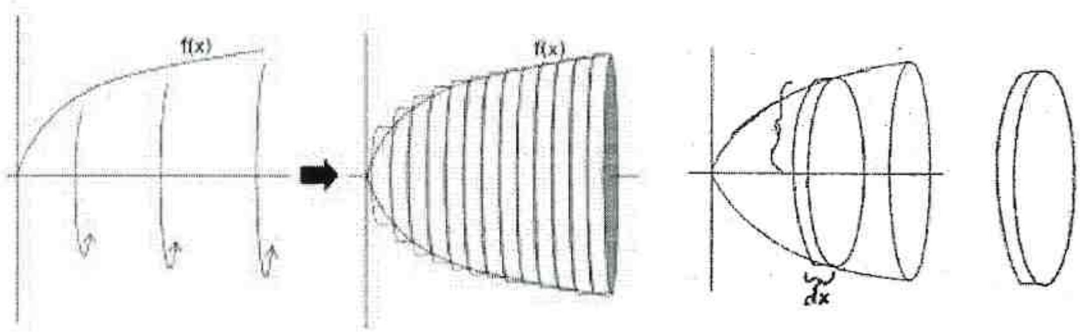
Recall finding area under the curve $y = \sqrt{x}$ between [0, 4]. $Area = \int_a^b (Top\ graph - bottom\ graph)dx$



*Essentially, the Integral Notation allows us to add infinite numbers of differently sized rectangles to form area calculation.

With **Disc Method**, we are going to take this region created by f(x) and the x-axis and rotate this function 360° around the x-axis. What shapes do you see if we were to separate the resulting object into thin slices?

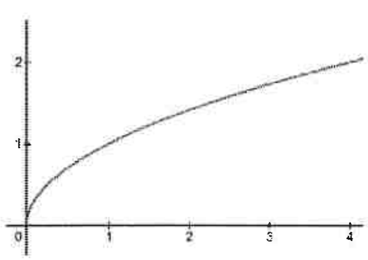
____

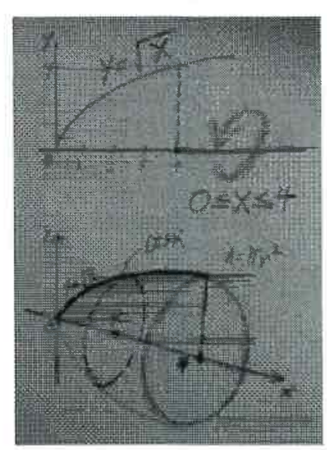


Disc Method: (Top – Bottom) – Vertical Radius	Disc Method: (Right – Left) – Horizontal Radius
$V = \pi \int_{x_1}^{x_2} [R(x)]^2\, dx$	$V = \pi \int_{y_1}^{y_2} [R(y)]^2\, dy$
(expression(s) used above has form: " y = ___")	(expression(s) used above has form: " x = ___")

Radius [R(x) or R(y)] - distance from the AOR (Axis of Revolution) to the **boundary** of shaded region

Example 1: Find the volume of the solid formed by rotating the curve $y = \sqrt{x}$ around the x-axis between [0, 4]





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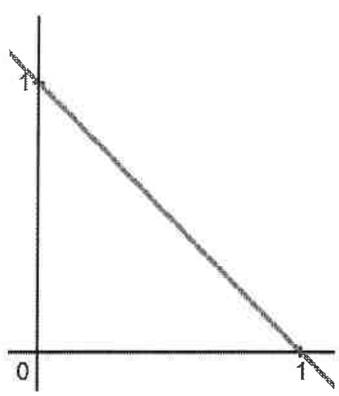
Disc Method: (Top – Bottom) – Vertical Radius – Horizontal AOR	**Disc Method: (Right – Left) – Horizontal Radius Vertical AOR**
$V = \pi \int_{x_1}^{x_2} [R(x)]^2 \, dx$	$V = \pi \int_{y_1}^{y_2} [R(y)]^2 \, dy$
(expression(s) used above has form: " y = ___")	(expression(s) used above has form: " x = ___")

Radius [R(x) or R(y)] : distance from the AOR(Axis of Revolution) to the **outer boundary** of shaded region

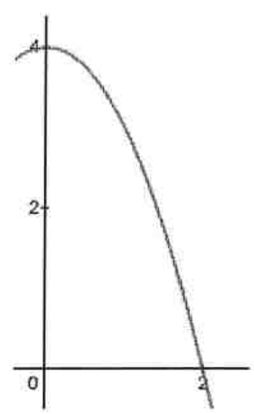
Example 2: Find the volume of the solid created by $f(x) = 2 - x^2$ revolved about the line y = 1.

Example 3: Given the region is formed by the function, x-axis, and y-axis. Find the volume of the solid formed by revolving the region about the **y-axis**

a) $y = -x + 1$



b) $y = 4 - x^2$

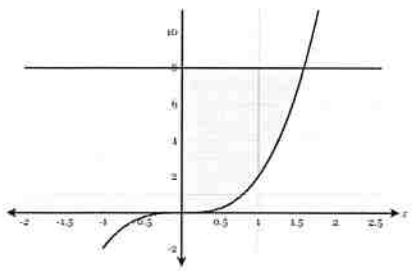


8.2a Disc Method Practice Problems Worksheet

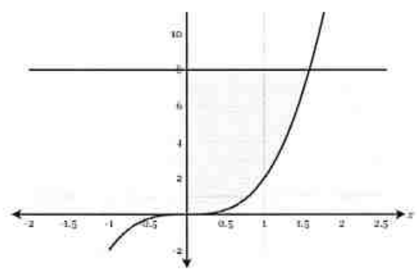
Disc Method: (Top – Bottom)	Disc Method: (Right – Left)
$V = \pi \int_{x_1}^{x_2} [R(x)]^2 \, dx$	$V = \pi \int_{y_1}^{y_2} [R(y)]^2 \, dy$
(expression(s) used above has form: " y = ___ ")	(expression(s) used above has form: " x = ___ ")

1. Let the region R be the area enclosed the function $f(x) = 2x^3$ the horizontal line y=8, and the y-axis. Find the volume of the solid generated when the shaded region is:

a) rotated about the line y = 8

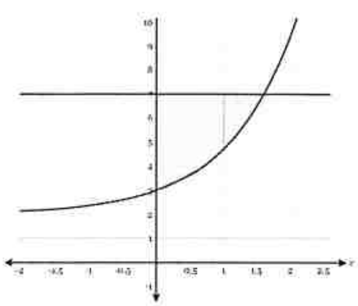


b) rotated about the y-axis

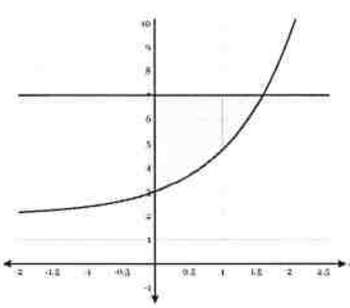


2) Let the region R be the area enclosed the function $f(x) = e^x + 2$, the horizontal line y=7, and the y-axis. Find the volume of the solid generated when the shaded region is:

a) rotated about the line y = 7



b) rotated about the y-axis



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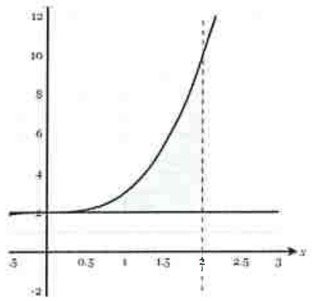
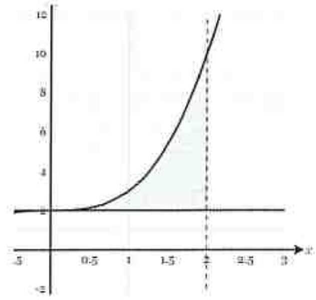
Disc Method: (Top – Bottom)	**Disc Method: (Right – Left)**
$V = \pi \int_{x_1}^{x_2} [R(x)]^2 \, dx$	$V = \pi \int_{y_1}^{y_2} [R(y)]^2 \, dy$
(expression(s) used above has form: " y = ____")	(expression(s) used above has form: " x = ____")

3) Let the region R be the area enclosed by the function $f(x) = x^3 + 2$, the horizontal line y=2, and the vertical lines x=0 and x=2. Find the volume of the solid generated when shaded region is:

a) rotated about the line y = 2

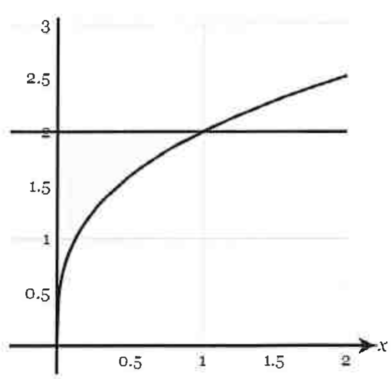


b) rotated about x = 2

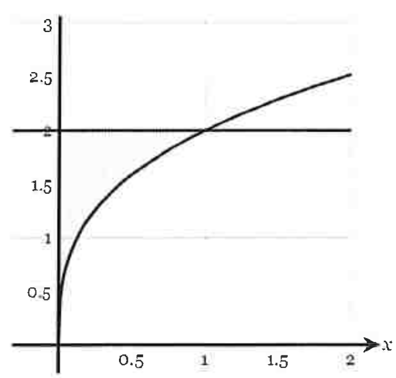


4. Let the region R be the area enclosed the function $f(x) = 2x^{\frac{1}{3}}$, the horizontal line y=2, and the y-axis. Find the volume of the solid generated when shaded region is

a) rotated about the line y = 2



b) rotated about y-axis



8.2a Disc Method Practice Problems Worksheet

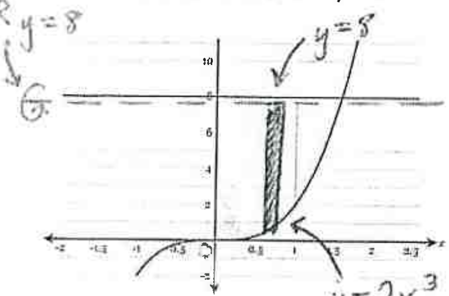
Key (11)

Disc Method: (Top – Bottom)	Disc Method: (Right – Left)
$V = \pi \int_{x_1}^{x_2} [R(x)]^2\, dx$	$V = \pi \int_{y_1}^{y_2} [R(y)]^2\, dy$
(expression(s) used above has form: " y = ____ ")	(expression(s) used above has form: " x = ____ ")

1. Let the region R be the area enclosed the function $f(x) = 2x^3$ the horizontal line y=8, and the y-axis. Find the volume of the solid generated when the shaded region is:

a) rotated about the line y = 8

AOR y=8



$R(x) = 8 - 2x^3$

* intersection:

$2x^3 = 8$

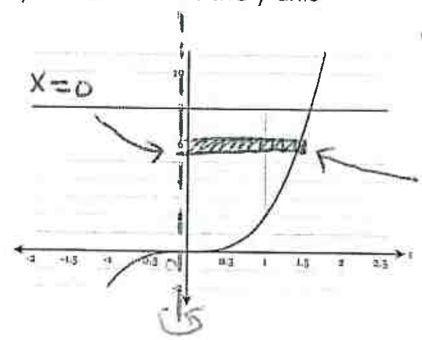
$x^3 = 4$

$x = \sqrt[3]{4} \approx 1.587$

$V = \pi \int_0^{1.587} [8 - 2x^3]^2\, dx$

$V = 65.310\pi$ units^3

b) rotated about the y-axis



$y = 2x^3$

$\frac{y}{2} = x^3$

$\sqrt[3]{\frac{y}{2}} = x$

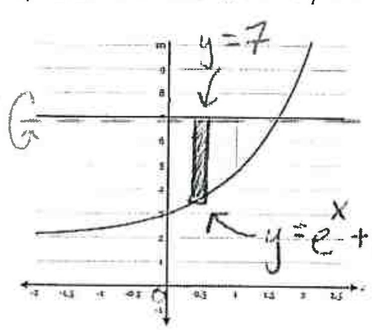
$R(y) = \sqrt[3]{\frac{y}{2}} - 0 = \sqrt[3]{\frac{y}{2}}$

$V = \pi \int_0^8 \left[\sqrt[3]{\frac{y}{2}}\right]^2 dy = 12.095\pi$ units^3

2) Let the region R be the area enclosed the function $f(x) = e^x + 2$, the horizontal line y=7, and the y-axis. Find the volume of the solid generated when the shaded region is:

a) rotated about the line y = 7

y=7



$V = \pi \int_0^{\ln 5} [5 - e^x]^2\, dx$

$V = 12.236\pi$ units^3

$R(x) = 7 - (e^x + 2) = 5 - e^x$

* intersection:

$e^x + 2 = 7$

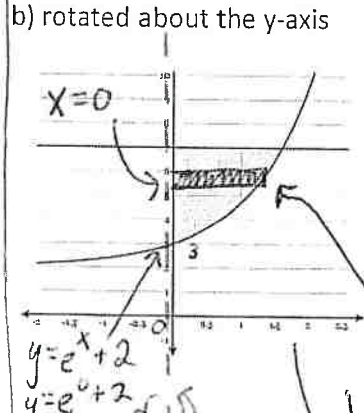
$e^x = 5$

$\ln e^x = \ln 5$

$x \ln e = \ln 5$

$x = \ln 5$

b) rotated about the y-axis



$y = e^x + 2$

$y - 2 = e^x$

$\ln(y-2) = \ln e^x$

$\ln(y-2) = x \cdot \ln e$

$x = \ln(y-2)$

$V = \pi \int_3^7 [\ln(y-2)]^2\, dy$

$R(y) = \ln(y-2) - 0$

$R(y) = \ln(y-2)$

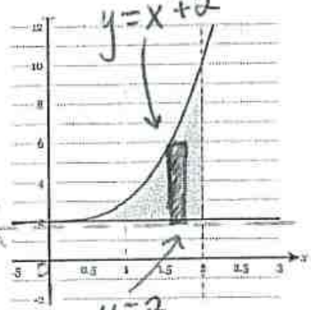
$V = 4.857\pi$ units^3

(12)

Disc Method: (Top – Bottom)	Disc Method: (Right – Left)
$V = \pi \int_{x_1}^{x_2} [R(x)]^2 \, dx$	$V = \pi \int_{y_1}^{y_2} [R(y)]^2 \, dy$
(expression(s) used above has form: " y = ____ ")	(expression(s) used above has form: " x = ____ ")

3) Let the region R be the area enclosed by the function $f(x) = x^3 + 2$, the horizontal line y=2, and the vertical lines x=0 and x=2. Find the volume of the solid generated when shaded region is:

a) rotated about the line y = 2



$R(x) = x^3 + 2 - (2) = x^3$

$V = \pi \int_0^2 [x^3]^2 \, dx$

$V = \frac{128}{7}\pi$ units^3

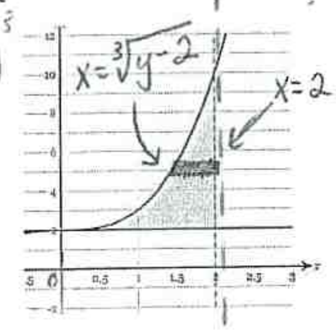
*intersection:

$(\sqrt[3]{y-2})^3 = (2)^3$

$y - 2 = 8$

$y = 10$

b) rotated about x = 2



$f(x) = x^3 + 2$

$y = x^3 + 2$

$y - 2 = x^3$

$\sqrt[3]{y-2} = \sqrt[3]{x^3}$

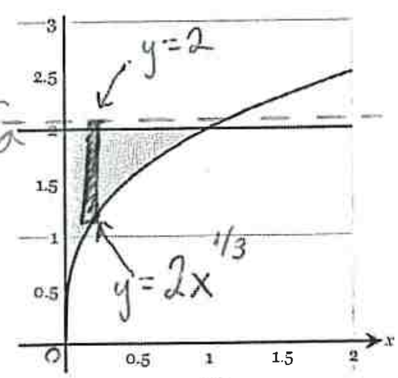
$\sqrt[3]{y-2} = x$

$R(y) = 2 - \sqrt[3]{y-2}$

$V = \pi \int_2^{10} \left[2 - \sqrt[3]{y-2}\right]^2 dy = 3.199\pi$ units^3

4. Let the region R be the area enclosed the function $f(x) = 2x^{\frac{1}{3}}$, the horizontal line y=2, and the y-axis. Find the volume of the solid generated when shaded region is

a) rotated about the line y = 2

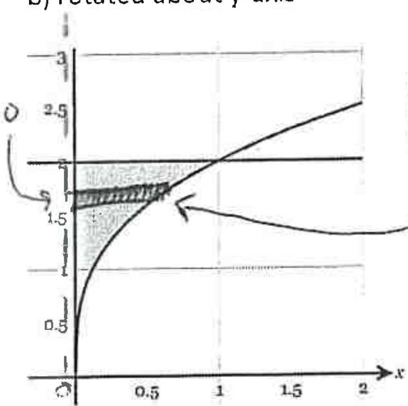


$R(x) = 2 - 2x^{1/3}$

$V = \pi \int_0^1 \left[2 - 2x^{1/3}\right]^2 dx$

$V = 0.399\pi$ units^3

b) rotated about y-axis



$y = 2x^{1/3}$

$\left(\frac{y}{2}\right)^3 = \left(x^{1/3}\right)^3$

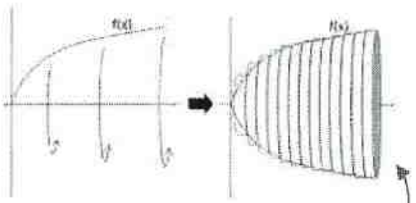
$\frac{y^3}{8} = x$

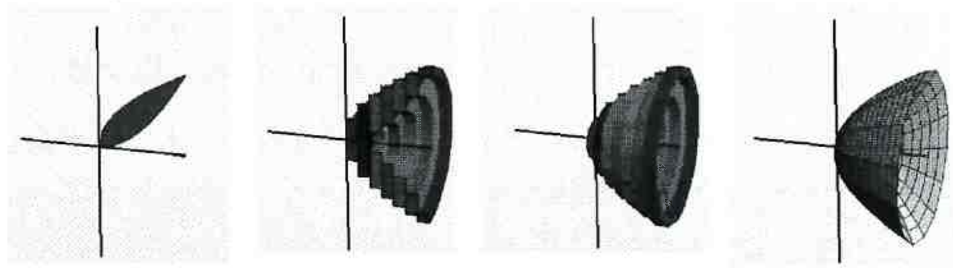
$R(y) = \frac{y^3}{8} - 0 = \frac{y^3}{8}$

$V = \pi \int_0^2 \left(\frac{y^3}{8}\right)^2 dy = \frac{2}{7}\pi$ units^3

AP Calculus Ch. 8.2b: Volume by Washer Method Notes

Reviewing **Disc Method** | Illustration of **Washer Method**





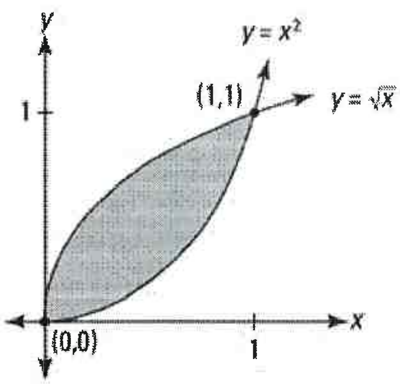
Disc Method: $V = \pi \int_{x_1}^{x_2} [R(x)]dx$

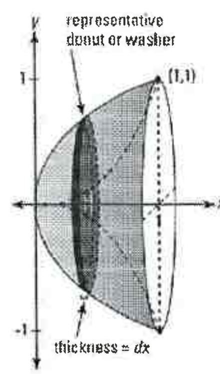
Washer Method: (Top – Bottom) , Vertical Radius (Horizontal AOR)	Washer Method: (Right – Left) , Horizontal Radius (Vertical AOR)
$V = \pi \int_{x_1}^{x_2} [R(x)]^2 - [r(x)]^2 \, dx$	$V = \pi \int_{y_1}^{y_2} [R(y)]^2 - [r(y)]^2 dy$
(expression(s) used above has form: " y = ___ ")	(expression(s) used above has form: " x = ___ ")

Radius [R(x) or R(y)] - distance from the AOR (Axis of Revolution) to the **outer**(further)curve

radius[r(x) or r(y)] - distance from the AOR (Axis of Revolution) to the **inner**(closer) curve

Example 1: Find the volume of the solid enclosed by the graphs of $y = x^2$ and $y = \sqrt{x}$, and revolving about the x-axis.





Example 2: Find the volume of the solid created by revolving the function y = x² + 1 bounded by the line y = 2 revolved about the x-axis.

(14)

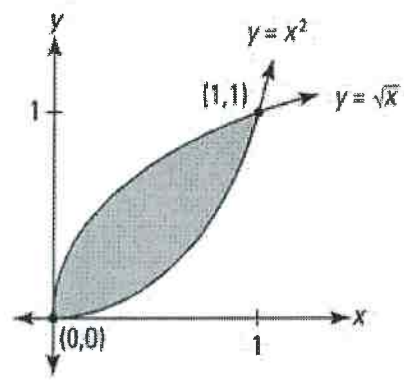
Washer Method: (Top – Bottom), Vertical Radius (Horizontal AOR)	Washer Method: (Right – Left), Horizontal Radius (Vertical AOR)
$V = \pi \int_{x_1}^{x_2} [R(x)]^2 - [r(x)]^2\, dx$	$V = \pi \int_{y_1}^{y_2} [R(y)]^2 - [r(y)]^2 dy$
(expression(s) used above has form: " y = ___ ")	(expression(s) used above has form: " x = ___ ")

Radius [R(x) or R(y)] - distance from the AOR (Axis of Revolution) to the **outer**(further)curve

radius[r(x) or r(y)] - distance from the AOR (Axis of Revolution) to the **inner**(closer) curve

Example 3: Find the volume of the solid created by revolving the function $y = x^2 + 1$ bounded by the line $y = 2$ and the y-axis about the line $y = 4$

Example 4: Find the volume of the solid created enclosed region of $y = x^2$ and $y = \sqrt{x}$ revolving about the line $x = -2$

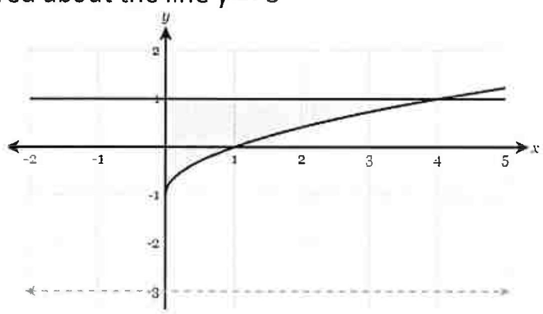


8.2b Volume - Washer Method Practice Problems Worksheet

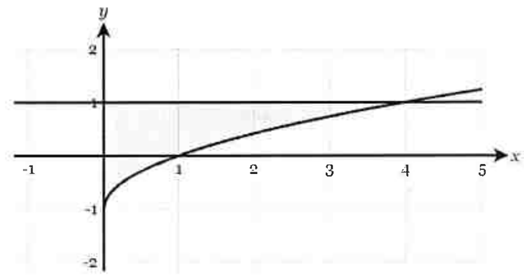
Washer Method: (Top – Bottom) – Vertical Radius	Washer Method: (Right – Left) – Horizontal Radius
$V = \pi \int_{x_1}^{x_2} [R(x)]^2 - [r(x)]^2 \, dx$	$V = \pi \int_{y_1}^{y_2} [R(y)]^2 - [r(y)]^2 dy$
(expression(s) used above has form: " y = ____ ")	(expression(s) used above has form: " x = ____ ")

1. Let the region R be the area enclosed the the function $f(x) = \sqrt{x} - 1$, the horizontal line y=1, and the *y*-axis. Find the volume of the solid generated when the region is:

a) revolved about the line y = -3

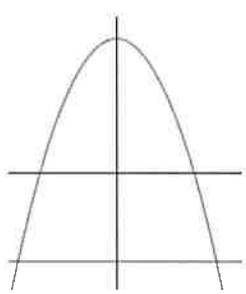


b) revolved about the line x = - 1

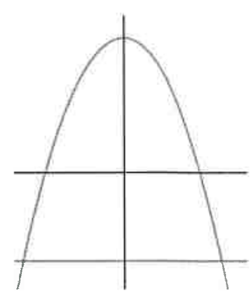


2. Let the region R be the area enclosed the the function $f(x) = 3 - x^2$ the line y = -2. Find the volume of the solid generated when the region is:

a) revolved about the line y = 3



b) revolved about the line y = - 2



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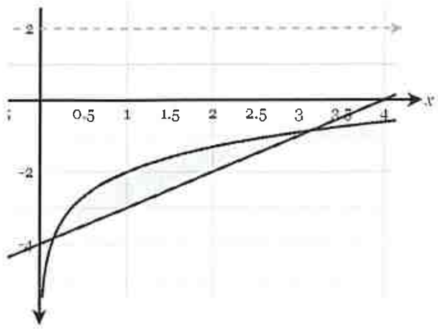
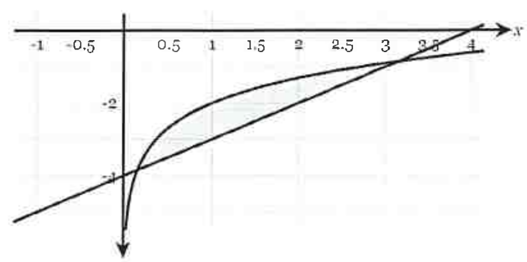
Washer Method: (Top – Bottom) – Vertical Radius	Washer Method: (Right – Left) – Horizontal Radius
$V = \pi \int_{x_1}^{x_2} [R(x)]^2 - [r(x)]^2 \, dx$	$V = \pi \int_{y_1}^{y_2} [R(y)]^2 - [r(y)]^2 dy$
(expression(s) used above has form: " y = ___")	(expression(s) used above has form: " x = ___")

3. Let the region R be the area enclosed the function $f(x) = \ln x - 2$ and $g(x) = x - 4$. Find the volume of the solid generated when the region is:

a) revolved about the line $y = 2$

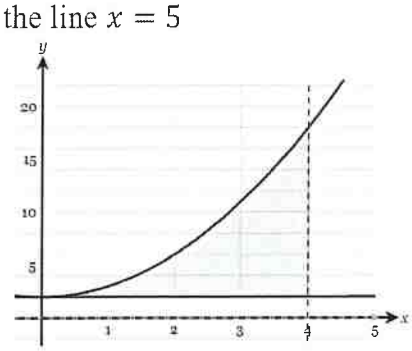


b) revolved about the line $x = -1$

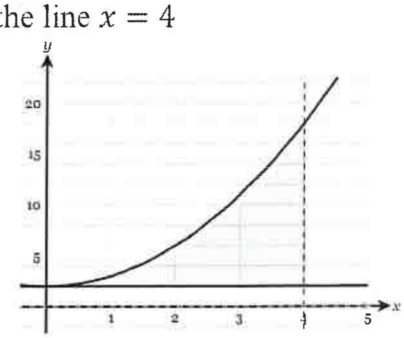


4. Let the region R be the area enclosed by the function $f(x) = x^2 + 2$, the horizontal line y=2, & the vertical lines x=0 & x=4. Find volume of the solid generated when region is:

a) revolved about the line $x = 5$



b) revolved about the line $x = 4$



8.2b Volume - Washer Method Practice Problems Worksheet

Key (17)

Washer Method: (Top – Bottom) – Vertical Radius	Washer Method: (Right – Left) – Horizontal Radius
$V = \pi \int_{x_1}^{x_2} [R(x)]^2 - [r(x)]^2\, dx$	$V = \pi \int_{y_1}^{y_2} [R(y)]^2 - [r(y)]^2 dy$
(expression(s) used above has form: " y = ___ ")	(expression(s) used above has form: " x = ___ ")

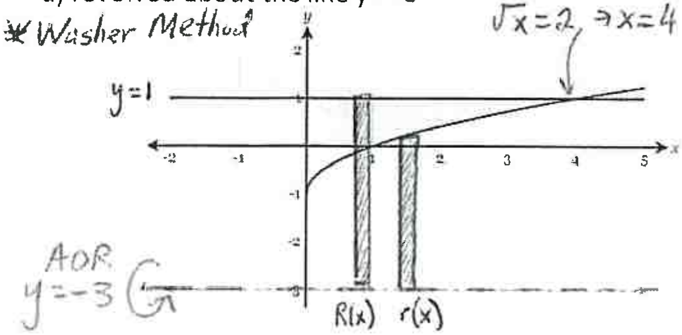
1) Let the region R be the area enclosed the the function $f(x) = \sqrt{x} - 1$, the horizontal line y=1, and the y-axis. Find the volume of the solid generated when the region is:

a) revolved about the line y = -3

*Washer Method

$\sqrt{x} - 1 = 1$

$\sqrt{x} = 2 \Rightarrow x = 4$



$R(x) = 1 - (-3) = 4$

$r(x) = \sqrt{x} - 1 - (-3) = \sqrt{x} + 2$

$V = \pi \int_0^4 [4]^2 - [\sqrt{x}+2]^2 dx$

$V = 18.667\pi$ units^3

b) revolved about the line x = - 1

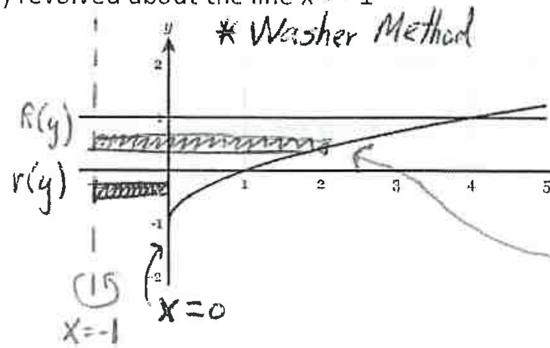
*Washer Method

$y = \sqrt{x} - 1$

$y + 1 = \sqrt{x}$

$(y+1)^2 = (\sqrt{x})^2$

$x = (y+1)^2$



$R(y) = (y+1)^2 - (-1) = (y+1)^2 + 1$

$r(y) = 0 - (-1) = 1$

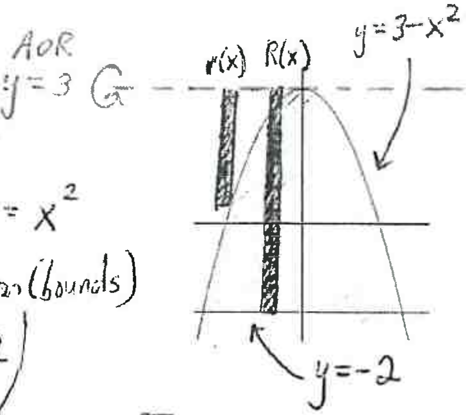
$V = \pi \int_{-1}^{1} [(y+1)^2 + 1]^2 - [1]^2 dy$

$V = \frac{176}{15}\pi$ units^3

2) Let the region R be the area enclosed the the function $f(x) = 3 - x^2$ the line y = -2. Find the volume of the solid generated when the region is:

a) revolved about the line y = 3

*Washer Method



$R(x) = 3 - (-2) = 5$

$r(x) = 3 - (3 - x^2) = x^2$

*Find intersection (bounds)

$3 - x^2 = -2$

$5 = x^2$

$\sqrt{5} = \sqrt{x^2}$

$\pm\sqrt{5} = x$

$V = \pi \int_{-\sqrt{5}}^{\sqrt{5}} (5)^2 - (x^2)^2 dx$

$V = 89.443\pi$ units^3

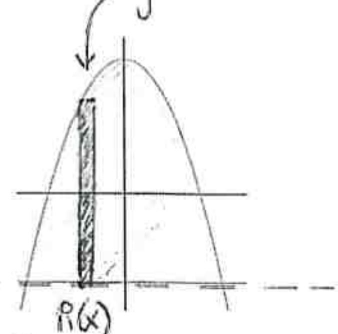
b) revolved about the line y = - 2

*Disc Method

$R(x) = 3 - x^2 - (-2)$

$R(x) = 5 - x^2$

$V = \pi \int_{-\sqrt{5}}^{\sqrt{5}} [5 - x^2]^2 dx$



$V = 59.628\pi$ units^3

*intersections:

$3 - x^2 = -2$

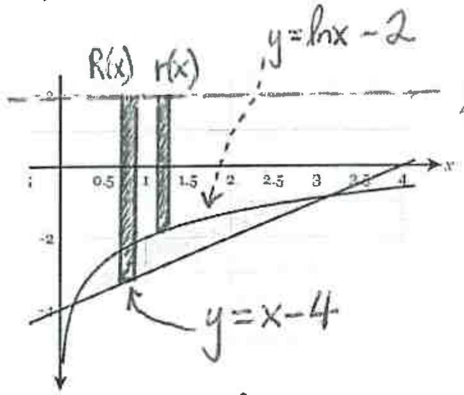
$x = \pm\sqrt{5}$

(18)

Washer Method: (Top – Bottom) – Vertical Radius	Washer Method: (Right – Left) – Horizontal Radius
$V = \pi \int_{x_1}^{x_2} [R(x)]^2 - [r(x)]^2 \, dx$	$V = \pi \int_{y_1}^{y_2} [R(y)]^2 - [r(y)]^2 dy$
(expression(s) used above has form: " y = ___ ")	(expression(s) used above has form: " x = ___ ")

3) Let the region R be the area enclosed the function $f(x) = \ln x - 2$ and $g(x) = x - 4$. Find the volume of the solid generated when the region is:

a) revolved about the line $y = 2$ *Washer Method



$R(x) = 2-(x-4) = 2-x+4 = 6-x$

$r(x) = 2-(\ln x-2) = 2-\ln x+2 = 4-\ln x$

*find bounds:

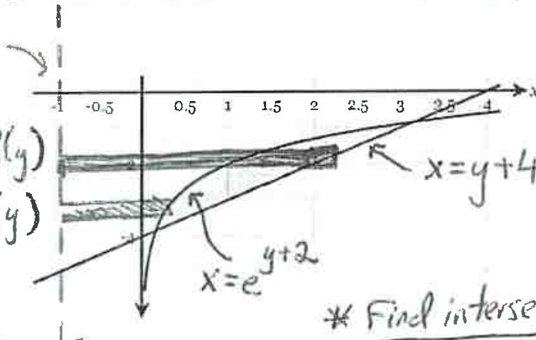
set $\ln x - 2 = x - 4$

$x = 0.1586, x = 3.146$

$V = \pi \int_{0.159}^{3.146} (6-x)^2 - (4-\ln x)^2 dx$

$V = 16.402\pi$ units^3

b) revolved about the line x = –1 *Washer Method



AOR x=-1

*Rewrite equations:

$y = \ln x - 2$ | $y = x - 4$

$y+2 = \ln x$ | $y+4 = x$

$e^{y+2} = e^{\ln x}$

$e^{y+2} = x$

$x = e^{y+2}$

*Find intersections:

$e^{y+2} = y+4$

$y = -0.853, y = -3.8$

$R(y) = y+4-(-1) = y+5$

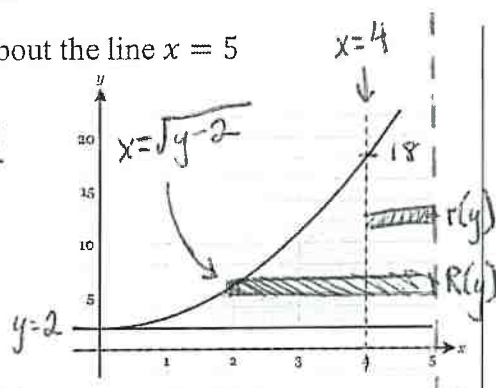
$r(y) = e^{y+2}-(-1) = e^{y+2}+1$

$V = \pi \int_{-3.841}^{-0.853} (y+5)^2 - (e^{y+2}+1)^2 dy$

$V = 9.341\pi$ units^3

4) Let the region R be the area enclosed by the function $f(x) = x^2 + 2$, the horizontal line y=2, & the vertical lines x=0 & x=4. Find volume of the solid generated when region is:

a) revolved about the line $x = 5$ *Washer Method



*rewrite equation:

$y = x^2+2$

$y-2 = x^2$

$\pm\sqrt{y-2} = \sqrt{x^2}$

$x = \sqrt{y-2}$

*find intersection:

$\sqrt{y-2} = 4$

$(\sqrt{y-2})^2 = (4)^2$

$y-2 = 16$

$y = 18$

AOR x=5

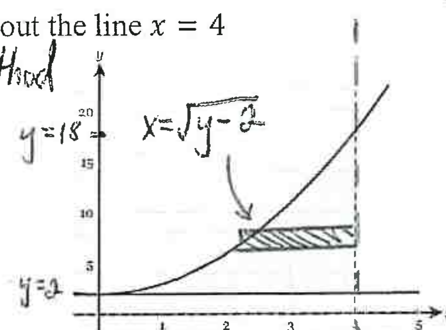
$R(y) = 5-\sqrt{y-2}$

$r(y) = 5-(4) = 1$

$V = \pi \int_2^{18} [5-\sqrt{y-2}]^2 - [1]^2 dy$

$V = 85.333\pi$ units^3

b) revolved about the line $x = 4$ *Disc Method



AOR x=4

$R(y) = 4-\sqrt{y-2}$

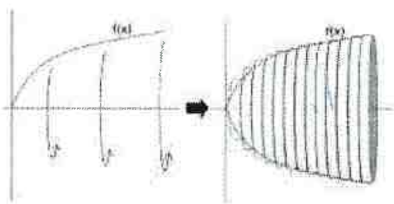
$V = \pi \int_2^{18} [4-\sqrt{y-2}]^2 dy$

$V = 42.667\pi$ units^3

AP Calculus Ch. 8.4 Volumes with Known Cross Section

Cross-section is the shape that results if we cut the object in half and look at the resulting shape sideways:

Disc Method

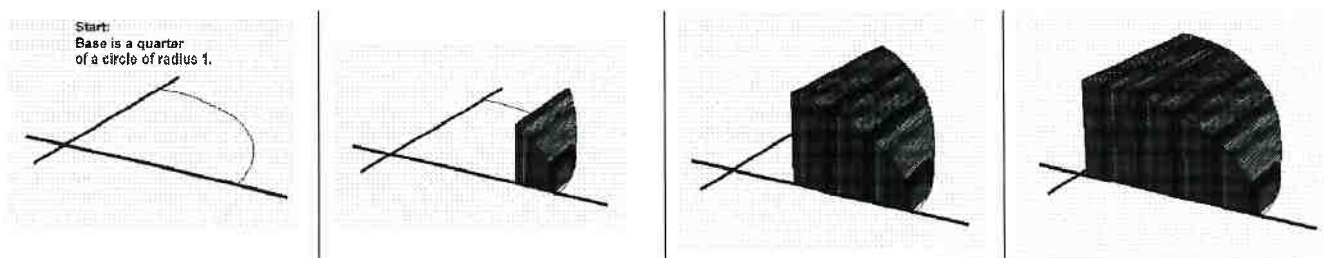


Washer Method

$V = \pi \int [Area \text{ of cross section}]\, dx$

The volume problems we have covered so far(Disc, Washer) have involved taking the shaded region and rotate it about a center, (AOR), resulting in circular Area, either $Area = \pi[R(x)]^2$ or $Area = \pi[R(x)]^2 - \pi[r(x)]^2$

Now, we still have a shaded region, but now it will act as the base of the 3D object. We will now build the cross-section area on top of this base. (no longer rotating around an axis)



Top-Bottom Vertical base	Right-Left Horizontal base
$V = \int_{x_1}^{x_2} [\textbf{Area of cross section}]\, dx$	$V = \int_{y_1}^{y_2} [\textbf{Area of cross section}]\, dy$
*Note: All values in integral are in terms of x (in the form of " y = ____")	*Note: All values in integral are in terms of y (in the forms of " x = _______")

Area formulas for Cross sections:

1. Square: $A = (base)^2$	2. Isosceles Right Triangle (**leg** on base): $A = \frac{1}{2}(base)^2$	3. Isosceles Right Triangle (**hypotenuse** on base): $A = \frac{1}{4}(base)^2$
4. Rectangle: A = (base)(height)	5. Equilateral Triangle: $A = \frac{\sqrt{3}}{4}(base)^2$	6. Semicircle: $A = \frac{\pi}{8}(base)^2$

Example 1: Find the volume of the solid if the base is bounded by curve $y = \sqrt{1 - x^2}$, y = 0, x = 0 (in the first quadrant) and the cross sections are squares parallel to the y-axis.

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Top-Bottom Vertical base	Right-Left Horizontal base
$V = \int_{x_1}^{x_2} [\textit{Area} \textbf{ of cross section}]dx$ *Note: All values in integral are in terms of x (equations in the form of " y = ____ ")	$V = \int_{y_1}^{y_2} [\textit{Area} \textbf{ of cross section}]dy$ *Note: All values in integral are in terms of y (equations in the form of " x = ____ ")

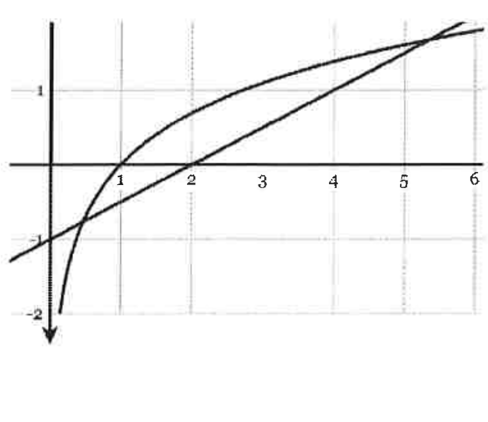
Area formulas for Cross sections:

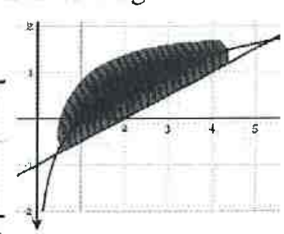
1. Square: $A = (base)^2$	2. Isosceles Right Triangle (**leg** on base): $A = \frac{1}{2}(base)^2$	3. Isosceles Right Triangle (**hypotenuse** on base): $A = \frac{1}{4}(base)^2$
4. Rectangle: A = (base)(height)	5. Equilateral Triangle: $A = \frac{\sqrt{3}}{4}(base)^2$	6. Semicircle: $A = \frac{\pi}{8}(base)^2$

Example 2: Find the volume of the solid if the base is bounded by the curve $y = x^2$ and the line $y = 4$ and the cross sections are isosceles right triangles whose hypotenuse lie on the base and are parallel to the x-axis.

Example 3: Find the volume of the solid if the base is bounded by the curve $y = x^2$ and the line $y = 4$ and the cross sections are semicircles whose base are perpendicular to the y-axis.

Example 4: Let the region R be the area enclosed by the function $f(x) = \ln x$ and $g(x) = \frac{1}{2}x - 1$. If the region R is the base of a solid such that each cross section perpendicular to the x-axis is an isosceles right triangle with a leg in the region R, find the volume of the solid.

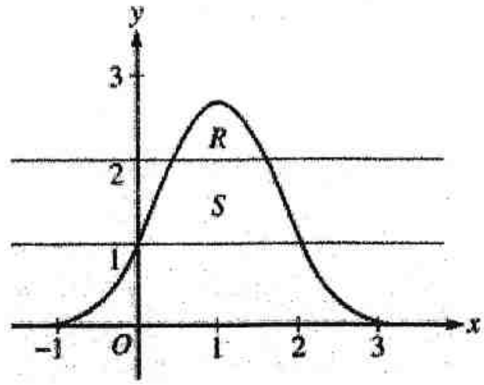




Ch. 8 Area/Volume   FRQ Problems Worksheet #1

1)

Let R be the region bounded by the graph of $y = e^{2x-x^2}$ and the horizontal line $y = 2$, and let S be the region bounded by the graph of $y = e^{2x-x^2}$ and the horizontal lines $y = 1$ and $y = 2$, as shown above.



(a) Find the area of R.

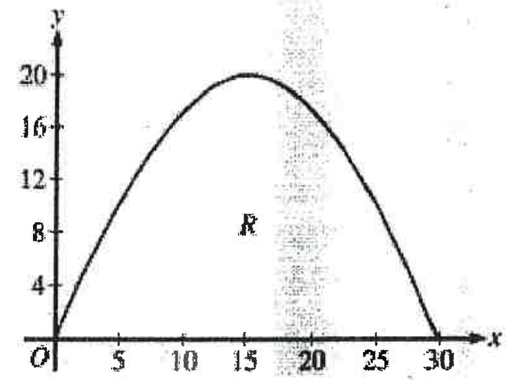
(b) Find the area of S.

(c) Write, but do not evaluate, an integral expression that gives the volume of the solid generated when R is rotated about the horizontal line $y = 1$.

22

2)

A baker is creating a birthday cake. The base of the cake is the region R in the first quadrant under the graph of $y = f(x)$ for $0 \le x \le 30$, where $f(x) = 20\sin\left(\frac{\pi x}{30}\right)$. Both x and y are measured in centimeters. The region R is shown in the figure above. The derivative of f is $f'(x) = \frac{2\pi}{3}\cos\left(\frac{\pi x}{30}\right)$.

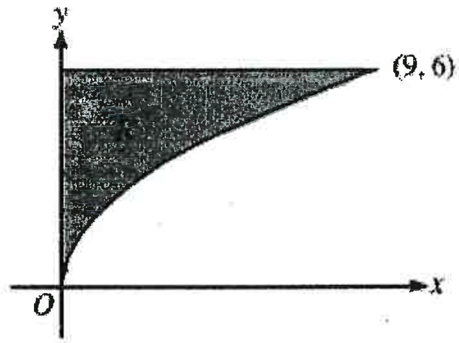


(a) The region R is cut out of a 30-centimeter-by-20-centimeter rectangular sheet of cardboard, and the remaining cardboard is discarded. Find the area of the discarded cardboard.

(b) The cake is a solid with base R. Cross sections of the cake perpendicular to the x-axis are semicircles. If the baker uses 0.05 gram of unsweetened chocolate for each cubic centimeter of cake, how many grams of unsweetened chocolate will be in the cake?

23

3) (Non-Calculator)



Let R be the region in the first quadrant bounded by the graph of $y = 2\sqrt{x}$, the horizontal line $y = 6$, and the y-axis, as shown in the figure above.

(a) Find the area of R.

(b) Write, but do not evaluate, an integral expression that gives the volume of the solid generated when R is rotated about the horizontal line $y = 7$.

(c) Region R is the base of a solid. For each y, where $0 \le y \le 6$, the cross section of the solid taken perpendicular to the y-axis is a rectangle whose height is 3 times the length of its base in region R. Write, but do not evaluate, an integral expression that gives the volume of the solid.

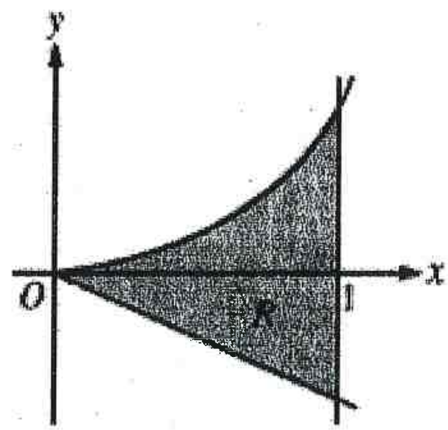
(24)

4)

Let R be the shaded region bounded by the graph of $y = xe^{x^2}$, the line $y = -2x$, and the vertical line $x = 1$, as shown in the figure above.

(a) Find the area of R.

(b) Write, but do not evaluate, an integral expression that gives the volume of the solid generated when R is rotated about the horizontal line $y = -2$.



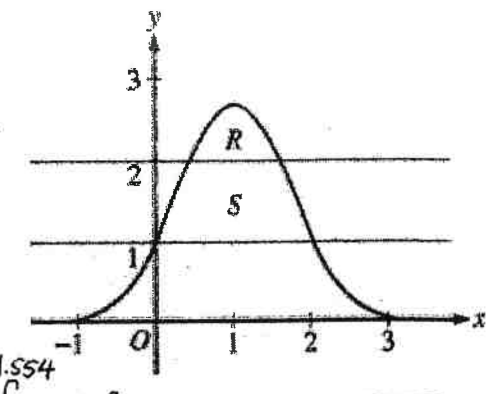
Ch. 8 Area/Volume FRQ Problems Worksheet #1

Key

(25)

1)

Let R be the region bounded by the graph of $y = e^{2x-x^2}$ and the horizontal line $y = 2$, and let S be the region bounded by the graph of $y = e^{2x-x^2}$ and the horizontal lines $y = 1$ and $y = 2$, as shown above.

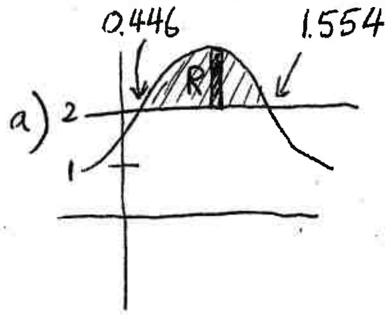


(a) Find the area of R.

(b) Find the area of S.

(c) Write, but do not evaluate, an integral expression that gives the volume of the solid generated when R is rotated about the horizontal line $y = 1$.

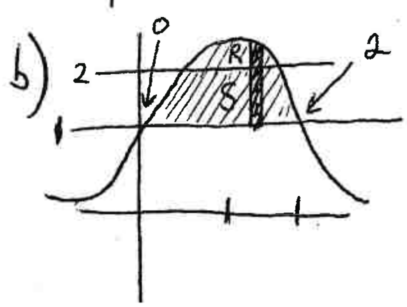
a)



Top/bottom
$y = e^{2x-x^2}$
$y = 2$

$$\text{Area} = \int_{0.446}^{1.554} e^{2x-x^2} - 2\,dx = \boxed{0.514}$$

b)

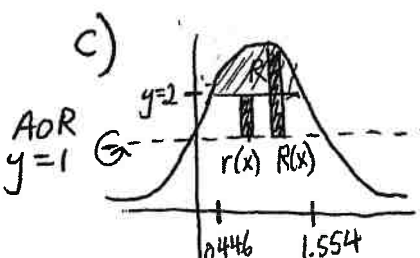


Area of S = Area of R+S − Area of R

$$\text{Area}(R+S) = \int_0^2 e^{2x-x^2} - 1\,dx = 2.06016$$
(Top/bottom)

$$\text{Area of } S = 2.06016 - 0.514 = \boxed{1.546}$$

c)



$R(x) = e^{2x-x^2} - 1$

$r(x) = 2 - 1 = 1$

$$V = \pi \int_{x_1}^{x_2} R(x)^2 - r(x)^2\,dx$$

$$V = \pi \int_{0.446}^{1.554} \left[e^{2x-x^2} - 1\right]^2 - [1]^2\,dx$$

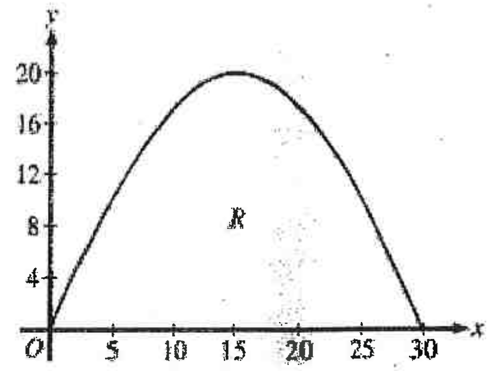
Washer Method

Top/bottom
$y = e^{2x-x^2}$
$y = 2$

26

2)

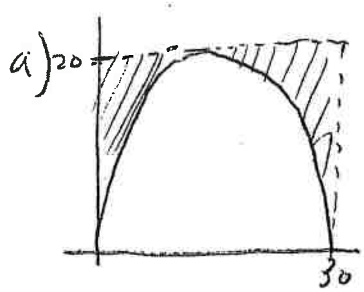
A baker is creating a birthday cake. The base of the cake is the region R in the first quadrant under the graph of $y = f(x)$ for $0 \le x \le 30$, where $f(x) = 20\sin\left(\frac{\pi x}{30}\right)$. Both x and y are measured in centimeters. The region R is shown in the figure above. The derivative of f is $f'(x) = \frac{2\pi}{3}\cos\left(\frac{\pi x}{30}\right)$.



(a) The region R is cut out of a 30-centimeter-by-20-centimeter rectangular sheet of cardboard, and the remaining cardboard is discarded. Find the area of the discarded cardboard.

(b) The cake is a solid with base R. Cross sections of the cake perpendicular to the x-axis are semicircles. If the baker uses 0.05 gram of unsweetened chocolate for each cubic centimeter of cake, how many grams of unsweetened chocolate will be in the cake?

a)

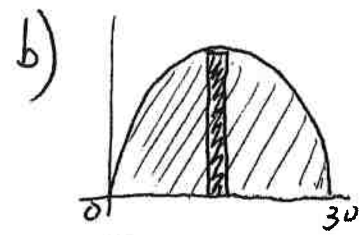


Area = Area of Box − Area under parabola (Top - bottom) $y = 20\sin\left(\frac{\pi x}{30}\right)$, $y = 0$

$= 30(20) - \int_0^{30} 20\sin\left(\frac{\pi x}{30}\right) - 0\,dx$

$= 600 - 381.972 = \boxed{218.028\ cm^2}$

b)



Top/bottom
$y = 20\sin\left(\frac{\pi x}{30}\right)$
$y = 0$

$base = 20\sin\left(\frac{\pi x}{30}\right) - 0$

$base = 20\sin\left(\frac{\pi x}{30}\right)$

Area (semicircle) $= \frac{\pi}{8}[base]^2$

$= \frac{\pi}{8}\left[20\sin\left(\frac{\pi x}{30}\right)\right]^2$

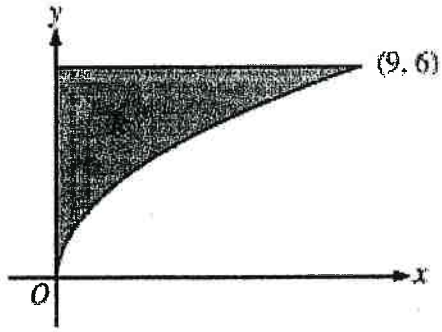
Volume $= \int_0^{30} \frac{\pi}{8}\left[20\sin\left(\frac{\pi x}{30}\right)\right]^2 dx$

$V = 2356.194\ cm^3$

$\frac{0.05\ grams}{1\ cm^3} \cdot 2356.194\ cm^3 = \boxed{117.809\ \text{grams (of chocolate)}}$

27

3) (Non-calculator)



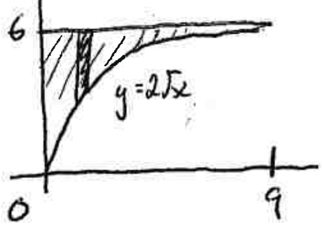
Let R be the region in the first quadrant bounded by the graph of $y = 2\sqrt{x}$, the horizontal line $y = 6$, and the y-axis, as shown in the figure above.

(a) Find the area of R.

(b) Write, but do not evaluate, an integral expression that gives the volume of the solid generated when R is rotated about the horizontal line $y = 7$.

(c) Region R is the base of a solid. For each y, where $0 \le y \le 6$, the cross section of the solid taken perpendicular to the y-axis is a rectangle whose height is 3 times the length of its base in region R. Write, but do not evaluate, an integral expression that gives the volume of the solid.

a)



Top-bottom
$y = 2\sqrt{x}$
$y = 6$

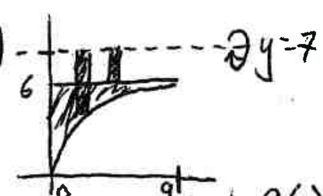
$$\text{Area} = \int_0^9 6 - 2\sqrt{x}\,dx$$

$$= \int_0^9 6 - 2x^{1/2}\,dx = 6x - \frac{2x^{3/2}}{3/2}\Bigg]_0^9$$

$$6x - \frac{4}{3}x^{3/2}\Bigg]_0^9 = 6(9) - \frac{4}{3}(9)^{3/2} - (0 - 0)$$

$$= 54 - \frac{4}{3}(3)^3 = 54 - 4(9) = \boxed{18}$$

b)



washer method

Top/bottom

$y = 6$

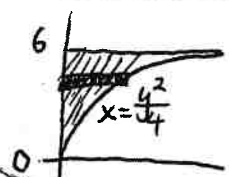
$y = 2\sqrt{x}$

$R(x) = 7 - 2\sqrt{x}$

$r(x) = 7 - 6 = 1$

$$\boxed{V = \pi \int_0^9 \left[7 - 2\sqrt{x}\right]^2 - \left[1\right]^2 dx}$$

c)



$y = 2\sqrt{x} \rightarrow \frac{y}{2} = \sqrt{x}$

$\left(\frac{y}{2}\right)^2 = x$

Right/Left

$x = y^2/4$

$x = 0$

base $= \frac{y^2}{4} - 0 = \frac{y^2}{4}$

height $= 3(\text{base}) = 3\left(\frac{y^2}{4}\right)$

Area $=$ base $\times$ height

$= \left(\frac{y^2}{4}\right) \times 3\left(\frac{y^2}{4}\right) = \frac{3}{16}y^4$

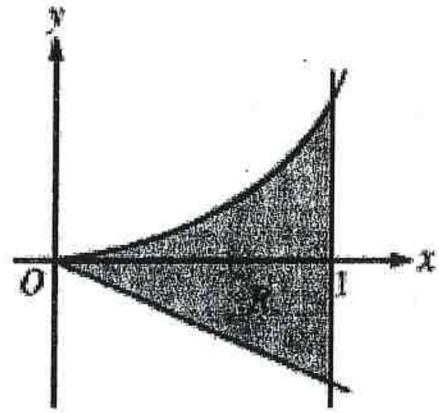
$$\boxed{V = \int_0^6 \frac{3}{16}y^4\,dy}$$

(28) 4)

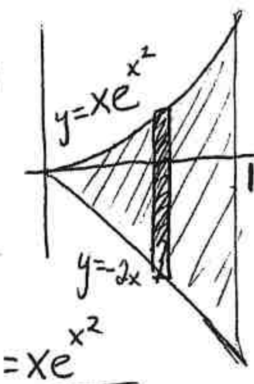
Let R be the shaded region bounded by the graph of $y = xe^{x^2}$, the line $y = -2x$, and the vertical line $x = 1$, as shown in the figure above.

(a) Find the area of R.

(b) Write, but do not evaluate, an integral expression that gives the volume of the solid generated when R is rotated about the horizontal line $y = -2$.



a)



$y = \underline{xe^{x^2}}$

$y = \underline{-2x}$

Top/bottom

u-substitution

$$\text{Area} = \int_0^1 xe^{x^2} - (-2x)\,dx = \int_0^1 xe^{x^2} + 2x\,dx$$

$u = x^2$

$\frac{du}{dx} = 2x$

$dx = \frac{du}{2x}$

$\int xe^u \frac{du}{2x}$

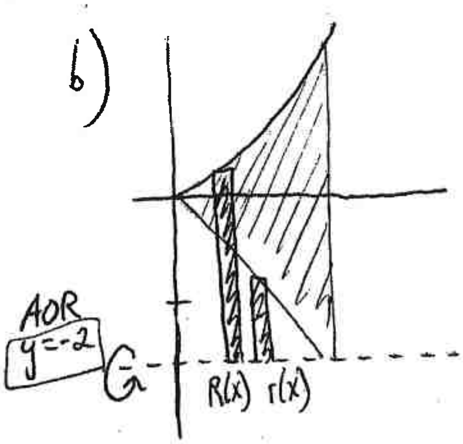
$\frac{1}{2}\int e^u du = \frac{1}{2}e^{x^2}$

$\left[\frac{1}{2}e^{x^2} + \frac{2x^2}{2}\right]_0^1$

$\frac{1}{2}e^1 + 1 - \left(\frac{1}{2}e^0 + 0^2\right)$

$\frac{1}{2}e + 1 - \frac{1}{2} = \boxed{\frac{1}{2}e + \frac{1}{2}}$

b)



washer method

Top/bottom

$y = \underline{xe^{x^2}}$

$y = \underline{-2x}$

$R(x) = xe^{x^2} - (-2)$

$r(x) = -2x - (-2)$

$= -2x + 2$

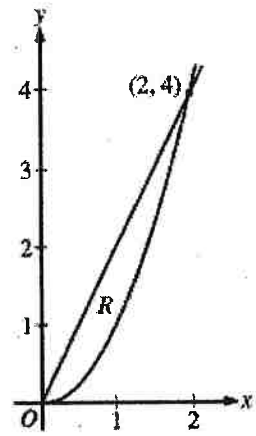
$$V = \pi \int_{x_1}^{x_2} [R(x)]^2 - [r(x)]^2\,dx$$

$$\boxed{V = \pi \int_0^1 \left[xe^{x^2} + 2\right]^2 - \left[-2x + 2\right]^2 dx}$$

Calculus Chapter 8 Review Worksheet #2

1. (Non-Calculator)

Let R be the region in the first quadrant enclosed by the graphs of $y = 2x$ and $y = x^2$, as shown in the figure above.



(a) Find the area of R.

(b) The region R is the base of a solid. For this solid, at each x the cross section perpendicular to the x-axis has area $A(x) = \sin\left(\frac{\pi}{2}x\right)$. Find the volume of the solid.

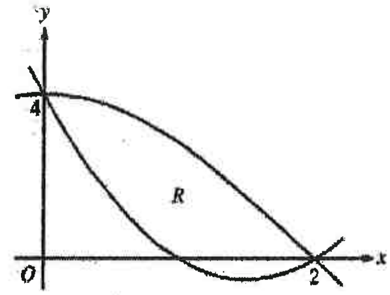
(c) Another solid has the same base R. For this solid, the cross sections perpendicular to the y-axis are squares. Write, but do not evaluate, an integral expression for the volume of the solid.

(d) Write an Integral Expression that gives the Volume of the Solid generated when R is rotated about $x = -1$.

3a

2) **(Non-Calculator)**

Let $f(x) = 2x^2 - 6x + 4$ and $g(x) = 4\cos\left(\frac{1}{4}\pi x\right)$. Let R be the region bounded by the graphs of f and g, as shown in the figure above.



(a) Find the area of R.

(b) Write, but do not evaluate, an integral expression that gives the volume of the solid generated when R is rotated about the horizontal line $y = 4$.

(c) The region R is the base of a solid. For this solid, each cross section perpendicular to the x-axis is a square. Write, but do not evaluate, an integral expression that gives the volume of the solid.

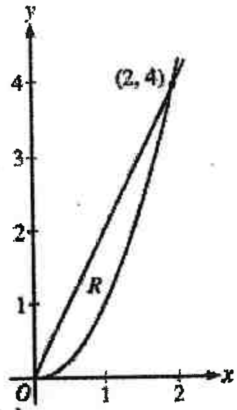
AB Calculus **Chapter 8** **Review Worksheet #4**

Key

(31)

1. **(Non-Calculator)**

Let R be the region in the first quadrant enclosed by the graphs of $y = 2x$ and $y = x^2$, as shown in the figure above.



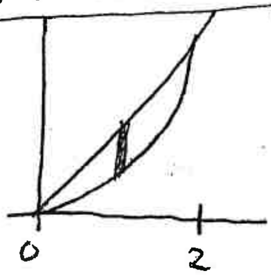
(a) Find the area of R.

(b) The region R is the base of a solid. For this solid, at each x the cross section perpendicular to the x-axis has area $A(x) = \sin\left(\frac{\pi}{2}x\right)$. Find the volume of the solid.

(c) Another solid has the same base R. For this solid, the cross sections perpendicular to the y-axis are squares. Write, but do not evaluate, an integral expression for the volume of the solid.

d) Find Volume of solid by rotating R about line $x=-1$ (write expression)

a)



Top-bottom

$y = 2x$

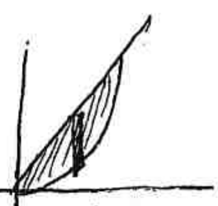
$y = x^2$

$$\text{Area} = \int_0^2 2x - x^2\,dx$$

$$= \frac{2x^2}{2} - \frac{x^3}{3}\Big]_0^2 = 2^2 - \frac{2^3}{3} - (0-0)$$

$$= 4 - \frac{8}{3} = \frac{12}{3} - \frac{8}{3} = \boxed{\frac{4}{3}}$$

b)



$A(x) = \sin\left(\frac{\pi}{2}x\right)$

Cross-section

$$V = \int [\text{Area}]\,dx$$

$$V = \int_0^2 \sin\left(\frac{\pi}{2}x\right)dx$$

$u = \frac{\pi}{2}x$

$\frac{du}{dx} = \frac{\pi}{2}$

$\pi\,dx = 2\,du$

$dx = \frac{2}{\pi}du$

$\int \sin u \cdot \frac{2}{\pi}du$

$\frac{2}{\pi}\int \sin u\,du$

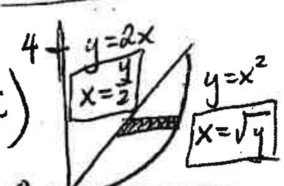
$= \frac{2}{\pi} \cdot -\cos u$

$-\frac{2}{\pi}\cos\left(\frac{\pi}{2}x\right)\Big]_0^2$

$-\frac{2}{\pi}\cos(\pi) - \left(-\frac{2}{\pi}\cos(0)\right)$

$-\frac{2}{\pi}(-1) + \frac{2}{\pi} = \boxed{\frac{4}{\pi}}$

c)



Right/Left

$x = \sqrt{y}$

$x = y/2$

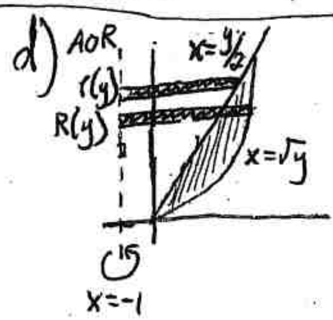
base $= \sqrt{y} - \frac{y}{2}$

$\text{Area}_{\text{square}} = [\text{base}]^2$

$\text{Area} = \left[\sqrt{y} - \frac{y}{2}\right]^2$

$$\boxed{V = \int_0^4 \left[\sqrt{y} - \frac{y}{2}\right]^2 dy}$$

d)



* washer method

* Right/Left

$x = \sqrt{y}$

$x = y/2$

$R(y) = \sqrt{y} - (-1)$

$r(y) = y/2 - (-1)$

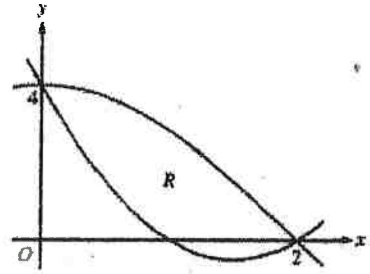
$$V = \pi\int_{y_1}^{y_2} [R(y)]^2 - [r(y)]^2\,dy$$

$$\boxed{V = \pi\int_0^4 \left[\sqrt{y}+1\right]^2 - \left[\frac{y}{2}+1\right]^2 dy}$$

(32)

2) **(Non-Calculator)**

Let $f(x) = 2x^2 - 6x + 4$ and $g(x) = 4\cos\left(\frac{1}{4}\pi x\right)$. Let R be the region bounded by the graphs of f and g, as shown in the figure above.

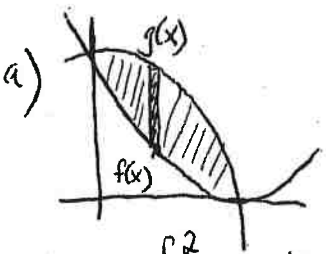


(a) Find the area of R.

(b) Write, but do not evaluate, an integral expression that gives the volume of the solid generated when R is rotated about the horizontal line $y = 4$.

(c) The region R is the base of a solid. For this solid, each cross section perpendicular to the x-axis is a square. Write, but do not evaluate, an integral expression that gives the volume of the solid.

a)



$$\text{Area} = \int_0^2 g(x) - f(x)\,dx$$

$$\text{Area} = \int_0^2 4\cos\left(\tfrac{\pi}{4}x\right) - \left(2x^2 - 6x + 4\right)dx$$

$$\text{Area} = \int_0^2 4\cos\left(\tfrac{\pi}{4}x\right) - 2x^2 + 6x - 4\,dx$$

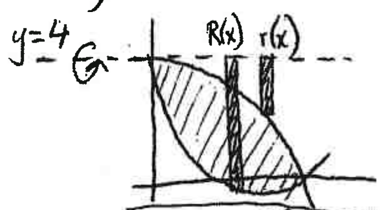
$u = \frac{\pi}{4}x$, $\frac{du}{dx} = \frac{\pi}{4}$, $\pi dx = 4du$ | $dx = \frac{4}{\pi}du$, $4\int \cos u \cdot \frac{4}{\pi}du$

$$4 \cdot \tfrac{4}{\pi}\sin\left(\tfrac{\pi}{4}x\right) - \tfrac{2x^3}{3} + \tfrac{6x^2}{2} - 4x \Big]_0^2$$

$$\tfrac{16}{\pi}\sin\left(\tfrac{\pi}{4}\cdot 2\right) - \tfrac{2(2)^3}{3} + \tfrac{6(2)^2}{2} - 4(2) - \left[\tfrac{16}{\pi}\sin(0) - 0 + 0 - 0\right]$$

$$\boxed{\tfrac{16}{\pi}(1) - \tfrac{16}{3} + \tfrac{24}{2} - 8} \quad \text{or} \quad \boxed{\tfrac{16}{\pi} - \tfrac{4}{3}}$$

b) AOR: $y = 4$



* washer method
* Top/bottom
$y = 4\cos\left(\frac{\pi}{4}x\right)$
$y = 2x^2 - 6x + 4$

$R(x) = 4 - (2x^2 - 6x + 4) = 4 - 2x^2 + 6x - 4 = \boxed{-2x^2 + 6x}$

$r(x) = 4 - 4\cos\left(\frac{\pi}{4}x\right)$

$$\boxed{V = \pi\int_0^2 \left[-2x^2 + 6x\right]^2 - \left[4 - 4\cos\left(\tfrac{\pi}{4}x\right)\right]^2 dx}$$

c)

Top/bottom
$y = 4\cos\left(\frac{\pi}{4}x\right)$
$y = 2x^2 - 6x + 4$

base $= 4\cos\left(\frac{\pi}{4}x\right) - \left(2x^2 - 6x + 4\right)$

Area square $= [\text{base}]^2$

$$\boxed{V = \int_0^2 \left[4\cos\left(\tfrac{\pi}{4}x\right) - 2x^2 + 6x - 4\right]^2 d}$$

To express $y=\sqrt{4-x}$ as a function $x=g(y)$, solve for x:

$$y=\sqrt{4-x}$$
$$y^2=4-x$$
$$x=g(y)=4-y^2$$

The graph of $x=g(y)=4-y^2$ is to the right of the graph of $x=f(y)=1-\dfrac{y^2}{4}$, $0\le y\le 2$. So, $g(y)\ge f(y)$. Partitioning the y-axis, we have

$$A=\int_0^2[g(y)-f(y)]\,dy=\int_0^2\left[(4-y^2)-\left(1-\frac{y^2}{4}\right)\right]dy=\int_0^2\left(3-\frac{3y^2}{4}\right)dy$$
$$=\left[3y-\frac{y^3}{4}\right]_0^2=6-\frac{8}{4}=4 \text{ square units}$$

■

NOW WORK Problem 31 and AP® Practice Problem 1.

When the graphs of functions of x form the top and bottom borders of the region, partitioning the x-axis is usually easier, provided it is easy to find the integrals. If the graphs of functions of y form the left and right borders of the region, partitioning the y-axis is usually easier, provided the integrals with respect to y are easily found.

8.1 Assess Your Understanding

Concepts and Vocabulary

1. Express the area of the region bounded by the graphs of $y=x^2$ and $y=\sqrt{x}$ as an integral using a partition of the x-axis. Do not find the integral.
2. Express the area of the region bounded by the graph of $x=y^2$ and the line $x=1$ as an integral using a partition of the y-axis. Do not find the integral.

Skill Building

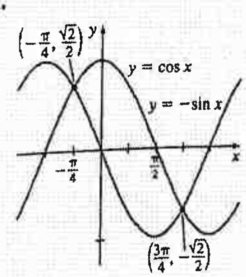
In Problems 3–12, find the area of the region bounded by the graphs of the given equations by partitioning the x-axis.

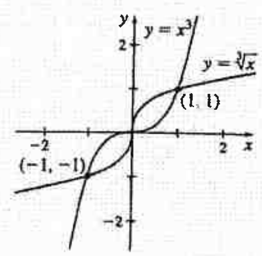
3. $y=x,\ y=2x,\ x=1$
4. $y=x,\ y=3x,\ x=3$
5. $y=x^2,\ y=x$
6. $y=x^2,\ y=4x$
7. $y=e^x,\ y=e^{-x},\ x=\ln 2$
8. $y=e^x,\ y=-x+1,\ x=2$
9. $y=x^2,\ y=x^4$
10. $y=x,\ y=x^3$
11. $y=\cos x,\ y=\dfrac{1}{2},\ 0\le x\le\dfrac{\pi}{3}$
12. $y=\sin x,\ y=\dfrac{1}{2},\ \dfrac{\pi}{6}\le x\le\dfrac{5\pi}{6}$

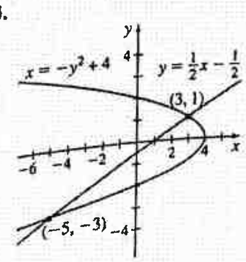
In Problems 13–20, find the area of the region bounded by the graphs of the given equations by partitioning the y-axis.

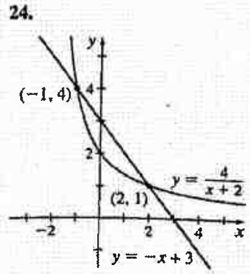
13. $x=y^2,\ x=2-y$
14. $x=y^2,\ x=y+2$
15. $x=9-y^2,\ x=5$
16. $x=16-y^2,\ x=7$
17. $x=y^2+4,\ y=x-6$
18. $x=y^2+6,\ y=8-x$
19. $y=\ln x,\ x=1,\ y=2$
20. $y=\ln x,\ x=e,\ y=0$

In Problems 21–24, find the area of the shaded region in the graph.

21.


22.


23.


24.


In Problems 25–32, find the area A of the region bounded by the graphs of the given equations:

(a) by partitioning the x-axis.

(b) by partitioning the y-axis.

25. $y=\sqrt{x},\ y=x^3$

26. $y=\sqrt{x},\ y=x^2$

27. $y=x^2+1,\ y=x+1$

28. $y=x^2+1,\ y=4x+1$

29. $y=\sqrt{9-x},\ y=\sqrt{9-3x},\ y=0$

30. $y=\sqrt{16-2x},\ y=\sqrt{16-4x},\ y=0$

PAGE 579 **31.** $y=\sqrt{2x-6},\ y=\sqrt{x-2},\ y=0$

32. $y=\sqrt{2x-5},\ y=\sqrt{4x-17},\ y=0$

In Problems 33–46, find the area of the region bounded by the graphs of the given equations.

33. $y=4-x^2,\ y=x^2$

34. $y=9-x^2,\ y=x^2$

35. $x=y^2-4,\ x=4-y^2$

36. $x=y^2,\ x=16-y^2$

37. $y=\ln x^2$, $y=0$, and the line $x=e$

38. $y=\ln x$, $y=1-x$, and the line $y=1$

39. $y=\cos x,\ y=1-\dfrac{3}{\pi}x,\ x=\dfrac{\pi}{3}$

40. $y=\sin x,\ y=1,\ 0\le x\le \dfrac{\pi}{2}$

41. $y=e^{2x}$ and the lines $x=1$ and $y=1$

42. $y=e^x,\ \ y=e^{3x},\ x=2$

43. $y^2=4x,\ 4x-3y-4=0$

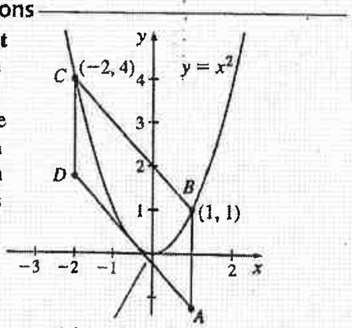
44. $y^2=4x+1,\ x=y+1$

45. $y=\sin x,\ y=\dfrac{2x}{\pi},\ x\ge 0$

46. $y=\cos x,\ x\ge 0,\ y=\dfrac{3x}{\pi}$

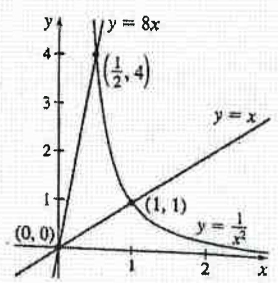
Applications and Extensions

47. An Archimedean Result Show that the area of the shaded region in the figure is two-thirds of the area of the parallelogram *ABCD*. (This illustrates a result due to Archimedes concerning sectors of parabolas.)

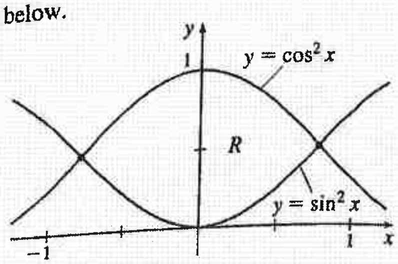


The line segment joining *A* and *D* is parallel to the line segment joining (−2, 4) and (1, 1) and is tangent to the graph of $y=x^2$.

48. Equal Areas Find h so that the area of the region bounded by the graphs of $y=x$, $y=8x$, and $y=\dfrac{1}{x^2}$ is equal to that of an isosceles triangle of base 1 and height h. See the figure.



49. The region R bounded by the graphs of $y=\cos^2 x$ and $y=\sin^2 x$ is shown below.



(a) Find the points of intersection of the two graphs.

(b) Find the area of the region in the first quadrant bounded by the graphs of $y=\cos^2 x$ and $y=\sin^2 x$ and the y-axis.

50. Area Find the area of the region in the first quadrant bounded by the graphs of $y=\sin(2x)$ and $y=\cos(2x)$, $0\le x\le \dfrac{\pi}{8}$.

In Problems 51 and 52, find the area of the region bounded by the graphs of f and g.

51. $f(x)=3\ln x$ and $g(x)=x\ln x,\ x\ge 1$

52. $f(x)=4x\ln x$ and $g(x)=x^2\ln x,\ x\ge 1$

53. Area Find the area of the region bounded by the hyperbola $\dfrac{y^2}{9}-x^2=1$ and

(a) the line $y=4$. (b) the line $y=-7$.

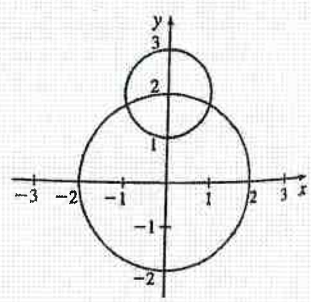
54. Area Find the area of the region bounded by the hyperbola $\dfrac{x^2}{9}-\dfrac{y^2}{16}=1$ and

(a) the line $x=6$. (b) the line $x=-8$.

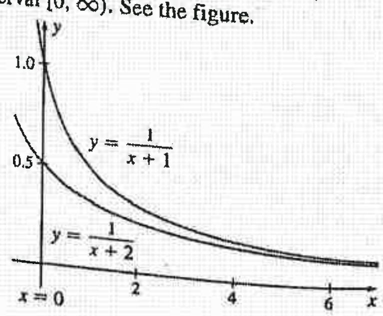
55. Area of a Lune A **lune** is a crescent-shaped region formed when two circles intersect.

(a) Find the area of the smaller lune formed by the intersection of the two circles $x^2+y^2=4$ and $x^2+(y-2)^2=1$ (shaded in the figure).

(b) What is the area of the larger lune?



56. Area between Graphs Find the area, if it is defined, of the region bounded by the graphs of $y=\dfrac{1}{x+1}$ and $y=\dfrac{1}{x+2}$ on the interval $[0,\infty)$. See the figure.



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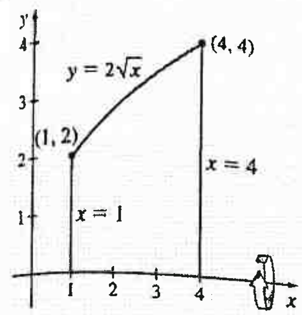
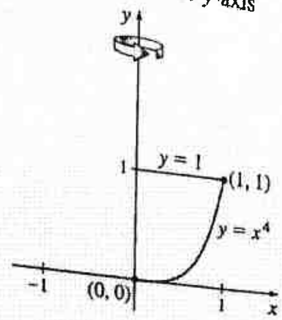
Skill Building

In Problems 5–10, find the volume of the solid of revolution generated by revolving the shaded region about the indicated axis.

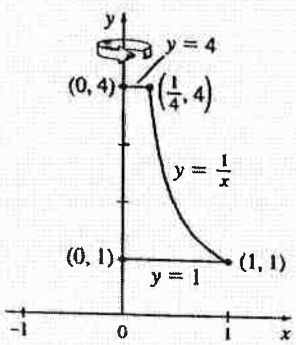
5. $y=2\sqrt{x}$ about the x-axis



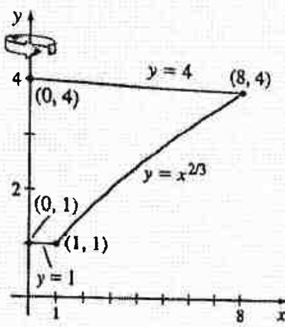
6. $y=x^4$ about the y-axis



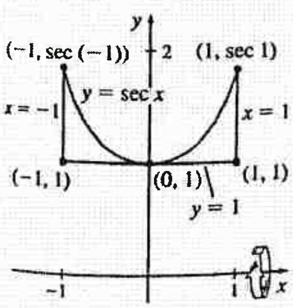
7. $y=\dfrac{1}{x}$ about the y-axis



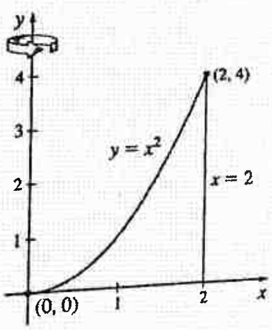
8. $y=x^{2/3}$ about the y-axis



9. $y=\sec x$ about the x-axis



10. $y=x^2$ about the y-axis



In Problems 11–16, use the disk method to find the volume of the solid of revolution generated by revolving the region bounded by the graphs of the given equations about the indicated axis.

11. $y=2x^2$, the x-axis, $x=1$; about the x-axis
12. $y=\sqrt{x}$, the x-axis, $x=4$, $x=9$; about the x-axis
13. $y=e^{-x}$, the x-axis, $x=0$, $x=2$; about the x-axis
14. $y=e^x$, the x-axis, $x=-1$, $x=1$; about the x-axis
15. $y=x^2$, $x\geq 0$, $y=1$, $y=4$; about the y-axis
16. $y=2\sqrt{x}$, the y-axis, $y=4$; about the y-axis

In Problems 17–22, use the washer method to find the volume of the solid of revolution generated by revolving the region bounded by the graphs of the given equations about the indicated axis.

17. $y=x^2$, $x\geq 0$, the y-axis, $y=4$; about the x-axis
18. $y=2x^2$, $x\geq 0$, the y-axis, $y=2$; about the x-axis
19. $y=2\sqrt{x}$, the y-axis, $y=4$; about the x-axis
20. $y=x^{2/3}$, the x-axis, $x=8$; about the y-axis
21. $y=x^3$, the x-axis, $x=1$; about the y-axis
22. $y=2x^4$, the x-axis, $x=1$; about the y-axis

In Problems 23–38, find the volume of the solid of revolution generated by revolving the region bounded by the graphs of the given equations about the indicated axis.

23. $y=\dfrac{1}{x}$, the x-axis, $x=1$, $x=2$; about the x-axis
24. $y=\dfrac{1}{x}$, the x-axis, $x=1$, $x=2$; about the y-axis
25. $y=\sqrt{x}$, the y-axis, $y=9$; about the y-axis
26. $y=\sqrt{x}$, the y-axis, $y=9$; about the x-axis
27. $y=(x-2)^3$, the x-axis, $x=0$, $x=3$; about the x-axis
28. $y=(x-2)^3$, the x-axis, $x=0$, $x=3$; about the y-axis
29. $y=(x+1)^2$, $x\geq 0$, $y=16$; about the y-axis
30. $y=(x+1)^2$, $x\leq 0$, $y=16$; about the x-axis
31. $x=y^4-1$, the y-axis; about the y-axis
32. $y=x^4-1$, the x-axis; about the x-axis
33. $y=4x$, $y=x^3$, $x\geq 0$; about the x-axis
34. $y=2x+1$, $y=x$, $x=0$, $x=3$; about the x-axis
35. $y=1-x$, $y=e^x$, $x=1$; about the x-axis
36. $y=\cos x$, $y=\sin x$, $x=0$, $x=\dfrac{\pi}{4}$; about the x-axis
37. $y=\csc x$, $y=0$, $x=\dfrac{\pi}{2}$, $x=\dfrac{3\pi}{4}$; about the x-axis
38. $y=\sec x$, $y=0$, $x=0$, $x=\dfrac{\pi}{3}$; about the x-axis

In Problems 39–46, find the volume of the solid of revolution generated by revolving the region bounded by the graphs of the given equations about the indicated line.

39. $y=e^x$, $y=0$, $x=0$, $x=2$; about $y=-1$
40. $y=\dfrac{1}{x}$, $y=0$, $x=1$, $x=4$; about $y=4$
41. $y=x^2$, the x-axis, $x=1$; about $x=1$
42. $y=x^3$, $x=0$, $y=1$; about $x=-1$
43. $y=\sqrt{x}$, the x-axis, $x=4$; about $x=-4$
44. $y=\dfrac{1}{\sqrt{x}}$, the x-axis, $x=1$, $x=4$; about $x=4$
45. $y=\dfrac{1}{x^2}$, $y=0$, $x=1$, $x=4$; about $y=4$
46. $y=\sqrt{x}$, $y=0$, $0\leq x\leq 4$; about $y=-4$

In Problems 47–50, find the volume of the solid of revolution generated by revolving the indicated region about each line.

47. The region bounded by $y=x^2$, the x-axis, and $x=3$
  (a) About the x-axis
  (b) About the line $y=-1$
  (c) About the line $y=10$
  (d) About the line $y=a$, $a\geq 9$

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48. The region bounded by $y=x^2$, the x-axis, and $x=3$

(a) About the y-axis
(b) About the line $x=-5$
(c) About the line $x=5$
(d) About the line $x=b$, $b\ge 3$

49. The region bounded by $y=x^2$, the y-axis, and $y=4$

(a) About the y-axis
(b) About the line $x=-5$
(c) About the line $x=5$
(d) About the line $x=b$, $b\ge 2$

50. The region bounded by $y=x^2$, the y-axis, and $y=4$

(a) About the x-axis
(b) About the line $y=-1$
(c) About the line $y=4$
(d) About the line $y=a$, $a\ge 4$

Applications and Extensions

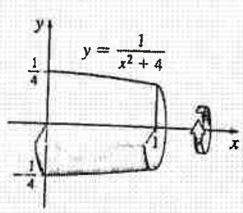
51. Volume of a Sphere

(a) Graph $y=\sqrt{a^2-x^2}$.
(b) Revolve the region bounded by $y=\sqrt{a^2-x^2}$ and the x-axis about the x-axis to generate a sphere of radius a. Use the Disk Method to show that the volume V of the sphere is $\frac{4}{3}\pi a^3$.

52. Volume of a Cone

(a) Find an equation of the line segment from the point $(0,h)$, $h>0$, to the point $(a,0)$, $a>0$.
(b) Graph the line segment from (a).
(c) Revolve the region in the first quadrant bounded by the line segment from (a), the x-axis, and the y-axis, about the y-axis to generate a cone of height h and radius a. Use the Disk Method to show that the volume V of the cone is $\frac{1}{3}\pi a^2 h$.

53. Volume of a Solid of Revolution The figure shows the solid of revolution generated by revolving the region bounded by the graph of $y=\frac{1}{x^2+4}$ and the x-axis from $x=0$ to $x=1$ about the x-axis.



(a) Express the volume of the solid of revolution as an integral.
(b) Use technology to find the volume.

54. Volume of a Solid of Revolution A solid of revolution is generated by revolving the region bounded by the graph of $y=\ln x$, the line $x=e$, and the x-axis about the x-axis.

(a) Express the volume of the solid of revolution as an integral.
(b) Use technology to find the volume.

55. Volume of an Ellipsoid

(a) Graph $y=\sqrt{9-4x^2}$, the upper half of an ellipse.
(b) Find the volume of the solid of revolution generated by revolving the region bounded by $y=\sqrt{9-4x^2}$ and the x-axis about the x-axis.

56. Volume of a Solid of Revolution

(a) Graph $y=\sqrt{4x^2+1}$, the upper portion of a hyperbola.
(b) Find the volume of the solid of revolution generated by revolving the region bounded by $y=\sqrt{4x^2+1}$, the lines $x=-1$ and $x=1$, and the x-axis about the x-axis.

57. Mixed Practice A region in the first quadrant is bounded by the x-axis and the graph of $y=kx-x^2$, where $k>0$.

(a) In terms of k, find the volume generated when the region is revolved around the x-axis.
(b) In terms of k, find the area of the region.

58. Mixed Practice

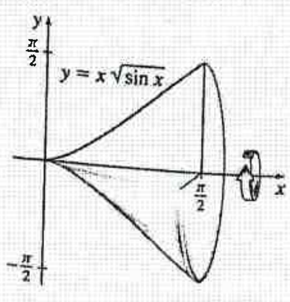
(a) Find all numbers b for which the graphs of $y=2x+b$ and $y^2=4x$ intersect in two distinct points.
(b) If $b=-4$, find the area bounded by the graphs of $y=2x-4$ and $y^2=4x$.
(c) If $b=0$, find the volume of the solid generated by revolving about the x-axis the region bounded by the graphs of $y=2x$ and $y^2=4x$.

59. Volume of a Solid of Revolution Find the volume of the solid of revolution generated by revolving the region bounded by the graphs of $y=\cos x$ and $x=0$ from $x=0$ to $x=\frac{\pi}{2}$ about the line $y=1$. *Hint:* $\cos^2 x=\frac{1+\cos(2x)}{2}$.

60. Volume of a Solid of Revolution Find the volume of the solid of revolution generated by revolving the region bounded by the graphs of $y=\cos x$ and $x=0$ from $x=0$ to $x=\frac{\pi}{2}$ about the line $y=-1$. (See the hint in Problem 59.)

61. The Volume of a Solid of Revolution Find the volume of the solid of revolution generated by revolving the region bounded by the graph of $y=\ln x$ and the x-axis from $x=1$ to $x=e$ about the x-axis.

62. The Volume of a Solid of Revolution Find the volume of the solid of revolution generated by revolving the region bounded by the graph of $y=x\sqrt{\sin x}$ and the x-axis from $x=0$ to $x=\frac{\pi}{2}$ about the x-axis. See the figure below.



63. The Volume of a Solid of Revolution Find the volume of the solid of revolution generated by revolving the region bounded by the graph of $y=x\sqrt{\ln x}$, the x-axis, and the lines $x=1$ and $x=e^2$ about the x-axis.

The volume V of the solid is

$$V=\int_0^2 A(y)dy=\int_0^2\left(\sqrt{8y}-y^2\right)^2dy=\int_0^2\left(8y-4\sqrt{2}y^{5/2}+y^4\right)dy$$

$$=\left[4y^2-\frac{8\sqrt{2}}{7}y^{7/2}+\frac{y^5}{5}\right]_0^2=16-\frac{128}{7}+\frac{32}{5}=\frac{144}{35}\text{ cubic units}$$

NOW WORK Problem 5 and AP® Practice Problems 1-4, 6, and 7.

8.4 Assess Your Understanding

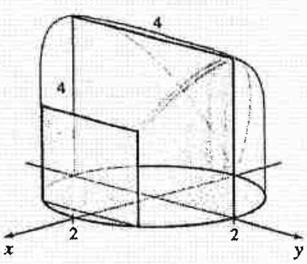
Concepts and Vocabulary

1. In your own words, explain the method of slicing.
2. *True or False* The slicing method works only with solids of revolution.

Skill Building

In Problems 3–10, find the volume of each solid by the method of slicing.

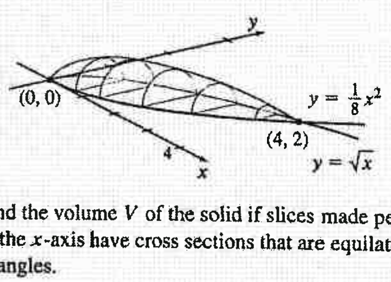
PAGE 607 3. Find the volume of the solid whose base is a circle of radius of 2, if slices made perpendicular to the base are squares. See the figure.



4. Find the volume of the solid whose base is a circle of radius 2, if slices made perpendicular to the base are isosceles right triangles with one leg on the base.

PAGE 610 5. The base of a solid is the region bounded by the graphs of $y=\sqrt{x}$ and $y=\frac{1}{8}x^2$.

(a) Find the volume V of the solid if slices made perpendicular to the x-axis have cross sections that are semicircles. See the figure.



(b) Find the volume V of the solid if slices made perpendicular to the x-axis have cross sections that are equilateral triangles.

6. The base of a solid is the region bounded by the graphs of $y=\sqrt{x}$ and $y=\frac{1}{8}x^2$.

(a) Find the volume V of the solid if slices made perpendicular to the y-axis have cross sections that are semicircles.

(b) Find the volume V of the solid if slices made perpendicular to the y-axis have cross sections that are equilateral triangles.

7. The base of a solid is the region bounded by the graphs of $y=x^2$, $x=2$, and $y=0$.

(a) Find the volume V of the solid if slices made perpendicular to the x-axis are squares.

(b) Find the volume V of the solid if slices made perpendicular to the x-axis are semicircles.

(c) Find the volume V of the solid if slices made perpendicular to the x-axis are equilateral triangles.

8. The base of a solid is the region bounded by the graphs of $y=x^2$, $x=2$, and $y=0$.

(a) Find the volume V of the solid if slices made perpendicular to the y-axis are squares.

(b) Find the volume V of the solid if slices made perpendicular to the y-axis are semicircles.

(c) Find the volume V of the solid if slices made perpendicular to the y-axis are equilateral triangles.

9. The base of a solid is the region bounded by the graphs of $y=3\sqrt{3x}$ and $y=x^2$.

(a) Find the volume V of the solid if slices made perpendicular to the y-axis are squares.

(b) Find the volume V of the solid if slices made perpendicular to the y-axis are semicircles.

(c) Find the volume V of the solid if slices made perpendicular to the y-axis are equilateral triangles.

10. The base of a solid is the region bounded by the graphs of $y=3\sqrt{3x}$ and $y=x^2$.

(a) Find the volume V of the solid if slices made perpendicular to the x-axis are squares.

(b) Find the volume V of the solid if slices made perpendicular to the x-axis are semicircles.

(c) Find the volume V of the solid if slices made perpendicular to the x-axis are equilateral triangles.

Applications and Extensions

In Problems 11 and 12, find the volume of each solid.

PAGE 608 11. The solid is a pyramid 40 m high whose horizontal cross section h meters from the top is a square with sides of length $2h$ meters.

12. The solid is a pyramid 20 m high whose horizontal cross section h meters from the top is a rectangle with sides of length $2h$ and h meters.

PAGE 607 13. **Verifying a Geometry Formula** Use slicing to verify that the volume V of a sphere of radius R is $V=\frac{4}{3}\pi R^3$.

14. The base of a solid is the region enclosed by the graphs of $y=\sqrt{x}$ and $y=\frac{1}{8}x^2$.

(a) Find the volume V of the solid if slices made perpendicular to the x-axis have cross sections that are triangles whose base is the distance between the graphs and whose height is three times the base.

(b) Find the volume V of the solid if slices made perpendicular to the y-axis have cross sections that are triangles whose base is the distance between the graphs and whose height is three times the base.

In Problems 15 and 16, find the volume of each solid.

15. The solid is horn-shaped; slices taken perpendicular to the x-axis are circles whose diameters extend from the graph of $y=x^{1/2}$ to the graph of $y=\frac{4}{3}x^{1/3}$, $0\le x\le 1$.

16. The solid is horn-shaped; slices taken perpendicular to the x-axis are circles whose diameters extend from the graph of $y=x^{1/3}$ to $y=\frac{3}{2}x^{1/3}$, $0\le x\le 1$.

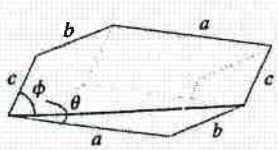
17. **Volume of a Solid** Find the volume of the cylindrical solid with a bulge in the middle, if slices taken perpendicular to the x-axis are circles whose diameters extend from the graph of $y=e^{-x^2}$ to $y=-e^{-x^2}$, $-1\le x\le 1$.

Challenge Problems

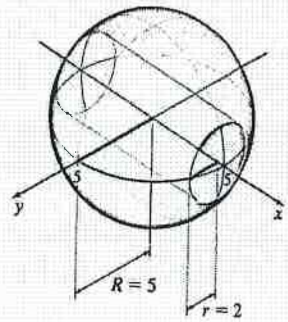
18. **Volume of Water** A hemispherical bowl of radius R contains water to the depth h. Find the volume of the water in the bowl.

19. **Volume of Water Left in a Glass** Suppose a cylindrical glass full of water is tipped until the water level bisects the base and touches the rim. What is the volume of the water remaining?

20. **Volume of a Solid** Find the volume of a parallelepiped with edge lengths a, b, and c, where the edges having lengths a and b make an acute angle θ with each other, and the edge of length c makes an acute angle of ϕ with the diagonal of the parallelogram formed by a and b.



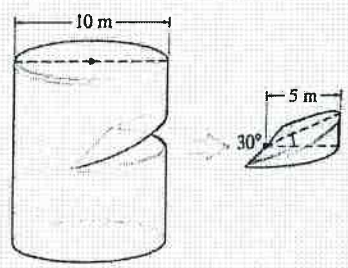
21. **Volume of a Bore** A hole of radius 2 cm is bored completely through a solid metal sphere of radius 5 cm. If the axis of the hole passes through the center of the sphere, find the volume of the metal removed by the drilling. See the figure.



22. **Volume** The axes of two pipes of equal radii r intersect at right angles. Find their common volume.

23. **Volume of a Cone** Find the volume of a cone with height h and an elliptical base whose major axis has length $2a$ and minor axis has length $2b$.
Hint: The area of this ellipse is πab.

24. **Volume of a Removed Sector** Suppose a wedge is cut from a solid right circular cylinder of diameter 10 m (like a wedge cut in a tree by an axe), where one side of the wedge is horizontal and the other is inclined at 30°. See the figure. If the horizontal part of the wedge penetrates 5 m into the cylinder and the two cuts meet along a vertical line through the center of the cylinder, find the volume of the wedge removed.
Hint: Vertical cross sections of the wedge are right triangles.



Preparing for the AP® Exam

AP® Practice Problems

1. A region R is bounded by the graphs of the functions $f(x)=3x$ and $g(x)=x^2$, $x\ge 0$. If R forms the base of a solid whose slices perpendicular to the x-axis are squares, then the volume of the solid is

(A) $\frac{9}{2}$ (B) $\frac{81}{10}$

(C) $\frac{45}{2}$ (D) $\frac{162}{5}$

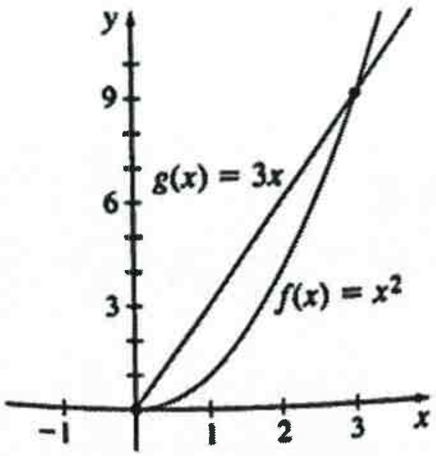
PAGE 610 2. A region R in the first quadrant is bounded by the graphs of $y=e^x$, $y=e^2$, and the y-axis. If R forms the base of a solid where slices perpendicular to the x-axis are semicircles, then the volume of the solid is given by

(A) $\frac{1}{2}\int_0^e \pi\left(\frac{e^2-e^x}{2}\right)^2 dx$ (B) $\int_0^{e^2} \pi\left(\frac{e^2-e^x}{2}\right)^2 dx$

(C) $\frac{1}{2}\int_0^2 \pi\left(\frac{e^2-e^x}{2}\right)^2 dx$ (D) $\frac{1}{2}\int_0^2 \pi(e^2-e^x)^2 dx$

8.1 AP Practice Problems (p.581-582)

1. The graphs of the functions $f(x) = x^2$ and $g(x) = 3x$ are shown below.



Find the area of the region bounded by the two graphs.

(A) $\frac{1}{3}$ (B) $\frac{7}{6}$ (C) $\frac{9}{2}$ (D) $\frac{27}{2}$

2. What is the area of the region bounded by the graphs of $y = e^{x/3}$, $y = 1$, and the line $x = 3$?

(A) $\frac{1}{3}(e - 10)$

(B) $e - 4$

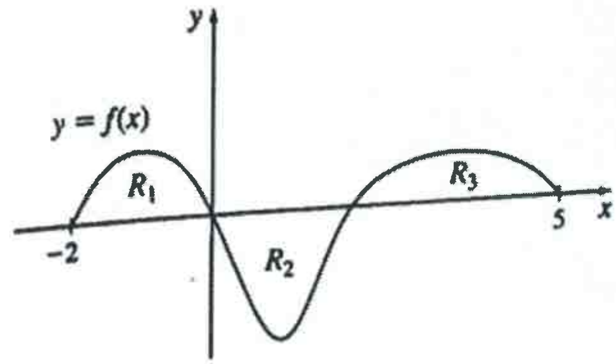
(C) $3e - 3$

(D) $3e - 6$

40

8.1

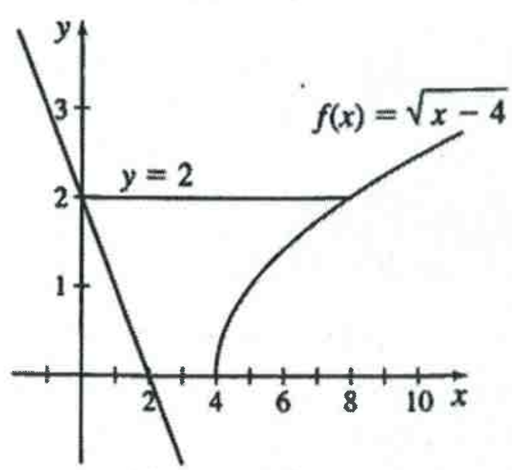
3. The figure below shows the graph of a function f over the closed interval $[-2, 5]$. If each of the regions R_1, R_2, and R_3 bounded by the graph of f and the x-axis has area equal to 4, then what is $\int_{-2}^{5} [f(x) + e^x]\, dx$?



(A) $e^5 - e^{-2} + 8$ (B) $e^5 - e^{-2} + 4$

(C) $e^5 - e^{-2} - 4$ (D) $e^5 - e^{-2} + 12$

4. The graphs of the functions $f(x) = \sqrt{x-4}$, $g(x) = -x + 2$, and $y = 2$ are shown in the figure.

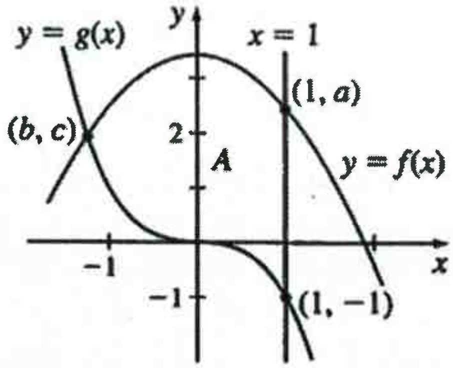


Find the area of the region bounded by the graphs and the x-axis.

(A) $\frac{4\sqrt{2} - 14}{3}$ (B) $\frac{26}{3}$ (C) $\frac{86}{3}$ (D) $\frac{112}{3}$

8.1

5. The graphs of the functions $y = f(x)$ and $y = g(x)$ intersect at the point (b, c) as shown in the figure below. The area A of the region bounded by the graphs of f, g and the line $x = 1$ is obtained by finding the integral



(A) $\int_1^b [f(x) - g(x)]dx$ (B) $\int_b^1 [f(x) - g(x)]dx$

(C) $\int_{-1}^c [f(x) - g(x)]dx$

(D) $\int_b^0 [f(x) - g(x)]dx + \int_0^1 [f(x) - |g(x)|]dx$

6. What is the area of the region bounded by the graphs of $y = x^2 - 4x + 2$ and $y = 7$?

(A) -36 (B) 36 (C) $\frac{64}{3}$ (D) 60

7. Find the area of the region bounded by the graphs of the functions $f(x) = e^x + 2$, $g(x) = -2x + 3$ and the line $x = 4$.

(A) $e^4 - 1$ (B) $e^4 + 11$

(C) $e^4 + 12$ (D) $2e^4 + 12$

42

8.1

8. The area of the region bounded by the graphs of $y = \sin x$, $y = \frac{1}{\pi}x + 1$, and the y-axis is

(A) $\frac{\pi}{2}$ (B) $1 + \frac{3\pi}{4}$ (C) $2 + \frac{\pi}{2}$ (D) $\frac{3\pi}{2} - 1$

9. Find the area of the region bounded by the graph of $x = y^2 + 2y - 3$ and the y-axis.

(A) $-\frac{32}{3}$ (B) $-\frac{20}{3}$ (C) $\frac{20}{3}$ (D) $\frac{32}{3}$

10. Consider the functions $f(x) = x^3 - 8x^2 + 15x + 2$ and $g(x) = 3x + 2$.

(a) Find the points (x, y) where the graphs of f and g intersect.

(b) Write the integral(s) that represent the area of the region(s) bounded by the graphs of f and g.

(c) Find the area of the region bounded by the graphs.

8.2 AP Practice Problems (p.595-596)

1. What is the volume of the solid of revolution generated when the region in the first quadrant bounded by the graph of $y = e^x$, the x-axis, and the line $x = 3$ is revolved about the x-axis?

(A) $2\pi e^6$ (B) $\frac{\pi}{2}(e^6 - 1)$

(C) $\pi(e^6 - 1)$ (D) $\frac{\pi}{2}(e^3 - 1)$

2. Find the volume of the solid of revolution generated when the region bounded by the graph of $y = \csc x$, the x-axis, and the lines $x = \frac{\pi}{4}$ and $x = \frac{3\pi}{4}$ is revolved about the x-axis.

(A) 2 (B) π (C) $\sqrt{2}\pi$ (D) 2π

3. The region bounded by the graph of $y = x^3$, the line $x = 2$, and the x-axis is revolved about the y-axis. The volume of the solid of revolution is

(A) 4π (B) $\frac{64}{5}\pi$ (C) $\frac{96}{5}\pi$ (D) $\frac{128}{7}\pi$

44

8.2

4. The volume of the solid of revolution generated by revolving the region under the graph of $y = \sin x$ from $x = 0$ to $x = \frac{\pi}{6}$ about the y-axis is given by

(A) $\pi \int_0^{\pi/6} \left[\left(\frac{\pi}{6} \right)^2 - \sin^2 x \right] dx$

(B) $\pi \int_0^{1/2} \left[\left(\frac{\pi}{6} \right)^2 - (\arcsin y)^2 \right] dy$

(C) $\pi \int_0^{1/2} (\arcsin y)^2 \, dy$

(D) $\pi \int_0^{1/2} \left[\left(\frac{\pi}{6} \right)^2 - \sin^2 y \right] dy$

5. Find the volume of the solid of revolution obtained by revolving the region bounded by the graph of $x = \sqrt{4 - 4y^2}$ and the y-axis about the y-axis.

(A) $\frac{4}{3}\pi$ (B) $\frac{8}{3}\pi$ (C) $\frac{11}{6}\pi$ (D) $\frac{16}{3}\pi$

8.2

6. The region bounded by the graphs of $y = 1$, $y = e^x$, and the line $x = 2$ is revolved about the x-axis. The volume of the resulting solid of revolution is

(A) $\frac{\pi}{2}(e^4 - 5)$ (B) $2\pi(e^4 - 2)$

(C) $\frac{\pi}{2}(e^4 - 2)$ (D) $\frac{\pi}{2}(e^4 - 4)$

7. The volume of the solid of revolution generated by revolving the region bounded by the graphs of $y = \sqrt{x}$, $y = 2$, and the y-axis about the line $x = 4$ is

(A) $\frac{18}{5}\pi$ (B) $\frac{224}{15}\pi$ (C) $\frac{256}{15}\pi$ (D) $\frac{328}{15}\pi$

(46)

8.2

8. The volume of the solid of revolution generated by revolving the region bounded by the graphs of $y = x^2$, $x \geq 0$, the y-axis, and $y = 4$ about the line $y = 5$ is given by

(A) $\pi \int_0^2 (5^2 - x^2)\, dx$ (B) $\pi \int_0^5 [25 - (x^2 - 5)]\, dx$

(C) $\pi \int_0^2 [(5 - x^2)^2 - 1]\, dx$ (D) $\pi \int_0^2 [1 - (5 - x^2)]\, dx$

9. The region in the first quadrant bounded by the graphs of $y = x + 2$, $y = 2x$, and the y-axis is revolved about the line $y = -1$. Find the volume of the solid generated.

(A) $\frac{52}{3}\pi$ (B) 12π (C) 28π (D) 32π

8.4 AP Practice Problems (p.611-612)

1. A region R is bounded by the graphs of the functions $f(x) = 3x$ and $g(x) = x^2, x \geq 0$. If R forms the base of a solid whose slices perpendicular to the x-axis are squares, then the volume of the solid is

(A) $\frac{9}{2}$ (B) $\frac{81}{10}$

(C) $\frac{45}{2}$ (D) $\frac{162}{5}$

2. A region R in the first quadrant is bounded by the graphs of $y = e^x$, $y = e^2$, and the y-axis. If R forms the base of a solid where slices perpendicular to the x-axis are semicircles, then the volume of the solid is given by

(A) $\frac{1}{2}\int_0^e \pi\left(\frac{e^2 - e^x}{2}\right)^2 dx$ (B) $\int_0^{e^2} \pi\left(\frac{e^2 - e^x}{2}\right)^2 dx$

(C) $\frac{1}{2}\int_0^2 \pi\left(\frac{e^2 - e^x}{2}\right)^2 dx$ (D) $\frac{1}{2}\int_0^2 \pi(e^2 - e^x)^2 dx$

48

8.4

3. The base of a solid is formed by the region bounded by the graphs of $y = e^{-x} + 1$, the x-axis, the y-axis, and the line $x = 4$. If slices perpendicular to the x-axis are squares, what is the volume of the solid?

(A) $\frac{1}{2}(5 - e^{-8} - 4e^{-4})$ (B) $8 - e^{-8} - 2e^{-4}$

(C) $\frac{1}{2}(9 - e^{-8})$ (D) $\frac{1}{2}(13 - e^{-8} - 4e^{-4})$

4. The base of a solid is the region in the first quadrant bounded by the coordinate axes and the parabola $y = -x^2 + 9$. Cross sections of the solid perpendicular to the x-axis are squares. What is the volume of the solid?

(A) 18 (B) $\frac{81}{4}$ (C) $\frac{648}{5}$ (D) $\frac{1458}{5}$

5. The base of a solid is the region in the first quadrant bounded by lines $y = 2x$, $x = 3$, and the x-axis. Every cross section of the solid perpendicular to the x-axis is an equilateral triangle. What is the volume of the solid?

(A) 9 (B) $\frac{9\sqrt{3}}{2}$ (C) $9\sqrt{3}$ (D) 27

8.4

6. Find the volume of the solid whose base is the region in the first quadrant that is bounded by the graphs of $y = x^2$, $y = 4$, and the y-axis if slices perpendicular to the y-axis are squares.

(A) $\frac{32}{5}$ (B) 8 (C) $\frac{256}{15}$ (D) $\frac{1024}{5}$

7. The base of a solid is the region bounded by the graph of $y = \ln x$, the line $x = e$, and the x-axis. If slices perpendicular to the y-axis are semicircles, then the volume of the solid is

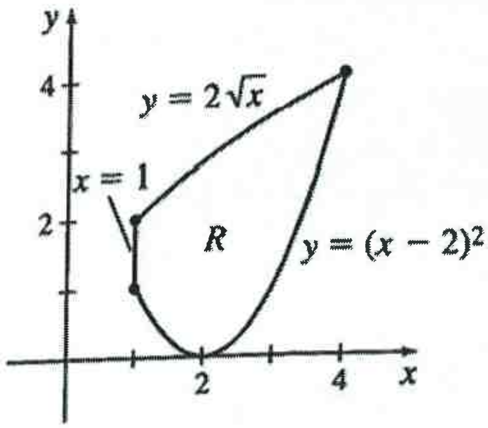
(A) $\frac{\pi}{16}\left[7e^2 - 4e - 1\right]$ (B) $\frac{\pi}{16}\left[-e^2 + 4e - 1\right]$

(C) $\frac{\pi}{8}\left[7e^2 - 4e - 1\right]$ (D) $\frac{\pi}{8}\left[-e^2 + 4e - 1\right]$

(50)

8.4

8. The region R bounded by the graphs of $y = (x-2)^2$, $y = 2\sqrt{x}$, and the line $x = 1$ is shown in the figure below.



(a) Find the area of R.

(b) Write, but do not evaluate, an integral expression that gives the volume of the solid of revolution that is generated when R is revolved about the x-axis.

(c) Write, but do not evaluate, an integral expression that gives the volume of the solid of revolution that is generated when R is revolved about the y-axis.

(d) The region forms the base of a solid. If cross sections of the solid perpendicular to the x-axis are squares, write, but do not evaluate, an expression that gives the volume of the solid.

Ch. 8 Unit Review AP Practice Problems (p.633)

1. The base of a solid is the region in the first quadrant bounded by the lines $y = 2x$, $x = 1$, and $x = 4$. Every cross section of the solid perpendicular to the x-axis is a semicircle. What is the volume of the solid?

(A) $\frac{21}{2}\pi$ (B) $\frac{32}{3}\pi$ (C) 21π (D) 42π

2. The area of the region bounded by the graphs of $y = e^x$ and $y = 2x + 4, 0 \le x \le 1$, is

(A) $4 - e$ (B) $5 - e$ (C) $6 - e$ (D) $e - 4$

3. What is the volume of the solid of revolution generated when the region bounded by the graph of $y = 4e^{-x^2}$, the x-axis, and the lines $x = 0$ and $x = 2$ is revolved about the y-axis?

(A) $2\pi(1 - e^{-4})$ (B) $4\pi e^{-4}$

(C) $4\pi(e^2 - 1)$ (D) $4\pi(1 - e^{-4})$

52

4. The region in the first quadrant bounded by the x-axis, the graph of $y = x^2$, and the line $x = 2$ is revolved about the line $y = -1$. The volume of the resulting solid of revolution is

(A) $\frac{22}{5}\pi$ (B) $\frac{52}{5}\pi$ (C) $\frac{176}{15}\pi$ (D) $\frac{232}{5}\pi$

5. A solid of revolution is formed by revolving the region bounded by the graphs of $y = \sin x$, $y = \cos x$, $0 \le x \le \frac{\pi}{4}$, and the line $x = 0$ about the x-axis. The volume of the solid is

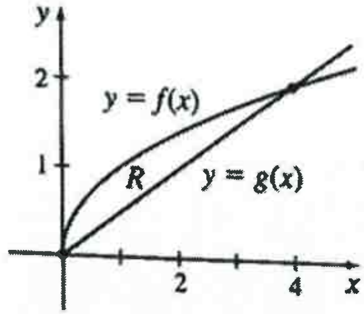
(A) $\frac{1}{4}\pi^2$ (B) $\frac{1}{2}\pi$ (C) $\frac{\sqrt{2}}{4}\pi$ (D) π

6. The region in the first quadrant bounded by the graph of $y = x^3$, the line $x = 2$, and the x-axis, is revolved about the y-axis. The volume of the solid of revolution is

(A) $\frac{32\pi}{5}$ (B) $\frac{64\pi}{5}$ (C) $\frac{256}{7}\pi$ (D) $\frac{65,536}{5}\pi$

7 (a, b)

7. The graphs of the functions $f(x) = \sqrt{x}$ and $g(x) = \frac{1}{2}x$ are shown in the figure below.

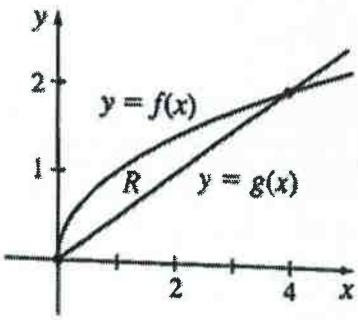


(a) Find the area of the region R bounded by the graphs of f and g.

(b) Find the volume of the solid of revolution generated by revolving the region R about the x-axis.

(54) 7 (c, d)

7. The graphs of the functions $f(x) = \sqrt{x}$ and $g(x) = \frac{1}{2}x$ are shown in the figure below.



(c) Find the volume of the solid of revolution generated by revolving the region R about the y-axis.

(d) The region R is the base of a solid. Find the volume of the solid if cross sections perpendicular to the base along the x-axis are squares.

* Need to be Memorized for Unit 8 Quiz *

55

Unit 8 Area & Volume Formula Sheet

$$Area = \int_{x_1}^{x_2} (Top\ \ graph - Bottom\ graph)dx$$

(in the forms of " y = __ ")

$$Area = \int_{y_1}^{y_2} (Right\ graph - Left\ graph)dy$$

(in the form of " x = ___ ")

Disc Method: (Top – Bottom) – Vertical Radius – Horizontal AOR

$$V = \pi \int_{x_1}^{x_2} [R(x)]^2\, dx$$

(expression(s) used above has form: " y = ___ ")

Disc Method: (Right – Left) – Horizontal Radius Vertical AOR

$$V = \pi \int_{y_1}^{y_2} [R(y)]^2\, dy$$

(expression(s) used above has form: " x = ___ ")

Washer Method: (Top – Bottom) , Vertical Radius (Horizontal AOR)

$$V = \pi \int_{x_1}^{x_2} [R(x)]^2 - [r(x)]^2\, dx$$

(expression(s) used above has form: " y = ___ ")

Washer Method: (Right – Left) , Horizontal Radius (Vertical AOR)

$$V = \pi \int_{y_1}^{y_2} [R(y)]^2 - [r(y)]^2 dy$$

(expression(s) used above has form: " x = ___ ")

Top-Bottom Vertical base

$$V = \int_{x_1}^{x_2} [\textbf{\textit{Area}}\ \textbf{of cross section}]dx$$

*Note: All values in integral are in terms of x
(in the form of " y = ____ ")

Right-Left Horizontal base

$$V = \int_{y_1}^{y_2} [\textbf{\textit{Area}}\ \textbf{of cross section}]dy$$

*Note: All values in integral are in terms of y
(in the forms of " x = _______ ")

Area formulas for Cross sections:

1. Square: $A = (base)^2$	2. Isosceles Right Triangle (leg on base): $A = \frac{1}{2}(base)^2$	3. Isosceles Right Triangle (hypotenuse on base): $A = \frac{1}{4}(base)^2$
4. Rectangle: A = (base)(height)	5. Equilateral Triangle: $A = \frac{\sqrt{3}}{4}(base)^2$	6. Semicircle: $A = \frac{\pi}{8}(base)^2$

56

Washer Method: Top 5 Student Mistakes

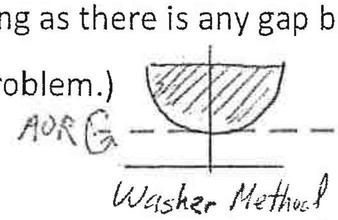
1) When drawing your radius, **Always include and connect to the Axis of Revolution (AOR, your dotted line)**. Don't connect between the 2 curves. That will not represent the radius. Remember, the AOR is the Center of the rotated object, so the radius needs to connect to the center.

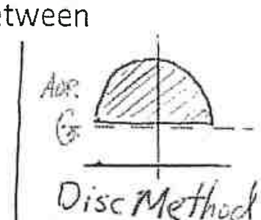
2) Use appropriate Washer Method Formula. Remember to square each of the radius separately.

Correct:	**Incorrect:**
$V = \pi \int_{x_1}^{x_2} [R(x)]^2 - [r(x)]^2 dx$	$V = \pi \int_{x_1}^{x_2} [R(x) - r(x)]^2 dx$

3) The Axis of Revolution has NO impact on the bounds of integration. Your bounds purely depends on the boundaries of your shaded region.

4) Just because the Axis of Revolution is touching the shaded region does not automatically mean this is a Disc Method problem. **Disc Method** is only when the Axis of Revolution is up against **a flat surface** of the shaded region. An Axis up against a **curved surface** means this is **a Washer Method** problem. (As long as there is any gap between AOR and shaded region, this will be a washer method problem.)





5) Remember when you have Horizontal Radius drawn on your diagram, you'll need to adjust your equations so that they start with "x = ___" in order to use them for your radius expressions. The original equations in the form of "y = ____" are suitable for Top-Bottom (Vertical Radius) but not for Right-Left (Horizontal Radius).

6) Remember to distribute the negative through when creating the Radius expressions: Use parentheses to help. Top - (Bottom) or Right - (Left)