Name: ______________________________ Period: __________

BC Calculus

Unit 9 Packet

Parametric,
Polar,
& Vector-Valued Functions

BC Calculus January 2024 Class Calendar

Monday	Tuesday	Wednesday	Thursday	Friday
15 **MLK Day** **No School**	16 **Teacher Workday** *(No School)*	17 Ch. 6-8 BC Topics Quiz Review	18 **Ch. 6-8 BC Topics Quiz**	19 9.1 – Defining and Differentiating Parametric Equations HW: 9.1 Packet Classwork Problems & 9.1 AP Practice (1-4)
22 9.2 – Equation of tangent line on curve, arc length & 2[nd] Derivative of Parametric Equations HW: 9.2 Packet Classwork Problems	23 9.3 –Parametric Arc Length HW: 9.3 Packet Classwork & 9.2 AP Practice (1-7 all)	24 9.5a -9.5b– Derivatives and Integrals of Vector Functions (arc length) HW: 9.5a-9.5b Packet Classwork Problems & 9.5 AP Practice (1-7 all)	25 9.6 – Motion along a Curve HW: 9.6 Packet Classwork Problems & 9.6 AP Practice (1-6 all)	26 9.4 prep – Graphing Polar Equations HW: 9.7 AP Practice (1-6 all)
29 9.4a – Area in Polar Coordinates HW: 9.4a Packet Classwork Problems	30 9.4b – Polar Area HW: 9.4b Packet Classwork Problems & 9.3 AP Practice (1-5 all) **pg. 55-56 in packet**	31 9.4c Polar Area Review HW: 9.4c Packet Classwork Problems & 9.4 AP Practice (1-5 all)	Feb 1 Unit 9 Test Review Day 1 HW: Complete Unit 9 Review WS #1 & Unit 9 Review AP Practice (1-11 all)	Feb 2 Unit 9 Test Review Day 2 HW: Complete Unit 9 Review WS #2 & Unit 9 Review AP Practice (12-25 all)

February 2024

Monday	Tuesday	Wednesday	Thursday	Friday
5 10.1 – Define Convergent and Divergent Infinite Series HW: Complete 10.1 Classwork Problems and AP Practice (1-5 all)	6 **Unit 9 Test**	7 10.2 – Working with Geometric Series HW: Complete 10.2 Classwork Problems & 10.2 AP Practice	8 10.3 – n^{th} term Test, Integral Test, P-Series Test for Convergence HW: Complete 10.3 Classwork Problems & 10.3 AP Practice	9 10.4 – Direct and Limit Comparison Tests for Convergence HW: Complete 10.4 Classwork Problems & 10.4 AP Pracice

BC Calculus – 9.1 Notes – Defining and Differentiating Parametric Equations

We have been looking at graphs of one equation with two variables, typically x and y. Now we are looking at three variables that will represent a curve in the plane.

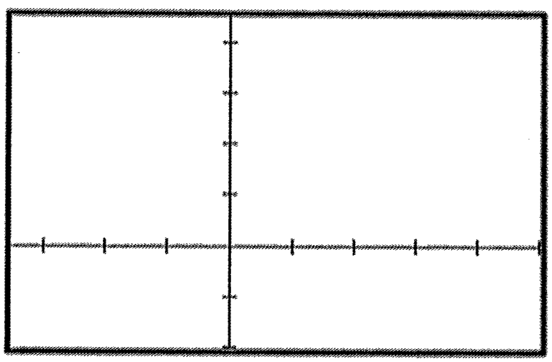
In the rectangular equation we are able to determine where the object is located at a point (x, y), but with the addition of the third variable (often t), we are able to determine when the object was at a point (x, y). NOTE: the third variable t is often time, but not always.

Parametric Equations

If f and g are continuous functions of t on an interval I, then the equations $x = f(t)$ and $y = g(t)$ are parametric equations and t is the parameter. You can sketch the curve of a parametric by substituting in values for t.

1. Sketch the curve with the following parametrization: $x(t) = 2t$ and $y(t) = t^2 - 1$, with $-1 \le t \le 2$.

t	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
x							
y							



WINDOW
Tmin=-1
Tmax=2
Tstep=0.2
Xmin=-3.5
Xmax=5
Xscl=1
Ymin=-2
Ymax=4.5
Yscl=1

To find the rectangular equation when you are given the parametric equations, eliminate the parameter t through substitution.

2. Given $x(t) = 2t, y(t) = t^2 - 1$. Find the rectangular equation by eliminating the parameter.

3. Given the parametric equations $x(t) = 2\cos t$ and $y(t) = 2\sin t$. Eliminate the parameter.

②

Derivative of a Parametric Equation

The derivative of a parametric given by $x = f(t)$ and $y = g(t)$ is found by the following:

4. Given $x(t) = t^{\frac{1}{2}}$ and $y(t) = \frac{1}{4}(t^2 - 4)$ for $t \geq 0$. Find $\frac{dy}{dx}$

5. Given $x(t) = e^{2t}$ and $y(t) = \cos t$ for $t \geq -1$. Find the equation of a tangent line when $t = \frac{\pi}{2}$.

9.1 Practice Problems:

1. For the given parametric equations, eliminate the parameter and write the corresponding rectangular equation. $x = e^{-t}$ and $y = e^{2t} - 1$.	2. Let C be a curve described by the parametrization $x = 5t$ and $y = t^4 + 3$. Find an expression for the slope of the line tangent to C at any point (x, y).
3. The position of a particle at any time $t \geq 0$ is given by $x(t) = 3t^2 + 1$ and $y(t) = \frac{2}{3}t^3$. Find $\frac{dy}{dx}$ as a function of x.	4. A particle moves along the curve $xy + y = 9$. If $x = 2$ and $\frac{dy}{dt} = 3$, what is the value of $\frac{dx}{dt}$?

5. A curve is described by the parametric equations $x = t\cos t$ and $y = t\sin t$. Find the equation of the line tangent to the curve at the point determined by $t = \pi$.

6. **Calculator active.** The coordinates $(x(t), y(t))$ of the position of a drone change at rates given by $x'(t) = 2t^3$ and $y'(t) = t^{\frac{1}{2}}$, where $x(t)$ and $y(t)$ are measured in meters and t is measured in seconds. At what time t, for $0 \le t \le 2$, does the slope of the line tangent to its path have a slope of 1.5?

7. A curve in the xy-plane is defined by the parametric equations $x(t) = \cos(3t)$ and $y(t) = \sin(3t)$ for $t \ge 0$. What is the value of $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$?

8. A curve is defined by the parametric equations $x(t) = at^2 + b$ and $y(t) = ct - b$, where a, b, and c are nonzero constants. What is the slope of the line tangent to the curve at the point $(x(t), y(t))$ when $t = 2$?

9. **No Calculator.** For $0 \le t \le 11$ the parametric equations $x = 3\sin t$ and $y = 2\cos t$ describe the elliptical path of an object. At the point where $t = 11$, the object travels along a line tangent to the path at that point. What is the slope of that line?

10. A particle moves in the xy-plane so that its position for $t \ge 0$ is given by the parametric equations $x(t) = 2kt^2$ and $y(t) = 3t$, where k is a positive constant. When $t = 2$ the line tangent to the particle's path has a slope of 4. What is the value of k?

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11. Find the equation of the line tangent to the curve defined parametrically by the equations $x(t) = t^3 + 2t$ and $y(t) = 2t^4 + 2t^2$ when $t = 1$.

12. For what values of t does the curve given by the parametric equations $x(t) = \frac{1}{4}t^4 - \frac{9}{2}t^2$ and $y(t) = 3t^3 + 2t$ have a vertical tangent?

13. Suppose a curve is given by the parametric equations $x = f(t)$ and $y = g(t)$, for all $t > 1$ and $\frac{dy}{dt} = \frac{t^2+2}{t-1} * \frac{dx}{dt}$. What is the value of $\frac{dy}{dx}$ when $t = 2$?

9.1 Parametric Equations

Test Prep

14. A curve is defined parametrically by $x(t) = t^2$ and $y(t) = t^3 - 3t$. Find the points on the graph where the tangent line is horizontal or vertical.

15. **Free Response.** Consider the curve given by the parametric equations $y = t^3 - 12t$ and $x = \frac{1}{2}t^2 - t$.

a. Find $\frac{dy}{dx}$ in terms of t.

b. Write an equation for the line tangent to the curve at the point where $t = -1$.

c. Find the x and y coordinates for each critical point on the curve and identify each point as having a vertical or horizontal tangent.

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16. A curve is given by the parametric equations $x(t) = 5t^3 - 5$ and $y(t) = t^2 + 7$. What is the equation of the tangent line to the curve when $t = 1$?

A. $x = 0$ B. $y = \frac{2}{15}x + 8$ C. $y = \frac{2}{15}x + 1$ D. $y = 8$ E. $y = \frac{15}{2}x + 7$

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9.1 AP Practice Problems (p.651) – Parametric Equations

1. Which pair of parametric equations represents a plane curve that is a circle with a radius of 3 and a center at (0, 0)?

(A) $x(t) = 9\cos t,\ y(t) = 9\sin t,\ 0 \le t \le 2\pi$

(B) $x(t) = 3\sin t,\ y(t) = 3\cos t,\ 0 \le t \le \pi$

(C) $x(t) = 3\sin(2t),\ y(t) = 3\cos(2t),\ 0 \le t \le \pi$

(D) $x(t) = \cos(3t),\ y(t) = \sin(3t),\ 0 \le t \le 2\pi$

2. A rectangular equation of the curve whose parametric equations are $x(t) = t + 1,\ y(t) = t^2 + 3t$ is

(A) $y = x^2 + x - 2$ (B) $y = x^2 + 3x$

(C) $y = x^2 + 5x + 4$ (D) $y = x^2 + 3x - 2$

3. A rectangular equation of the parametric equations $x(t) = 4\cos t$, $y(t) = \frac{1}{2}\sin t$ is

(A) $\frac{x^2}{16} + \frac{y^2}{4} = 1$ (B) $\frac{x^2}{16} + 2y^2 = 1$

(C) $16x^2 + 4y^2 = 1$ (D) $\frac{x^2}{16} + 4y^2 = 1$

4. An object is moving along a plane curve according to the parametric equations

$$x(t) = -5\cos\left(\frac{\pi}{2}t\right),\ y(t) = 2\sin\left(\frac{\pi}{2}t\right),\ 0 \le t \le 8$$

(a) Find a rectangular equation for the parametric equations.

(b) Describe the motion of the object from $t = 0$ to $t = 8$. Be sure to include where the object begins, where it ends, its path, and its orientation.

(8)

BC Calculus – 9.2 Notes – 2nd Derivative of Parametric Equations

Second Derivative of a Parametric Equation

The second derivative of a parametric given by $x = f(t)$ and $y = g(t)$ is

Given the following parametric equations, find $\frac{d^2y}{dx^2}$ in terms of t.

1. $x(t) = \sqrt{t}$ and $y(t) = \frac{1}{2}(t^2 - 2)$ for $t \geq 0$.

2. $x = 3\cos t$ and $y = 4\sin t$.

3. At $t = 1$, find the concavity of the graph defined parametrically by $x = t^3 + 1$ and $y = t^4 + t$.

9.2 practice problems

Given the following parametric equations, find $\frac{d^2y}{dx^2}$ in terms of t.

1. $x(t) = e^{-2t}$ and $y(t) = e^{2t}$.

2. $x(t) = t^3$ and $y(t) = t^4 + 1$ for $t > 0$.

3. $x(t) = at^3$ and $y(t) = bt$, where a and b are positive constants.

4. $\frac{dx}{dt} = 4$ and $\frac{dy}{dt} = \sin(t^2)$.

5. $x = e^t$ and $y = te^{-t}$.

6. $x = t^2 + 1$ and $y = 2t^3$.

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7. Given a curve defined by the parametric equations $x(t) = 2 - t^2$ and $y(t) = t^2 + t^3$. Determine the open t-intervals on which the curve is concave up or down.

8. If $x(\theta) = 2 + \sec\theta$ and $y(\theta) = 1 + 2\tan\theta$, Find the slope and the concavity at $\theta = \frac{\pi}{6}$.

9. If $x = \cos\theta$ and $y = 3\sin\theta$, find the slope and concavity at $\theta = 0$.

10. If $x(t) = t - \ln t$ and $y(t) = t + \ln t$, determine values of t where the graph is concave up.

9.2 Second Derivatives of Parametric Equations

Test Prep

11. If $x = 3t^2 - 1$ and $y = \ln t$, what is $\frac{d^2y}{dx^2}$ in terms of t?

A. $\frac{1}{6}t^2$ B. $-\frac{1}{3}t^{-3}$ C. $-\frac{1}{18}t^{-4}$ D. $-\frac{1}{2}t^{-4}$ E. $6t^4$

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12. If $x = \theta - \cos\theta$ and $y = 1 - \sin\theta$, find the slope and concavity at $\theta = \pi$.

A. Slope: -1, Concave down B. Slope: π, Concave up C. Slope: 1, Concave down

D. Slope: 1, Concave up E. Slope: $\frac{1}{\pi}$, Concave up

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9.2 AP Practice Problems (p.660) – Parametric Arc Length

1. The smooth curve C is represented by the parametric equations $x(t) = \tan t$, $y(t) = t^2 - 3t + 8$, $-\frac{\pi}{4} \leq t < \frac{\pi}{2}$. The slope of the tangent line to C at the point $(0, 8)$ is

   (A) -3 (B) $-\frac{3}{2}$ (C) 0 (D) $-\frac{1}{3}$

2. Which of the following is an equation of the tangent line to the plane curve represented by the parametric equations $x(t) = 2t - 5$, $y(t) = t^3$ when $t = 2$?

   (A) $y = -6x + 14$ (B) $y = 6x$
   (C) $y = 6x + 2$ (D) $y = 6x + 14$

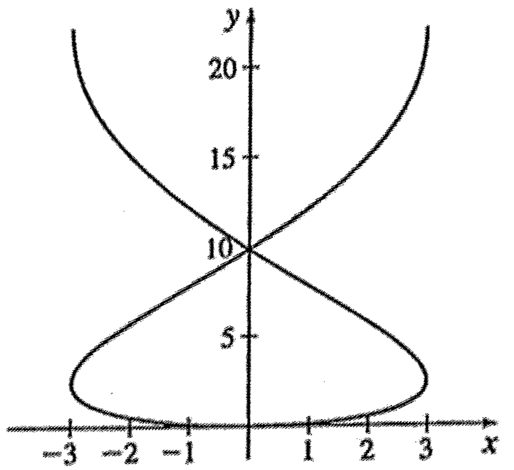
3. For the smooth curve, $x(t) = t^3 - 12t$, $y(t) = 4t^2 + t$, find all the points where the tangent line is either horizontal or vertical.

   (A) Horizontal: $t = \frac{1}{8}$; Vertical: $t = -2, 2$
   (B) Horizontal: $t = -2, 2$; Vertical: none
   (C) Horizontal: $t = -\frac{1}{8}$; Vertical: $t = -2, 2$
   (D) Horizontal: none, Vertical $t = -\frac{1}{8}$

4. The plane curve represented by the parametric equations

$$x(t) = 3\sin t,\ y(t) = t^2,\ -\frac{3}{2}\pi \le t \le \frac{3}{2}\pi$$

is shown below. The graph intersects itself at the point $(0, \pi^2)$.



(a) Find the numbers t that correspond to the point $(0, \pi^2)$.

(b) Find $\frac{dx}{dt}$ and $\frac{dy}{dt}$ and confirm that $\frac{dx}{dt} \neq 0$ at the point of intersection.

(c) Find the slope of the tangent lines at the point of intersection.

(d) Find an equation of each tangent line at the point of intersection.

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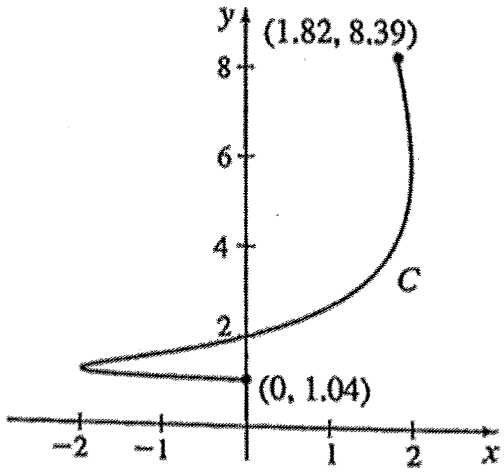
5. The smooth curve C represented by the parametric equations $x(t) = 2\sin t$, $y(t) = 1 + e^t$, $-\pi \le t \le 2$, is shown below.



The arc length of C is given by the integral

(A) $\int_{-2}^{1.8} \sqrt{4\cos^2 t + e^{2t}}\, dt$ (B) $\int_{-\pi}^{2} \sqrt{4\sin^2 t + (1 + e^t)^2}\, dt$

(C) $\int_{-\pi}^{2} \sqrt{4\cos^2 t + e^{2t}}\, dt$ (D) $\int_{0}^{1.8} \sqrt{4\cos^2 t + e^{2t}}\, dt$

6. The length of the curve represented by the parametric equations $x(t) = 2 + 4t^3$, $y(t) = 1 + 6t^2$, from $t = 0$ to $t = 1$ is

(A) $8\sqrt{2} - 4$ (B) $2^{7/2} - 1$ (C) 4 (D) $16\sqrt{2} - 8$

7. An object moves along the plane curve C represented by the parametric equations $x(t) = \cos(2t)$, $y(t) = \cos^2 t$. The distance the object travels from $t = \frac{\pi}{4}$ to $t = \frac{\pi}{3}$ is

(A) $\frac{1}{4}$ (B) $\frac{\sqrt{5}}{4}$ (C) $\frac{5}{4}$ (D) $\frac{\sqrt{5}}{2}$

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BC Calculus – 9.3 Notes – Finding Arc Lengths (Parametric Equations)

Recall: Arc Length

$$L = \int_a^b \sqrt{1 + \left(f'(x)\right)^2}\, dx$$

Arc Length in Parametric Form

For each set of parametric equations, find the length of the curve on the given interval.

1. $x(t) = \cos t$ and $y(t) = \sin t$ on the interval $0 \le t \le 2\pi$.	2. $x = 1 - 4t$ and $y = 7t$ on the interval $0 \le t \le 2$.

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9.3 Arc Length (Parametric Form)

Calculus

Practice

What is the length of the curve defined by the parametric equations? Solve without the use of a calculator.

1. $x(t) = 6t + 10$ and $y(t) = 14 - 4t$ for the interval $-1 \leq t \leq 3$?

2. $x = \frac{a}{2}t^2$ and $y = \frac{b}{2}t^2$, where a and b are constants. What is the length of the curve from $t = 0$ to $t = 1$?

3. $x(t) = 2t^2$ and $y(t) = \frac{2}{3}t^3$ for the interval $1 \leq t \leq 4$?

4. $x(\theta) = 5\cos\theta$ and $y(\theta) = 5\sin\theta$ for the interval $0 \leq \theta \leq 2\pi$.

5. $x(t) = 7t - 2$ and $y(t) = 4 - 8t$ for the interval $1 \leq t \leq 5$.

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6. If a curved is described by the parametric equations $x = t^2$ and $y = 2e^{2t}$, then which of the following gives the length of the path from $t = 0$ to $t = \ln 3$?

A. $\int_0^{\ln 3} \sqrt{4t^2 + 4e^{4t}}\, dt$

B. $\int_0^{\ln 3} \sqrt{t^4 + 4e^{4t}}\, dt$

C. $\int_0^{\ln 3} \sqrt{4t^2 + 16e^{4t}}\, dt$

D. $\int_0^{\ln 3} \sqrt{t^2 + 2e^{2t}}\, dt$

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7. Which of the following gives the length of the path described by the parametric equations $x = 2 + 4t$ and $y = 3 + t^2$ from $t = 0$ to $t = 1$?

A. $\int_0^1 \sqrt{4 + 2t}\, dt$

B. $\int_0^1 \sqrt{(2 + 4t)^2 + (3 + t^2)^2}\, dt$

C. $\int_0^1 \sqrt{16t^2 + t^4}\, dt$

D. $\int_0^1 \sqrt{16 + 4t^2}\, dt$

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8. Which of the following gives the length of the path described by the parametric equations $x = \cos t^3$ and $y = e^{5t}$ from $t = 0$ to $t = \pi$?

A. $\int_0^{\pi} \sqrt{9t^4 \sin^2(t^3) + 25e^{10t}}\, dt$

B. $\int_0^{\pi} \sqrt{-3t^2 \sin(t^3) + 5e^{5t}}\, dt$

C. $\int_0^{\pi} \sqrt{9t^4 \sin^2(t^3) + 25e^{5t}}\, d$

D. $\int_0^{\pi} \sqrt{(\cos(t^3))^2 + (e^{5t})^2}\, dt$

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9. Which of the following gives the length of the path described by the parametric equations $x = \sin 3t$ and $y = \cos 2t$ from $t = 0$ to $t = \pi$?

A. $\int_0^{\pi} \sqrt{\sin^2 3t + \cos^2 2t}\, dt$

B. $\int_0^{\pi} \sqrt{\cos^2 3t + \sin^2 2t}\, dt$

C. $\int_0^{\pi} \sqrt{9 \cos 3t + 4 \sin 2t}\, dt$

D. $\int_0^{\pi} \sqrt{9 \cos^2 3t + 4 \sin^2 2t}\, dt$

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10. Which of the following gives the length of the path described by the parametric equations $x = \sqrt{t}$ and $y = 3t - 1$ from $0 \le t \le 1$?

A. $\int_0^1 \sqrt{\frac{t}{4} + 9}\, dt$

B. $\int_0^1 \sqrt{\frac{1}{4}t^{-1} + 9}\, dt$

C. $\int_0^1 \sqrt{\frac{1}{4}t + 3}\, dt$

D. $\int_0^1 \sqrt{\frac{1}{2}t^{-\frac{1}{2}} + 3}\, dt$

No test prep. Problems 6-10 are great examples of problems you may see on the AP Exam.

BC Calculus – 9.5a & 9.5b - Derivatives and Integrals for Vector-Valued Functions

Vector basics:

- Vectors have magnitude (length) and direction.
- Vectors can be represented by directed line segments.
- Vectors are equal if they have the same direction and magnitude.
- Magnitude is designated by $||v||$
- Vectors have a horizontal and vertical component.
- Component form of a vector is $\langle x, y\rangle$

1. Find the component form and magnitude of the vector that has an initial point of (1,2) and terminal point (5,4).

   Component form:

   Magnitude:

Vector-Valued Functions: $r(t) = \langle f(t), g(t)\rangle$ where $f(t)$ and $g(t)$ are the component functions with the parameter t.

Differentiation of Vector-Valued Functions

If $r(t) = \langle f(t), g(t)\rangle$ then

Properties of the derivative for vector-valued functions

$\frac{d}{dt}[c \cdot r(t)] = c \cdot r'(t)$

$\frac{d}{dt}[r(t) \cdot s(t)] = r'(t)s(t) + r(t)s'(t)$

$\frac{d}{dt}[r(t) \pm s(t)] = r'(t) \pm s'(t)$

$\frac{d}{dt}[r(s(t))] = r'(s(t)) \cdot s'(t)$

1. $r(t) = \langle 2t^2 + 4t + 1,\ 3t^3 - 4t\rangle$ then $r'(t) =$

2. $r(t) = \langle t^3 + 5,\ 2t\rangle$ find $\frac{d}{dt}r(2t)$

3. The path of a particle moving along a path in the xy-plane is given by the vector-valued function, $f(t) = \langle t^2, \sin t\rangle$. Find the slope of the path of the particle at $t = \frac{3\pi}{4}$.

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9.5b - Integrals for Vector-Valued Functions

Integration of Vector-Valued Functions

If $r(t) = \langle f(t), g(t) \rangle$ then

1. Find $r(t)$ if $r'(t) = \langle 4e^{2t}, 2e^{t} \rangle$ and $r(0) = \langle 2, 0 \rangle$

2. Find $r(t)$ if $r'(t) = \langle \sec^2 t, \frac{1}{1+t^2} \rangle$

3. $\int_{-1}^{1} \langle t^3, t^{\frac{1}{5}} \rangle \, dt$

For problems 1-6, find the vector-valued function $f(t)$ that satisfies the given initial conditions.

1. $f(0) = \langle 2, 4 \rangle$, $f'(t) = \langle 2e^{t}, 3e^{3t} \rangle$

2. $f(0) = \langle \frac{1}{2}, -1 \rangle$, $f'(t) = \langle te^{-t^2}, -e^{-t} \rangle$

9.5a Derivatives of Vector-Valued Functions

Calculus

Practice

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Each problem contains a vector-valued function. Find the given first or second derivative.

1. $f(t) = \langle 4t^3 + 2t^2 + 7t,\ 4t^2 + 3t \rangle$, then $f'(t) =$

2. $f(t) = \langle 3\sin 2t, 4\cos 3t \rangle$, then $f'\left(\frac{\pi}{6}\right) =$

3. $f(t) = \langle 3e^{2t},\ 5e^{4t} \rangle$, then $f''(t) =$

4. $f(t) = \langle t^{-2},\ (t+1)^{-1} \rangle$, then $f''(-2) =$

5. $f(t) = \langle e^t + e^{-t},\ e^t - e^{-t} \rangle$, then $f'(t) =$

6. $f(t) = \langle 2\sin 4t, 2\cos 3t \rangle$, then $f'(t) =$

7. $f(t) = \langle t\sin t,\ t\cos t \rangle$, then $f'\left(\frac{\pi}{2}\right) =$

8. $f(t) = \langle 3t^2 + 6t + 1,\ 4t^3 - 2t^2 + 6t \rangle$, then $f'(1) =$

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9. The path of a particle moving along a path in the xy-plane is given by the vector-valued function, $f(t) = \langle t^3 + 2t^2 + t,\ 2t^3 - 4t \rangle$. Find the slope of the path of the particle at $t = 3$.

10. The position of a particle moving in the xy-plane is defined by the vector-valued function, $f(t) = \langle t^3 - 6t^2,\ 2t^3 - 9t^2 - 24t \rangle$. For what value of $t \geq 0$ is the particle at rest?

9.5a Derivatives of Vector-Valued Functions

Test Prep

11. **Calculator active.** The path of a particle moving along a path in the xy-plane is given by the vector-valued function f and f' is defined by $f'(t) = \langle t^{-1},\ 2ke^{kt} \rangle$ where k is a positive constant. The line $y = 4x + 5$ is parallel to the line tangent to the path of the particle at the point where $t = 2$. What is the value of k?

12. At time t, $0 \leq t \leq 2\pi$, the position of a particle moving along a path in the xy-plane is given by the vector-valued function, $f(t) = \langle t \sin t, \cos 2t \rangle$. Find the slope of the path of the particle at time $t = \frac{\pi}{2}$.

9.5b Find f(t)

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3. $f(0) = \langle 3, 1\rangle$, $f'(t) = \langle 6t^2, 4t\rangle$

4. $f(0) = \langle -2, 5\rangle$, $f'(t) = \langle 2\cos t, -3\sin t\rangle$

5. $f'(0) = \langle 3, 0\rangle, f(0) = \langle 0, 3\rangle,$
$f''(t) = \langle 5\cos t, -2\sin t\rangle$

6. $f'(0) = \langle 0, 2\rangle, f(0) = \langle 3, 0\rangle, f''(t) = \langle 4t^3, 3t^2\rangle$

7. **Calculator active.** For $t \geq 0$, a particle is moving along a curve so that its position at time t is $\big(x(t), y(t)\big)$. At time $t = 1$, the particle is at position $(2, 4)$. It is known that $\frac{dx}{dt} = \frac{\sqrt{t+3}}{e^t}$ and $\frac{dy}{dt} = \cos^2 t$. Find the x-coordinate of the particles position at time $t = 5$.

8. The instantaneous rate of change of the vector-valued function $f(t)$ is given by $f'(t) = \langle 8t^3 + 2t, 10t^4\rangle$. If $f(1) = \langle 3, 7\rangle$, what is $f(-1)$?

9. **Calculator active.** At time $t \geq 0$, a particle moving in the xy-plane has velocity vector given by $v(t) = \langle 3t^2, 3 \rangle$. If the particle is at point $(1, 2)$ at time $t = 0$, how far is the particle from the origin at time $t = 2$?

10. **Calculator active.** At time $t \geq 0$, a particle moving in the xy-plane has velocity vector given by $v(t) = \langle 2, \frac{\cos t}{e^t} \rangle$. If the particle is at point $(1, 2)$ at time $t = 0$, how far is the particle from the origin at time $t = 3$?

9.5b Integrating Vector-Valued Functions

Test Prep

11. **Calculator active.** A remote controlled car travels on a flat surface. The car starts at the point with coordinates $(7, 6)$ at time $t = 0$. The coordinates $(x(t), y(t))$ of the position change at rates given by $x'(t) = -10 \sin t^2$ and $y'(t) = 9 \cos(2 + \sqrt{t})$, where $x(t)$ and $y(t)$ are measured in feet and t is measured in minutes. Find the y-coordinate of the position of the car at time $t = 1$.

12. The instantaneous rate of change of the vector-valued function $f(t)$ is given by $f'(t) = \langle 2 + 20t - 4t^3,\ 6t^2 + 2t \rangle$. If $f(1) = \langle 5, -3 \rangle$, what is $f(-1)$?

9.5 AP Practice Problems (p.683) – Derivatives, Arc Length of Vector Functions

1. The domain of the vector function $\mathbf{r}(t) = \langle 3t^2 + t, -\ln(t+2) \rangle$ is

   (A) the set of all real numbers (B) $\{t \mid t > 0\}$

   (C) $\{t \mid t > -2\}$ (D) $\{t \mid t > -2,\ t \neq -1\}$

2. $\lim_{t \to 2} \left\langle t, \dfrac{t^2 - 2t}{t - 2} \right\rangle =$

   (A) $\langle 2, -2 \rangle$ (B) $\langle 2, 0 \rangle$

   (C) $\langle 2, 2 \rangle$ (D) The limit does not exist.

3. The derivative of the vector function $\mathbf{r}(t) = \langle \sin(3t), -\cos(3t) \rangle$ is

   (A) $\mathbf{r}'(t) = \langle \cos(3t), \sin(3t) \rangle$

   (B) $\mathbf{r}'(t) = \left\langle \frac{1}{3}\cos(3t), \frac{1}{3}\sin(3t) \right\rangle$

   (C) $\mathbf{r}'(t) = \langle 3\cos(3t), 3\sin(3t) \rangle$

   (D) $\mathbf{r}'(t) = \langle 3\cos(3t), -3\sin(3t) \rangle$

4. The arc length of the curve traced out by $\mathbf{r}(t) = \langle t^3 + 2t, \ln t \rangle$ from $t = 1$ to $t = 5$ is given by

   (A) $\displaystyle\int_1^5 \sqrt{(t^3 + 2t)^2 + (\ln t)^2}\, dt$

   (B) $\displaystyle\int_1^5 \sqrt{(3t^2 + 2)^2 + \frac{1}{t^2}}\, dt$

   (C) $\displaystyle\int_1^5 \sqrt{\frac{9t^4 + 12t^2 + 5}{t^2}}\, dt$

   (D) $\displaystyle\int_1^5 \sqrt{(3t^2 + 2)^2 - \frac{1}{t^2}}\, dt$

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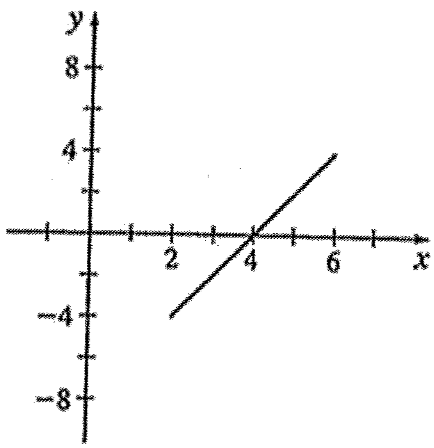
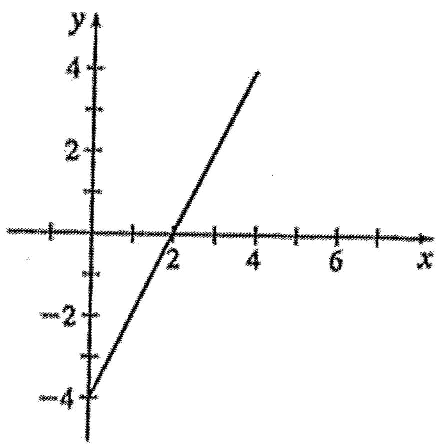
5. Which of the following is the second derivative of the vector function $\mathbf{r}(t) = \left\langle 4t, \frac{1}{2}t^2 + e^t \right\rangle$?

(A) $\langle 4, t + e^t \rangle$ (B) $\langle 0, 1 + e^t \rangle$
(C) $\langle 1, e^t \rangle$ (D) $\langle 0, e^t \rangle$

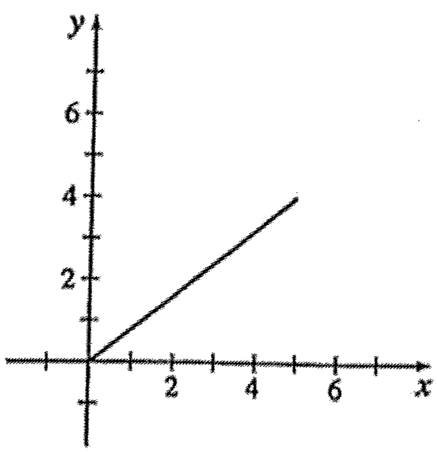
6. Which curve is traced out by the vector function $r(t) = \langle t, 2t - 4 \rangle, 0 \le t \le 4$?



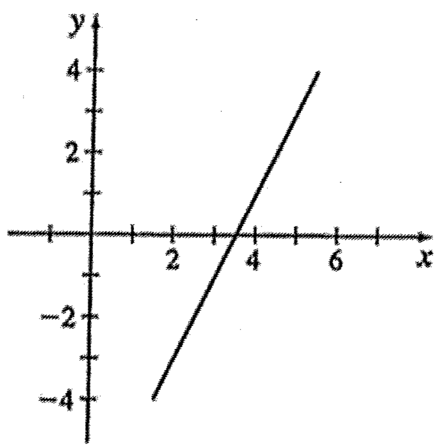
(A)



(B)



(C)



(D)

7. The tangent vector to the curve traced out by the vector function $\mathbf{r}(t) = \left\langle t^2 + 5, 8 - 3t \right\rangle$ at $t = 2$ is

(A) $\langle 4, -3 \rangle$ (B) $\langle 4, 3 \rangle$ (C) $\langle 4, -6 \rangle$ (D) $\langle 9, 2 \rangle$

BC Calculus – 9.6 Motion Using Parametrics and Vectors Notes

Position: $r(t) = \langle x(t), y(t) \rangle$

Velocity: $v(t) = r'(t) = \langle x'(t), y'(t) \rangle$

Acceleration: $a(t) = r''(t) = \langle x''(t), y''(t) \rangle$

Speed: $\|v(t)\| = \|r'(t)\| =$

1. Find the velocity vector, speed, and acceleration vector for the particle that moves in the xy-plane described by $r(t) = \langle 5\sin\frac{t}{5}, 5\cos\frac{t}{5} \rangle$

Quick review: When does a particle's speed increase or decrease?

Speeding up

Velocity & Acceleration have

Slowing down

Velocity & Acceleration have

2. If $r(t) = \langle 2t^3 + t, t^2 \rangle$, find velocity and acceleration at time t.

3. Find the speed at time $t = 2$ if $r(t) = \langle 3t, e^{-t^2} \rangle$

Total Distance Traveled by a Partice on $[a, b]$.

$$\int_a^b \|v(t)\|\, dt =$$

(28)

4. Given the velocity vector of the particle $v(t) = \langle 2t + 1,\ 5 \rangle$ and the position of the particle at time $t = 0$ is $(1, 2)$, find the position when $t = 3$. What is the total distance traveled on the interval $0 \leq t \leq 3$?

5. A particle moving along a curve so that its velocity for time $t \geq 0$ is given by $v(t) = \langle 2e^{-\frac{t}{4}}, \frac{t-4}{t+5} \rangle$.
   a. For what values of t is the particle moving to the right?

   b. For what values of t is the particle moving up?

Practice Problems:

For each problem, a particle moves in the xy-plane where the coordinates are defined at any time t by the position function given in parametric or vector form.

1. $x(t) = 4t^2$ and $y(t) = 2t - 1$. Find the velocity vector at time $t = 1$.	2. $x(t) = e^{-t}$ and $y(t) = e^{t}$. Find the acceleration vector at time $t = 1$.

3. $\left(x(t), y(t)\right) = (6 - 2t, t^2 + 3)$. In which direction is the particle moving as it passes through the point $(4, 4)$?

4. A position vector is $r(t) = \langle \frac{2}{t},\ e^{4t} \rangle$ for time $t > 0$. What is the velocity vector at time $t = 1$?

5. $r(t) = \langle \ln(t^2 + 1), 3t^2 \rangle$ for $t > 0$. Find the velocity vector at time $t = 2$.

6. $x(t) = 2\sin\frac{t}{2}$ and $y(t) = 2\cos\frac{t}{2}$ for time $t > 0$. Find the speed of the particle.

7. **Calculator active.** $x(t) = t^2 + 1$ and $y(t) = \frac{4}{3}t^3$ for time $t \geq 0$. Find the total distance traveled from $t = 0$ to $t = 3$.

8. $p(t) = \langle \cos 2t, 2\sin t \rangle$. Find the velocity vector $v(t)$.

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9. **Calculator active.** The velocity vector of a particle moving in the xy-plane has components given by $\frac{dx}{dt} = \cos t^2$ and $\frac{dy}{dt} = e^{t-2}$. At time $t = 3$, the position of the particle is $(1, 2)$. What is the y-coordinate of the position vector at time $t = 2$?

10. At time $t \geq 0$, a particle moving in the xy-plane has velocity vector given by $v(t) = \langle t^3, 4t \rangle$. What is the acceleration vector when $t = 2$?

11. The acceleration vector of a particle moving in the xy-plane is given by $a(t) = \langle 2, 3 \rangle$. When $t = 0$ the velocity vector is $\langle 3, 1 \rangle$ and the position vector is $\langle 1, 5 \rangle$. Find the position when time $t = 2$.

12. A particle moves on the curve $y = 2x$ so that the x-component has velocity $x'(t) = 3t^2 + 1$ for $t \geq 0$. At time $t = 0$, the particle is at the point $(2, 4)$. At what point is the particle when $t = 1$? [This one is tricky!]

For problems 13-15: At time t, $0 \leq t \leq 2\pi$, the position of a particle moving along a path in the xy-plane is given by parametric equations $x(t) = \cos 2t$ and $y(t) = \sin 2t$.

13. Find the speed of the particle when $t = 1$.

14. Find the acceleration vector at time $t = \frac{\pi}{4}$.

15. Find the distance traveled from $t = 0$ to $t = 3$.

9.6 Motion using Parametric and Vector-Valued Functions **Test Prep**

16. **Calculator active.** A remote-controlled car moves along a flat surface over the time interval $0 \leq t \leq 30$ seconds. The position of the remote-controlled car at time t is given by the parametric equations $x(t) = 2t + \sin t$ and $y(t) = 2\cos(t - \sin t)$, where $x(t)$ and $y(t)$ are measured in feet. The derivatives of these functions are given by $x'(t) = 2 + \cos t$ and $y'(t) = -2\sin(t - \sin t)(1 - \cos t)$.

a. Write the equation for the line tangent to the path of the remote-controlled car at time $t = 3$ seconds.

b. Find the speed of the remote-controlled car at time $t = 15$ seconds.

c. Find the acceleration vector of the remote-controlled car at the time when the car is at the point with x-coordinate 40.

9.6 AP Practice Problems (p.689) – Motion along a Curve of Vector Functions

1. If the velocity vector of a particle in motion is $\mathbf{v}(t) = \langle t+3, 6t^2 \rangle$, then the acceleration vector of the particle at $t = 2$ is

   (A) $\langle 0, 24 \rangle$ (B) $\langle 1, 24 \rangle$ (C) $\langle 1, 12 \rangle$ (D) $\langle 2, 24 \rangle$

2. A particle moves along the plane curve

$$\mathbf{r}(t) = \langle e^{2t}, t^4 + 2t^2 \rangle$$

   What is the acceleration vector of the particle at $t = 0$?

   (A) $\mathbf{a}(0) = \langle 1, 4 \rangle$ (B) $\mathbf{a}(0) = \langle 4, 0 \rangle$
   (C) $\mathbf{a}(0) = \langle 4, 4 \rangle$ (D) $\mathbf{a}(0) = \langle 4e^2, 4 \rangle$

3. A particle moves in the xy-plane along the curve

$$\mathbf{r}(t) = \langle 2\sqrt{t}, 3t^2 \rangle \quad t > 0$$

   What is the velocity of the particle at $t = 9$?

   (A) $\left\langle \frac{1}{3}, 54 \right\rangle$ (B) $\left\langle \frac{2}{3}, 54 \right\rangle$ (C) $\langle 3, 27 \rangle$ (D) $\left\langle \frac{1}{3}, 36 \right\rangle$

4. A particle of mass m is moving along the curve traced out by $\mathbf{r}(t) = \langle 2t^2, e^t \rangle$. Using Newton's Second Law of Motion, $\mathbf{F} = m\mathbf{a}$, the force acting on the particle at time t is

(A) $\langle 4m, e^t \rangle$ (B) $\langle 4m, me^t \rangle$

(C) $\langle 4m, e^m \rangle$ (D) $\langle 4, e^m \rangle$

5. A particle moves along the plane curve

$$\mathbf{r}(t) = \langle 1 - 2\cos t, 2\sin t \rangle$$

(a) Find the velocity of the particle.
(b) Find the acceleration of the particle.

6. A particle moves along the plane curve $\mathbf{r}(t) = \langle 2\sin t, 3\cos t \rangle$.

(a) Find the speed of the particle.

(b) What is the speed of the particle at $t = \frac{\pi}{2}$?

(34)

9.7 AP Practice Problems (p.696) – Integrals of Vector Functions & Projectile Motion

1. Find $\int_0^1 \mathbf{r}(t)\,dt$ where $\mathbf{r}(t) = \left\langle \frac{1}{t^2+1}, \frac{1}{t+1} \right\rangle$.

   (A) $\frac{\pi}{4} + \ln 2$ (B) $\left\langle \frac{\pi}{4}, \ln 2 \right\rangle$

   (C) $\left\langle -\frac{1}{2}, \ln 2 \right\rangle$ (D) $\langle \tan^{-1} t, \ln(t+1) \rangle$

2. $\int \left[\langle \sec^2 t, \cos t \rangle \right] dt =$

   (A) $\langle \tan t, \sin t \rangle$ (B) $\langle \tan t, -\sin t \rangle + \mathbf{c}$

   (C) $\langle \tan t, \sin t \rangle + \mathbf{c}$ (D) $\left\langle \frac{\sec^3 t}{3}, \sin t \right\rangle + \mathbf{c}$

3. A particle is moving with velocity

$$\mathbf{v}(t) = \langle \pi \cos(\pi t), 3t^2 + 1 \rangle \text{ m/s}$$

   for $0 \le t \le 10$ seconds. Given that the position of the particle at time $t = 2$ s is $\mathbf{r}(2) = \langle 3, -2 \rangle$, the position vector of the particle at t is

   (A) $\langle 3, -12 \rangle$ (B) $\langle 3 + \sin(\pi t), t^3 + t + 10 \rangle$

   (C) $\langle \sin(\pi t), t^3 + t \rangle$ (D) $\langle 3 + \sin(\pi t), t^3 + t - 12 \rangle$

4. The solution to the vector differential equation $\mathbf{r}'(t) = \langle 4e^{4t}, 3t^2 \rangle$ given $\mathbf{r}(0) = \langle 2, -1 \rangle$ is

(A) $\langle 1 + e^{4t}, -t^3 \rangle$ (B) $\langle 1 + e^{4t}, t^3 - 1 \rangle$

(C) $\langle 2 + e^{4t}, t^3 - 1 \rangle$ (D) $\langle 3 + e^{4t}, t^3 - 1 \rangle$

5. If an object travels in the xy-plane along the curve traced out by the vector function $\mathbf{r}(t) = \langle t^{3/2}, -t \rangle$ for $t \geq 0$, then the total distance traveled by the object from $t = 0$ to $t = 4$ is

(A) $\frac{16}{3}$ (B) $\frac{2}{3}10^{3/2}$ (C) $10^{3/2} - 1$ (D) $\frac{8}{27}[10^{3/2} - 1]$

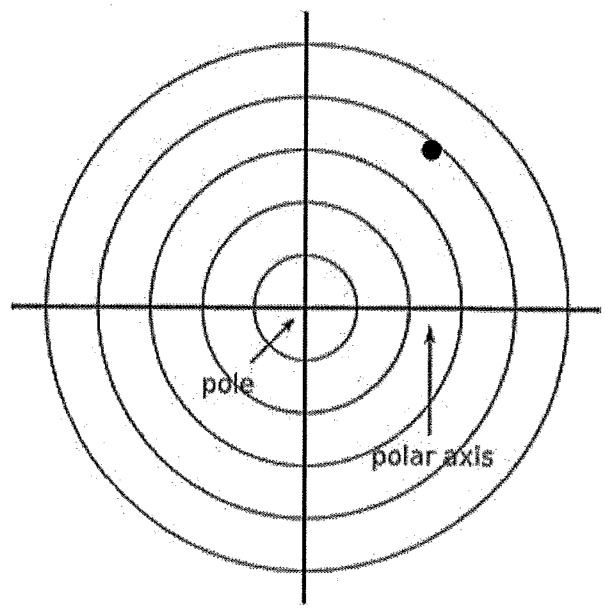
6. A flare is launched at an angle of elevation 60° ($\frac{\pi}{3}$ radians) with initial speed $\|\mathbf{v}(0)\| = 200$ ft/s from a stationary barge's deck which is three feet above the water's surface. The only external force acting on the flare is gravity, so $\mathbf{a}(t) = \langle 0, -32 \rangle$ ft/s^2.

(a) Find the velocity vector $\mathbf{v} = \mathbf{v}(t)$ of the flare.

(b) Find the position vector $\mathbf{r} = \mathbf{r}(t)$ of the flare.

BC Calculus – 9.4-prep Notes – Graphing Polar Equations

A rectangular coordinate system is only one way to navigate through a Euclidean plane. Such coordinates, (x, y), known as **rectangular coordinates**, are useful for expressing functions of y in terms of x. Curves that are not functions are often more easily expressing in an alternative coordinate system called **polar coordinates**.



In a polar coordinate system, we still have the traditional x- and y- axes. The intersection of these axes, the old origin, is called the **pole**. Similar to navigating on the Unit Circle, we can now get to any point in 2-D space by specifying an independent choice of an angle, θ, from the initial ray, **polar axis**, then walking out along that terminal ray a specified amount, r, in either direction.

Although the angle is the independent variable, we express the point in the polar plane as (r, θ). The point to the right would have coordinates of $\left(4, \frac{\pi}{4}\right)$.

Example 1:
Find several other equivalent polar coordinates for the point shown above, then find the equivalent rectangular coordinate.

Why use polar coordinates? Graphs that aren't functions in rectangular form $f(x)$ can still be functions in polar form $r(\theta)$. Some of these curves can be quite elaborate and are more easily expressed as polar, rather than rectangular equations, as the following calculator exploration will demonstrate.

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Example 2:

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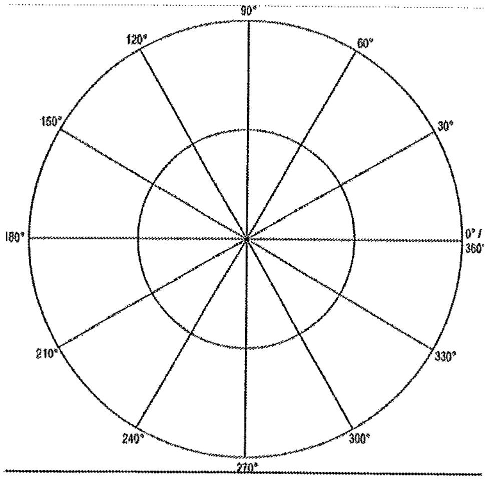
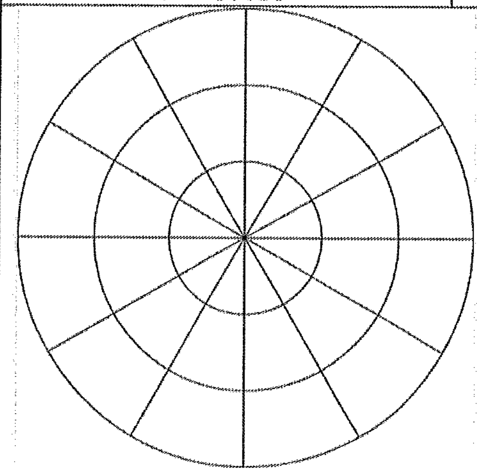
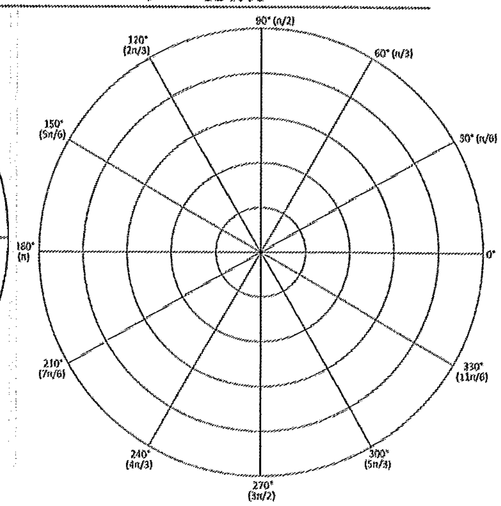
Put your graphing calculator in POLAR mode and RADIAN mode. Graph the following equations on your calculator, sketch the graphs on this sheet, and answer the questions.

$r = \cos\theta$	$r = 3\cos\theta$	$r = 4\sin\theta$

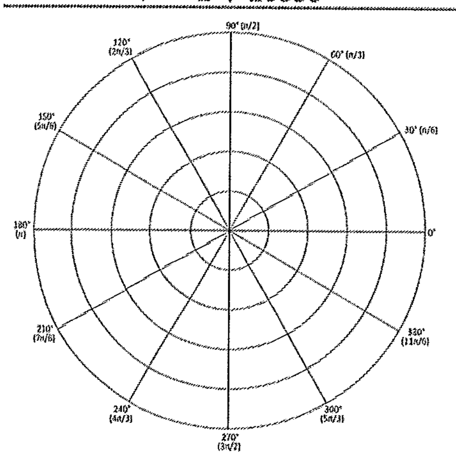


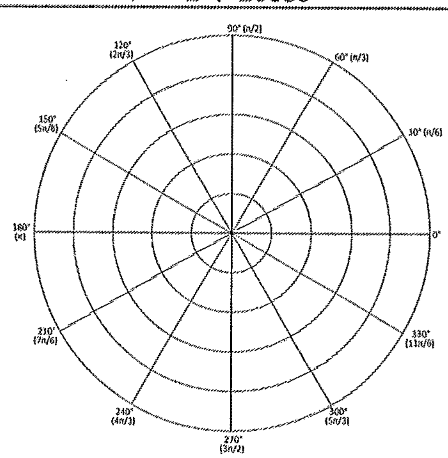



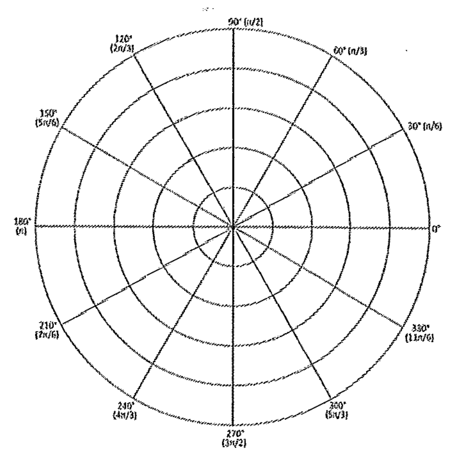
What do you notice about the above graphs?

Example 3a:

$r = 2 + 2\cos\theta$	$r = 1 + 2\cos\theta$	$r = 2 + \cos\theta$

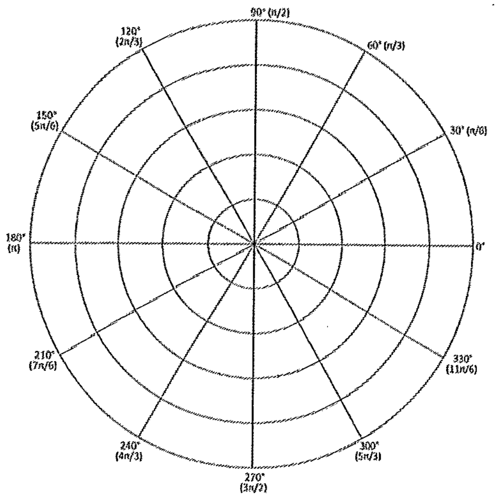


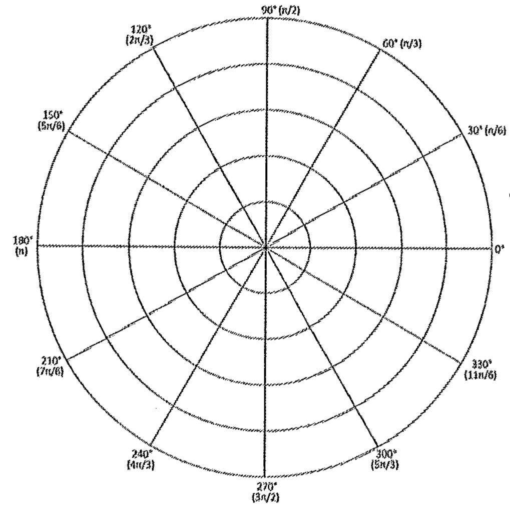


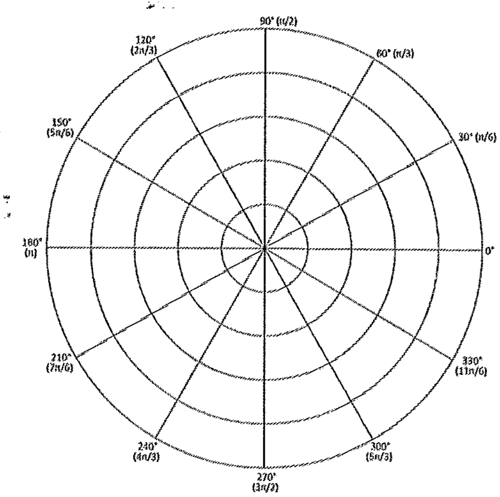


Example 3b:

$r = 2 + 2\sin\theta$	$r = 1 + 2\sin\theta$	$r = 2 + \sin\theta$



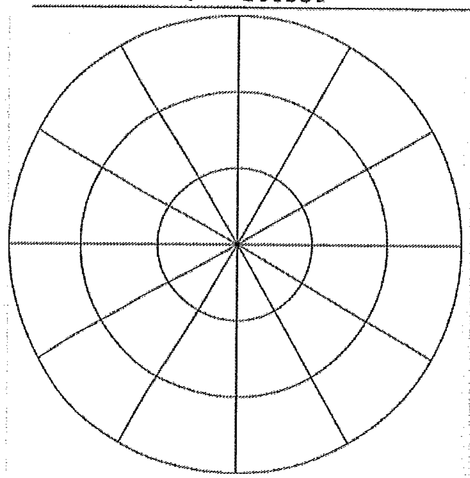


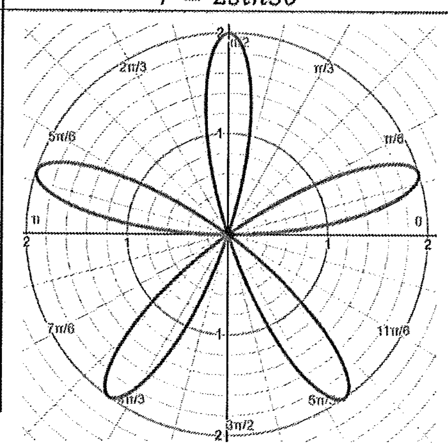


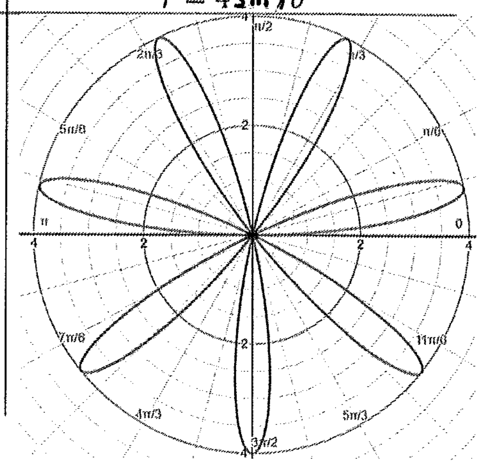
Which graphs go through the pole?
Which ones do not go through the pole?
Which ones have an inner loop?

Example 4:

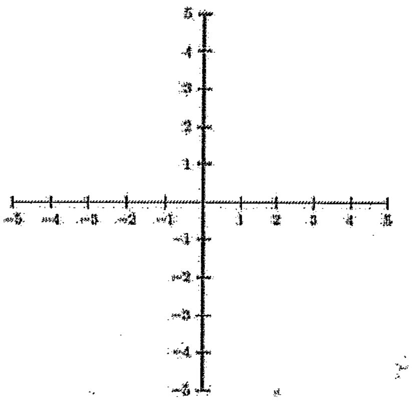
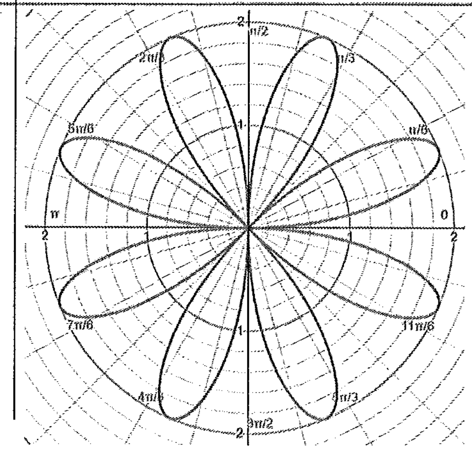
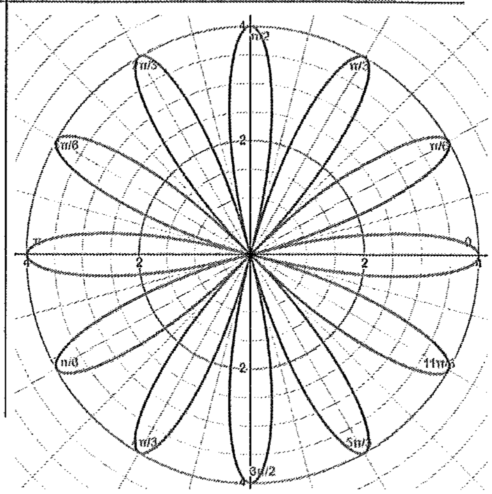
$r = 3\cos 3\theta$	$r = 2\sin 5\theta$	$r = 4\sin 7\theta$



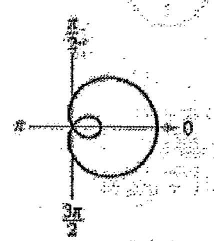
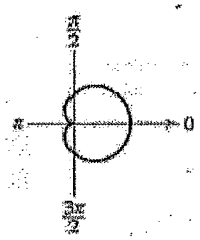
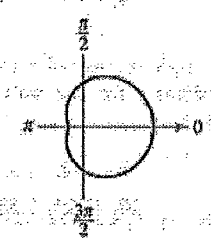
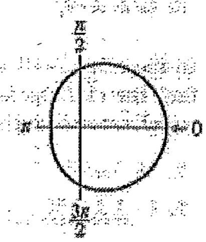
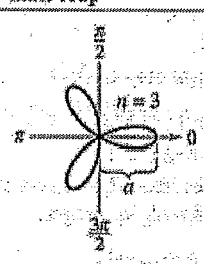
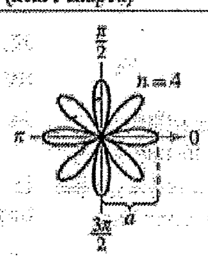
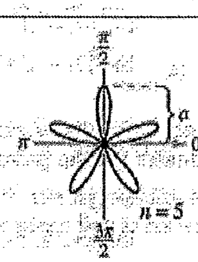
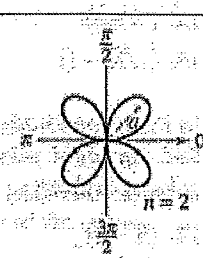
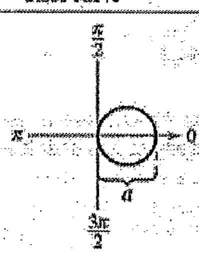
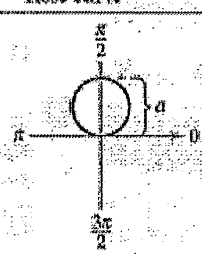
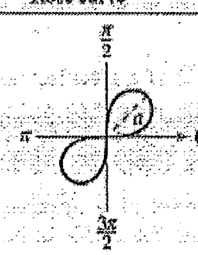
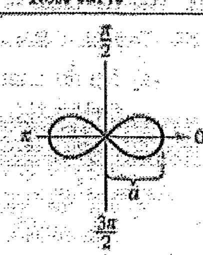




Example 5:

$r = 3cos2\theta$	$r = 2sin4\theta$	$r = 4cos6\theta$
		

What do you notice about the above graphs?

Limacons $r = a \pm b\cos\theta$ $r = a \pm b\sin\theta$ $(a > 0, b > 0)$	 $\frac{a}{b} < 1$ Limaçon with inner loop	 $\frac{a}{b} = 1$ Cardioid (heart-shaped)	 $1 < \frac{a}{b} < 2$ Dimpled limaçon	 $\frac{a}{b} \geq 2$ Convex limaçon
Rose Curves n petals if n is odd 2n petals if n is even	 $r = a\cos n\theta$ Rose curve	 $r = a\cos n\theta$ Rose curve	 $r = a\sin n\theta$ Rose curve	 $r = a\sin n\theta$ Rose curve
Circles and **Lemniscates**	 $r = a\cos\theta$ Circle	 $r = a\sin\theta$ Circle	 $r^2 = a^2\sin 2\theta$ Lemniscate	 $r^2 = a^2\cos 2\theta$ Lemniscate

BC Calculus – 9.4a Notes – Defining and Differentiating in Polar Form

(x, y) is for a **rectangular** coordinate system.

(r, θ) is for a **polar** coordinate system.

r is a directed distance from the origin to a point P.

θ is the directed angle

Polar ⟹ Rectangular	**Rectangular ⟹ Polar**
$x = r\cos\theta$ $y = r\sin\theta$	$\tan\theta = \dfrac{y}{x}$ $r^2 = x^2 + y^2$

Convert the following from polar form to rectangular form.

1. $r\cos\theta = -4$
2. $4r\cos\theta = r^2$
3. $\dfrac{4}{2\cos\theta - \sin\theta} = r$

Slope of a Curve in Polar Form

A curve in polar form is given by $r = f(\theta)$, then its rectangular coordinates are given by $\begin{cases} x = f(\theta)\cos\theta \\ y = f(\theta)\sin\theta \end{cases}$. The derivative $\frac{dy}{dx}$ is defined the same way as the derivative of a parametric equation.

$$\frac{dy}{dx} =$$

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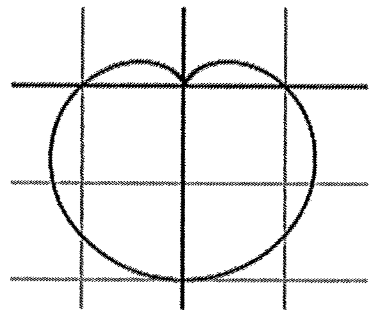
The following is an example of a common problem found on the AP Exam!

4. What is the slope of the line tangent to the polar curve $r = 1 + 2\sin\theta$ at $\theta = 0$?

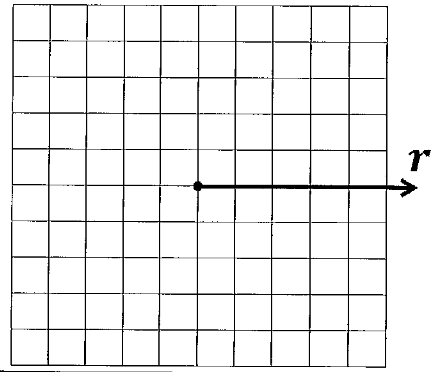
5. Find the value(s) of θ where the polar graph $r = 1 - \sin\theta$ on the interval $0 \le \theta \le 2\pi$ has horizontal and vertical tangent lines.



Practice Problems

Problems 1-5 are pre-calculus review on polar form.	
1. Find the corresponding rectangular coordinates for the polar coordinates $\left(7, \frac{5\pi}{4}\right)$.	2. **Calculator active.** Find two sets of polar coordinates for the rectangular coordinate $(4, -2)$. Limit your answers on the interval $0 \le \theta \le 2\pi$.
3. Convert the rectangular equation $x^2 + y^2 = 16$ to a polar equation.	4. Convert the polar equation $r = 3\sec\theta$ to a rectangular equation.

5. Sketch the polar curve $r = 2\cos 3\theta$ for $0 \leq \theta \leq \pi$ **without** a calculator, then check your answer.



Find the slope of the line tangent to the polar curve at the given value of θ.

6. $r = 3\theta$ at $\theta = \frac{\pi}{2}$.

7. $r = \frac{5}{3-\cos\theta}$ at $\theta = \frac{3\pi}{2}$.

8. $r = \cos\theta$ at $\theta = \frac{\pi}{3}$.

9. $r = 2(1 - \sin\theta)$ at $\theta = 0$.

10. A particle moves along the polar curve $r = 3\cos\theta$ so that $\frac{d\theta}{dt} = 2$. Find the value of $\frac{dr}{dt}$ at $\theta = \frac{\pi}{3}$. *Hint: remember implicit differentiation?*

11. A polar curve is given by the equation $r = \frac{15\theta}{\theta^2+1}$ for $\theta \geq 0$. What is the instantaneous rate of change of r with respect to θ when $\theta = 1$?

12. Find the value(s) of θ where the polar graph $r = 2 - 2\cos\theta$ has a horizontal tangent line on the interval $0 \leq \theta \leq 2\pi$. Use a graphing calculator to verify your answers.

13. Find the value(s) of θ where the polar graph $r = 3 - 3\sin\theta$ has a vertical tangent line on the interval $0 \leq \theta \leq 2\pi$. Use a graphing calculator to verify your answers.

14. **Calculator active.** For a certain polar curve $r = f(\theta)$, it is known that $\frac{dx}{d\theta} = \cos\theta - \theta\sin\theta$ and $\frac{dy}{d\theta} = \sin\theta + \theta\cos\theta$. What is the value of $\frac{d^2y}{dx^2}$ at $\theta = 3$?

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9.4a Differentiating in Polar Form

Test Prep

15. A polar curve is given by the differentiable function $r = f(\theta)$ for $0 \leq \theta \leq 2\pi$. If the line tangent to the polar curve at $\theta = \frac{\pi}{6}$ is vertical, which of the following must be true?

A. $f\left(\frac{\pi}{6}\right) = 0$

B. $f'\left(\frac{\pi}{6}\right) = 0$

C. $\frac{1}{2}f\left(\frac{\pi}{6}\right) - \frac{\sqrt{3}}{2}f'\left(\frac{\pi}{6}\right) = 0$

D. $\frac{\sqrt{3}}{2}f'\left(\frac{\pi}{6}\right) - \frac{1}{2}f\left(\frac{\pi}{6}\right) = 0$

---

16. **Calculator active.** For $0 \leq t \leq 8$, a particle moving in the xy-plane has position vector $\langle x(t),\ y(t)\rangle = \langle \sin(2t), t^2 - t\rangle$, where $x(t)$ and $y(t)$ are measured in meters and t is measured in seconds.

a. Find the speed of the particle at time $t = 3$ seconds. Indicate units of measure.

b. At time $t = 5$ seconds, is the speed of the particle increasing or decreasing? Explain your answer.

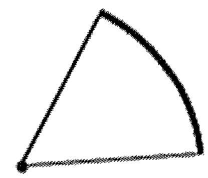
c. Find the total distance the particle travels over the time interval $0 \leq t \leq 6$ seconds.

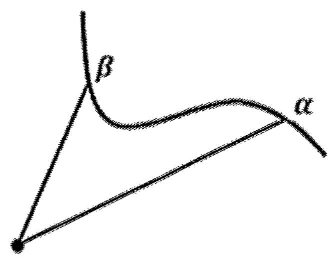
d. At time $t = 8$ seconds, the particle begins moving in a straight line. For $t \geq 8$, the particle travels with the same velocity vector that it had at time $t = 8$ seconds. Find the position of the particle at time $t = 11$ seconds.

BC Calculus – 9.4b Notes – Area Bounded by a single Polar Curve

9.4b Area Bounded by a Single Polar Curve

Recall: In geometry, we learned that the area of a sector is $A =$





The radius of a sector =
The central angle =

$A \approx$

Push the number of slices up to infinity, and we get

$A =$

This is the definition of integration!

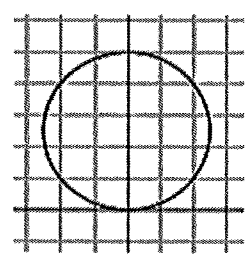
Area of a region bounded by a polar graph

If f is continuous and nonnegative on the interval $[\alpha, \beta]$, then the area of the region bounded by the graph of $r = f(\theta)$ between the radial lines $\theta = \alpha$ and $\theta = \beta$ is

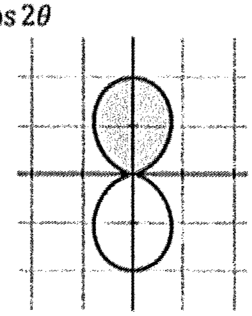
$A =$

The trick with polar graphs is to be careful with what interval it takes to trace out the polar graph. Watch what happens with this example.

1. Find the area bounded by $r = 5\sin\theta$.



2. Find the area of the shaded region of the polar curve for $r = 1 - \cos 2\theta$



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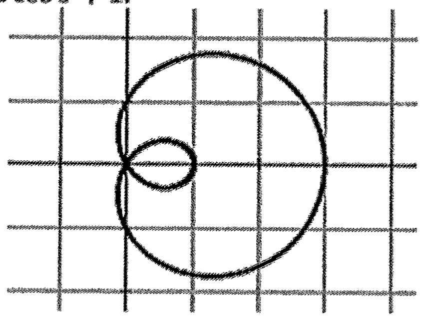
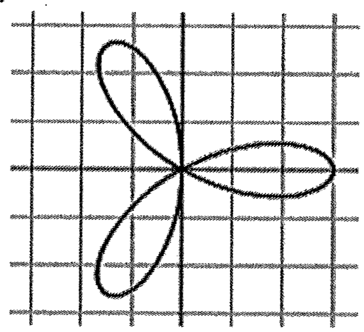
3. Find the area of the inner loop of the limaçon $r = 2\cos\theta + 1$.



4. Find the area of one petal of the rose curve $r = 3\cos(3\theta)$.



Find the area of the given region for each polar curve.

1. Inside the smaller loop of the limaçon $r = 2\sin\theta + 1$.

2. The region enclosed by the cardioid $r = 2 + 2\cos\theta$

3. Inside the graph of the limaçon $r = 4 + 2\cos\theta$.

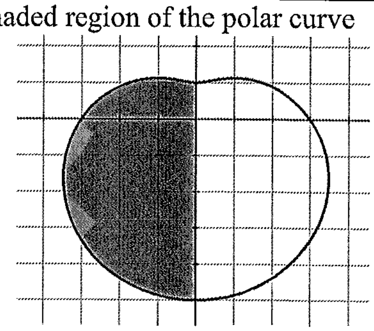
4. Inside one petal of the four-petaled rose $r = \cos 2\theta$.

5. Inside one loop of the lemniscate $r^2 = 4\cos 2\theta$.

6. Inside the inner loop of the limaçon $r = 2\sin\theta - 1$.

7. Write but do not solve, an expression that will give the area enclosed by one petal of the 3 petaled rose $r = 4\cos 3\theta$ found in the first and fourth quadrant.

8. Write but do not solve an expression that can be used to find the area of the shaded region of the polar curve $r = 3 - 2\sin\theta$.



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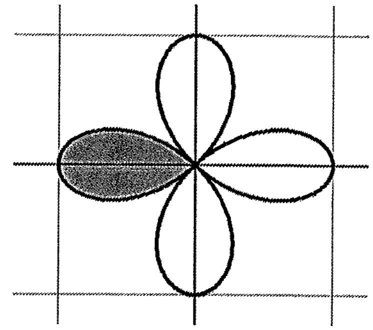
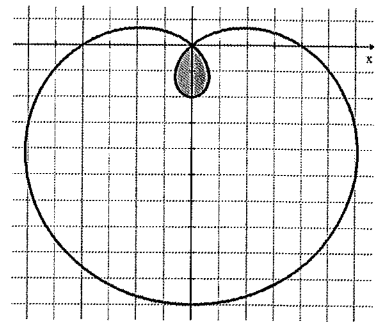
9. Write but do not solve an expression to find the area of the shaded region of the polar curve $r = \cos 2\theta$.



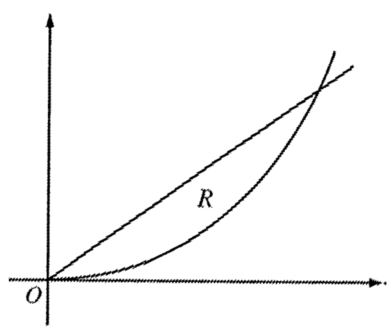
10. Find the area of the shaded region of the polar curve $r = 4 - 6\sin\theta$.



9.4b Area Bounded by a Polar Curve

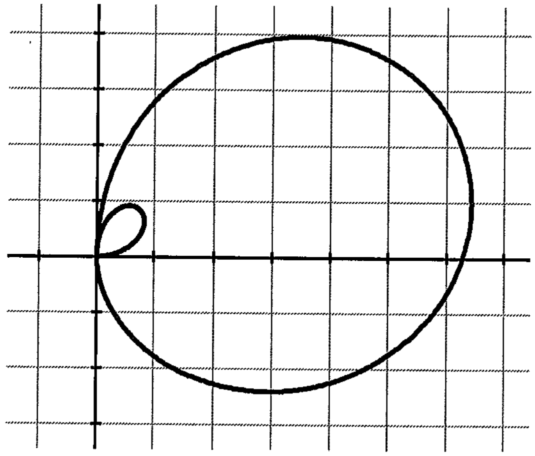
Test Prep

11.



Let R be the region in the first quadrant that is bounded by the polar curves $r = \frac{\theta}{2}$ and $\theta = k$, where k is a constant, $0 < k < \frac{\pi}{2}$, as shown in the figure above. What is the area of R in terms of k?

12.



Calculator active. Consider the polar curve defined by the function $r(\theta) = 2\theta \cos\theta$, where $0 \leq \theta \leq \frac{3\pi}{2}$. The derivative of r is given by $\frac{dr}{d\theta} = 2\cos\theta - 2\theta\sin\theta$. The figure above shows the graph of r for $0 \leq \theta \leq \frac{3\pi}{2}$.

a. Find the area of the region enclosed by the inner loop of the curve.

b. For $0 \leq \theta \leq \frac{3\pi}{2}$, find the greatest distance from any point on the graph of r to the origin. Justify your answer.

c. There is a point on the curve at which the slope of the line tangent to the curve is $\frac{2}{2-\pi}$. At this point, $\frac{dy}{d\theta} = \frac{1}{2}$. Find $\frac{dx}{d\theta}$ at this point.

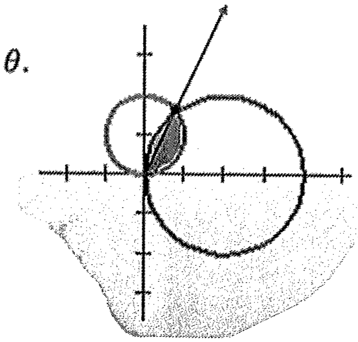
BC Calculus – 9.4c Notes – Area Bounded by two Polar Curves

Recall area bounded by a polar curve: $A =$

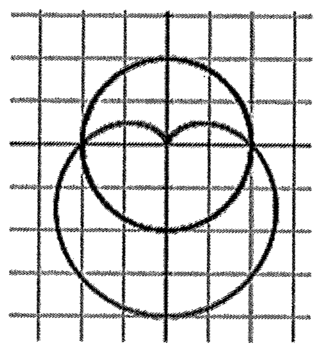
Things to watch for when using more than one polar curve for area.

- Points of intersection
- Symmetry

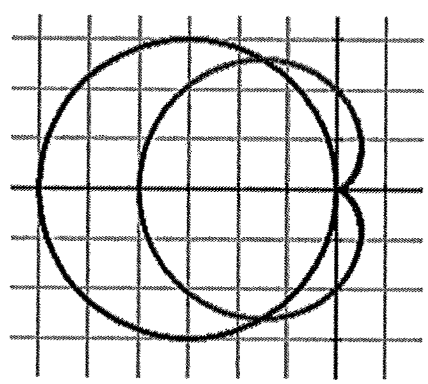
1. Find the area of the region common to the polar curve $r = 4\cos\theta$ and $r = 2\sin\theta$.

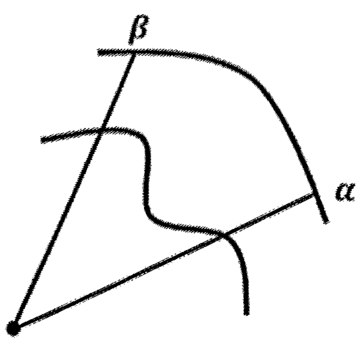


2. Find the area of the common region to the polar graphs of $r = 2$ and $r = 2 - 2\sin\theta$.



3. Find the area of the region common to the two polar curves $r = -6\cos\theta$ and $r = 2 - 2\cos\theta$.

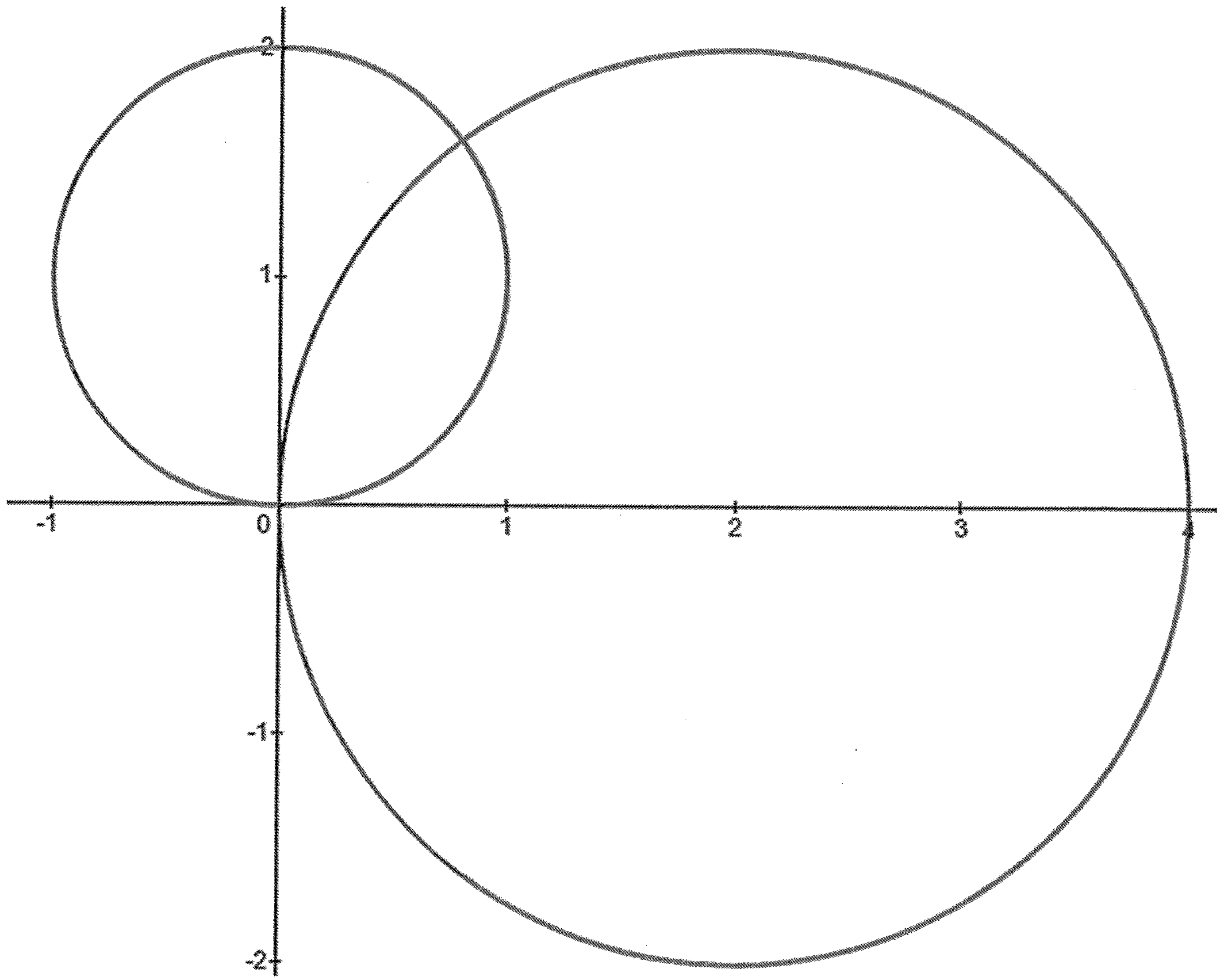




Area Bounded by Two Polar Curves

$A =$

$A =$


2
1
-1
-2
-1
0
1
2
3
4

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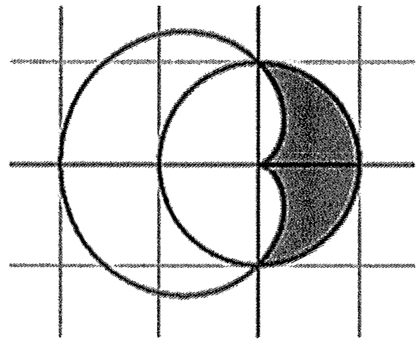
4. Find the area of the region bounded by the two polar curves $r = 1$ and $r = 1 - \cos\theta$ as shown in the graph below.

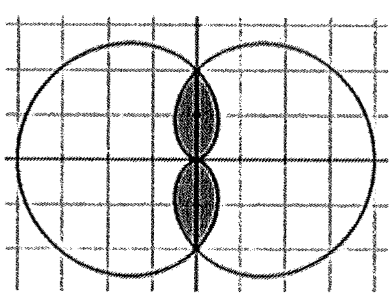


Practice Problems:

1. Find the area of the common interior of the polar graphs $r = 4\sin 2\theta$ and $r = 2$.

2. Find the area of the common interior of the polar graphs $r = 2\cos\theta$ and $r = 2\sin\theta$.

3. The polar curves $r = 2 - 2\cos\theta$ and $r = 2 + 2\cos\theta$ are shown below.



Which of the following gives the total area of the shaded regions?

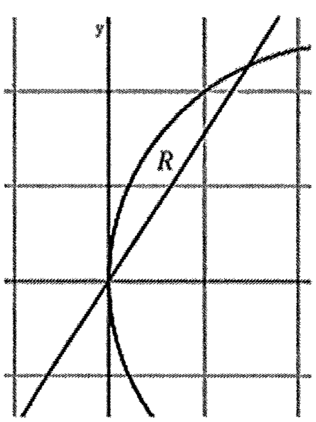
A. $\int_0^{\pi}(2 + 2\cos\theta)^2\, d\theta$

B. $\int_{\pi/2}^{\pi}(2 + 2\cos\theta)^2\, d\theta$

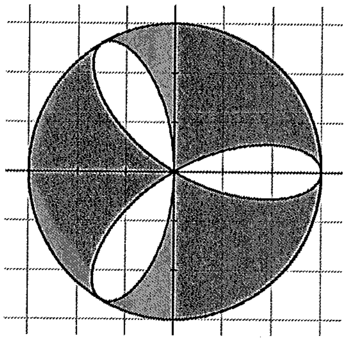
C. $8\int_0^{\frac{\pi}{2}}(1 - \cos\theta)^2\, d\theta$

D. $\int_0^{\frac{\pi}{2}}((2 - 2\cos\theta)^2 + (2 + 2\cos\theta)^2)\, d\theta$

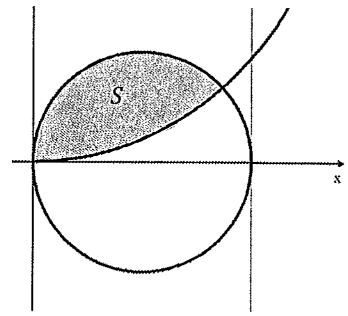
4. Let R be the region in the first quadrant that is bounded above by the polar curve $r = 5\cos\theta$ and below by the line $\theta = 1$, as shown in the figure below. What is the area of R?



5. The figure below shows the graphs of the polar curves $r = 3\cos 3\theta$ and $r = 3$. What is the sum of the areas of the shaded regions?



6. Let S be the region in the 1st Quadrant bounded above by the graph of the polar curve $r = \cos\theta$ and bounded below by the graph of the polar curve $r = \frac{5}{2}\theta$, as shown in the figure above. The two curves intersect when $\theta = 0.373$. What is the area of S?



7. Find the area inside the polar curve $r = 2\cos\theta$ and outside the polar curve $r = 1$.

8. Write an integral expression that represents the area of the region outside the polar curve $r = 3 + 2\sin\theta$ and inside the polar curve $r = 2$.

9. What is the total area outside the polar curve $r = 5\cos 2\theta$ and inside the polar curve $r = 5$?

10. Find the area of the common interior of the polar curves $r = 4\sin\theta$ and $r = 2$.

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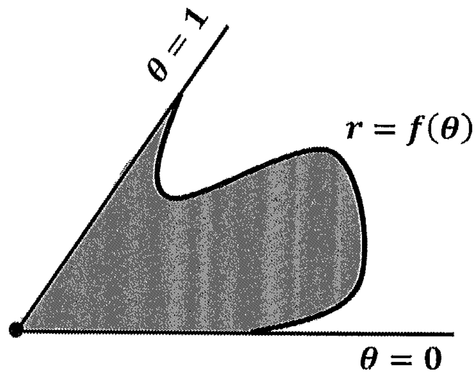
9.4 Area Bounded by Two Polar Curves

Test Prep

11.



θ	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
r	1	3	5	4	2

No calculator! Let R be the region bounded by the graph of the polar curve $r = f(\theta)$ and the lines $\theta = 0$ and $\theta = 1$, as shaded in the figure above. The table above gives values of the polar function $r = f(\theta)$ at selected values of θ. What is the approximation for the area of region R using a right Riemann sum with the four subintervals indicated by the data in the table?

9.3 AP Practice Problems (p.668) – Polar Equations

1. What is the slope of the tangent line to the cardioid $r = 1 + \cos\theta$ at $\theta = \frac{\pi}{6}$?

(A) -1 (B) $-\frac{\sqrt{3}+1}{2}$

(C) $\frac{\sqrt{3}+1}{2}$ (D) 1

2. Which integral gives the perimeter of one petal of the rose $r = 2\sin(3\theta)$?

(A) $\int_0^{\pi/2} \sqrt{\sin^2(3\theta) + 9\cos^2(3\theta)}\, d\theta$

(B) $2\int_0^{\pi/3} \sqrt{\sin^2(3\theta) + 9\cos^2(3\theta)}\, d\theta$

(C) $\int_0^{\pi/3} \sqrt{1 + 8\cos^2(3\theta)}\, d\theta$

(D) $\int_0^{\pi/3} [2\sin(3\theta) + 3\cos(3\theta)]\, d\theta$

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3. Parametric equations for the polar equation $r = 5\theta$ are

(A) $x = 5\theta \cos\theta;\ y = 5\theta \sin\theta$

(B) $x = 5\cos\theta;\ y = 5\sin\theta$

(C) $x = 5r\cos\theta;\ y = 5r\sin\theta$

(D) $x = \cos(5\theta);\ y = \sin(5\theta)$

4. The graph of the polar equation $r = 2 - 4\cos\theta$ is a

(A) limaçon without an inner loop.

(B) limaçon with an inner loop.

(C) a cardioid that has symmetry with respect to the x-axis.

(D) a cardioid that has symmetry with respect to the y-axis.

5. The arc length of the logarithmic spiral represented by $r = 4e^{3\theta/2}$ from $\theta = 0$ to $\theta = \ln 9$ is

(A) $\frac{26}{3}\sqrt{13}$

(B) $\frac{52}{3}\sqrt{13}$

(C) $\frac{104}{3}\sqrt{13}$

(D) $36\sqrt{13}$

9.4 AP Practice Problems (p.674) – Polar Area

1. The area in the second quadrant bounded by the graph of the polar equation $r = 2 - \cos\theta$ is

(A) π (B) $\frac{9}{4}\pi + 4$ (C) $\frac{5}{8}\pi + 2$ (D) $\frac{9}{8}\pi + 2$

2. Find the area of the region common to the graphs of $r = 1 + \sin\theta$ and $r = 3\sin\theta$.

(A) $2\left[\frac{1}{2}\int_0^{\pi/2}(3\sin\theta)^2 d\theta + \frac{1}{2}\int_0^{\pi/2}(1+\sin\theta)^2 d\theta\right]$

(B) $2\left[\frac{1}{2}\int_0^{\pi/2}(1+\sin\theta)^2 d\theta - \frac{1}{2}\int_0^{\pi/2}(3\sin\theta)^2 d\theta\right]$

(C) $2\left[\frac{1}{2}\int_0^{\pi/2}(1+\sin\theta)^2 d\theta - \int_0^{\pi/6}(3\sin\theta)^2 d\theta\right]$

(D) $2\left[\frac{1}{2}\int_0^{\pi/6}(3\sin\theta)^2 d\theta + \frac{1}{2}\int_{\pi/6}^{\pi/2}(1+\sin\theta)^2 d\theta\right]$

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3. The graph of the polar equation $r = \cos(2\theta)$ is a rose with four petals. Find the area of one petal.

(A) $\frac{\pi}{16}$ (B) $\frac{\pi}{8}$ (C) $\frac{\pi}{4}$ (D) $\frac{\pi}{2}$

4. The area of the inner loop of the limaçon $r = 2 - 4\sin\theta$ equals

I. $2 \cdot \frac{1}{2} \int_{\pi/6}^{\pi/2} (2 - 4\sin\theta)^2 d\theta$

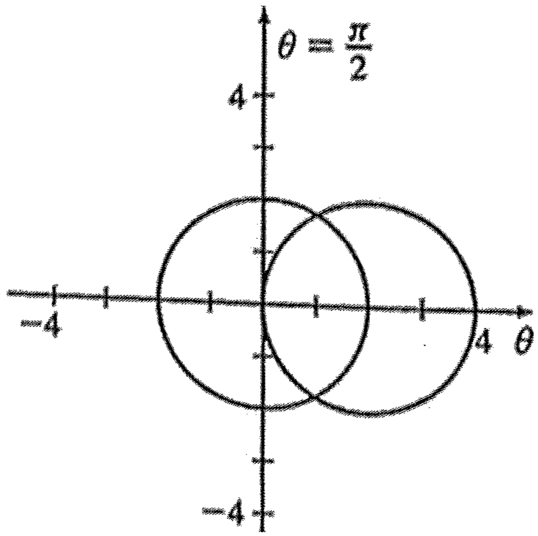
II. $\frac{1}{2} \int_{\pi/6}^{5\pi/6} (2 - 4\sin\theta)^2 d\theta$

III. $2 \cdot \frac{1}{2} \int_{\pi/2}^{5\pi/6} (2 - 4\sin\theta)^2 d\theta$

(A) I and II only (B) I and III only

(C) II and III only (D) I, II, and III

5. The graphs of the polar equations $r = 2$ and $r = 4\cos\theta$ are shown below.



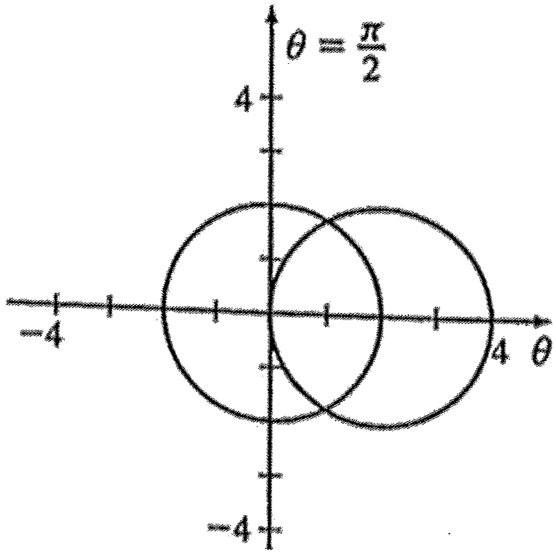
(a) Find the points of intersection of the two graphs.

(b) Find the area of the region that lies outside of the circle $r = 2$ and inside the circle $r = 4\cos\theta$.

(60)

(#5 continued)

5. The graphs of the polar equations $r = 2$ and $r = 4\cos\theta$ are shown below.



(c) Find the area of the region that lies inside the circle $r = 2$ but outside of the circle $r = 4\cos\theta$.

Ch. 9 Unit Review AP Practice Problems (p.701-702) – Parametric, Polar & Vector Functions

1. Which of the following parametric equations trace out a circle exactly once?

(A) $x(t) = 2 + \sin t,\ y(t) = 1 + 2\cos t,\ 0 \le t \le 2\pi$

(B) $x(t) = 2\cos t,\ y(t) = 1 - 2\sin t,\ -\pi \le t \le \pi$

(C) $x(t) = \sin(3t),\ y(t) = \cos(3t),\ 0 \le t \le 2\pi$

(D) $x(t) = \sin t,\ y(t) = \sin t,\ -\pi \le t \le \pi$

2. Which parametric equations correspond to the equation $x = 3 - y^2$?

I. $x(t) = 3 - t^2,\ y(t) = t$

II. $x(t) = 3 - t^{2/3},\ y(t) = t^{1/3}$

III. $x(t) = 3 - t,\ y(t) = t^{1/2}$

(A) I only (B) I and II only

(C) I and III only (D) I, II, and III

3. What is the slope of the tangent line to the curve of the polar equation $r = 2\cos\theta$ when $\theta = \frac{\pi}{3}$?

(A) $-\sqrt{3}$ (B) $\frac{\sqrt{3}}{2}$ (C) $\frac{\sqrt{3}}{3}$ (D) $-\frac{\sqrt{3}}{3}$

(62)

14. Find an equation of the tangent line to the plane curve represented by the parametric equations $x(t) = 1 + 2\ln t$, $y(t) = t^3 - 3$, at $t = 1$.

(A) $3x - 2y = 7$ (B) $3x - 2y = 2$

(C) $3x - 2y = -\frac{7}{2}$ (D) $2y + 3x = -\frac{7}{2}$

15. The plane curve C is represented by the parametric equations $x(t) = 2t^2 + 5$, $y(t) = 3t - t^3$. Find all the points on C where the tangent line is horizontal or vertical.

(A) vertical at $(5, 0)$; horizontal at $(7, 2)$

(B) vertical at $(0, 5)$; horizontal at $(7, 2)$ and $(-7, 2)$

(C) vertical at $(5, 0)$; horizontal at $(7, 2)$ and $(7, -2)$

(D) vertical at $(5, 0)$; horizontal at $(7, 2)$ and $(-7, 2)$

16. Find the distance traveled by an object that moves along the plane curve represented by the parametric equations

$x(t) = \frac{3}{4}t^2 + 5$, $y(t) = 3 + t^3$, from $t = 1$ to $t = 4$.

(A) $\frac{65^{3/2}}{8}$ (B) $\frac{60^{3/2}}{8}$

(C) $\frac{65^{3/2} + 5^{3/2}}{8}$ (D) $\frac{65^{3/2} - 5^{3/2}}{8}$

)7. The polar equation $r\sin\theta = \frac{5}{4}$ is

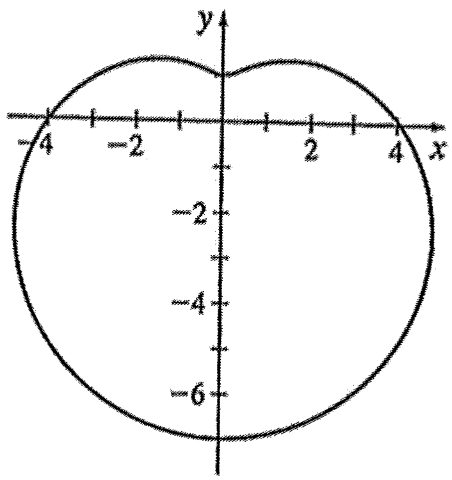
(A) a circle with radius $\frac{\sqrt{5}}{2}$.

(B) a horizontal line $\frac{5}{4}$ units above the polar axis.

(C) a vertical line perpendicular to the polar axis.

(D) a line through the pole with slope $\frac{5}{4}$.

8. Which polar equation has the graph shown below?



(A) $r = 4 - 4\sin\theta$ (B) $r = 2 + 3\sin\theta$

(C) $r = 4 - 3\cos\theta$ (D) $r = 4 - 3\sin\theta$

|9. Parametric equations for $r = 2^{\theta/3}$ are

(A) $x = 2^{\theta/3}\cos\theta,\ y = 2^{\theta/3}\sin\theta$

(B) $x = r\cos 2^{\theta/3},\ y = r\sin 2^{\theta/3}$

(C) $x = \cos 2^{\theta/3},\ y = \sin 2^{\theta/3}$

(D) $x = \frac{\theta}{3}\cos\theta,\ y = -\frac{\theta}{3}\sin\theta$

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10. One petal of the rose $r = \cos(2\theta)$ has perimeter given by the integral

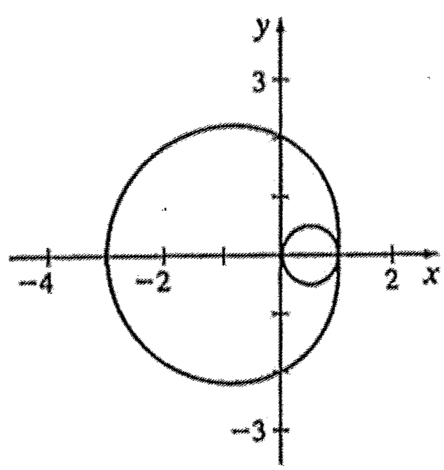
(A) $2\int_0^{\pi/4} \sqrt{\cos^2(2\theta) + \sin^2(\theta)}\, d\theta$

(B) $2\int_0^{\pi/4} \sqrt{\cos^2(2\theta) + 4\sin^2(2\theta)}\, d\theta$

(C) $2\int_0^{\pi/4} \sqrt{\cos^2(2\theta) - 4\sin^2(2\theta)}\, d\theta$

(D) $\int_0^{\pi/4} \sqrt{\cos^2(2\theta) + \sin^2(2\theta)}\, d\theta$

11. The graphs of the limaçon $r = 2 - \cos\theta$ and the circle $r = \cos\theta$ are shown below.



The area inside the limaçon but outside the circle is given by

I. $\frac{1}{2}\int_0^{2\pi} (2 - \cos\theta)^2 d\theta - \int_0^{\pi/2} \cos^2\theta d\theta$

II. $\frac{1}{2}\int_0^{2\pi} (2 - \cos\theta)^2 d\theta - \frac{\pi}{4}$

III. $\frac{1}{2}\int_0^{2\pi} (2 - 2\cos\theta)^2 d\theta$

(A) I only
(B) II only
(C) I and II only
(D) I, II, and III

12. The plane curve represented by
$x(t) = t - \sin t$, $y(t) = 1 - \cos t$
is a cycloid.

(a) Find the slope of the tangent line to the cycloid for $0 \le t \le 2\pi$.

(b) Find an equation of the tangent line to the cycloid at $t = \dfrac{\pi}{3}$.

(c) Find the length of the cycloid from $t = 0$ to $t = \dfrac{\pi}{2}$.

13. Which of the following is the tangent vector to the curve traced out by the vector function $\mathbf{r}(t) = \langle e^{2t}, 2\sin(3t)\rangle$ at $t = 0$?

(A) $\left\langle \dfrac{1}{6}, \dfrac{2}{3} \right\rangle$ (B) $\langle 1, 6\rangle$

(C) $\langle 2, 6\rangle$ (D) $\langle 2, -6\rangle$

14. Determine the domain of the vector function

$$\mathbf{r}(t) = \langle \sqrt{t-1}, -\ln(5-t)\rangle$$

(A) $\{t \mid 1 < t < 5\}$ (B) $\{t \mid t > 1\}$

(C) $\{t \mid t < 5\}$ (D) $\{t \mid 1 \le t < 5\}$

15. The vector function $\mathbf{r}(t) = \langle 4 - 3t, -t \rangle$ traces out

(A) Two lines: $y = 4 - 3x$ and $y = x$.
(B) Two lines: $x = 4 - 3t$ and $y = -t$.
(C) A line that contains the points (4, 0) and (1, 1).
(D) A line that contains the points (4, 0) and (1, −1).

16. The derivative $\mathbf{r}'(t)$ of the vector function $\mathbf{r}(t) = \langle \cos(t^3 + t), \sin(t^3 + t) \rangle$ is

(A) $\langle -\sin(3t^2 + 1), \cos(3t^2 + 1) \rangle$

(B) $\langle (3t^2 + 1)\sin(t^3 + t), (3t^2 + 1)\cos(t^3 + t) \rangle$

(C) $\langle 3t^2 + 1 - \sin(t^3 + t), 3t^2 + 1 + \cos(t^3 + t) \rangle$

(D) $\langle -(3t^2 + 1)\sin(t^3 + t), (3t^2 + 1)\cos(t^3 + t) \rangle$

17. A particle moves along the curve traced out by the vector function $\mathbf{r}(t) = \langle 10 \ln t, t^3 + 8 \rangle$, $t > 0$. What is the speed of the object at $t = 2$?

(A) $\frac{5}{12}$ (B) $\frac{12}{5}$ (C) 13 (D) $\langle 5, 12 \rangle$

18. A particle moves along the curve traced out by the vector function $\mathbf{r}(t) = \langle e^{3t}, 5t - t^2 \rangle$. What is the acceleration of the particle at $t = 0$?

(A) $\langle 9, 0 \rangle$ (B) $\langle 6, -2 \rangle$ (C) $\langle 9, -2 \rangle$ (D) $\langle 9, 2 \rangle$

19. The arc length of the smooth curve traced out by $\mathbf{r}(t) = \langle e^t, \sin t\rangle$ from $t = 0$ to $t = \pi$ is given by

(A) $\int_0^\pi \sqrt{(e^t + \sin t)^2}\,dt$ (B) $\int_0^\pi \sqrt{e^{2t} + \cos^2 t}\,dt$

(C) $\int_0^\pi \sqrt{e^{2t} + \sin^2 t}\,dt$ (D) $\int_0^\pi \sqrt{(e^t \cos t)^2 + (e^t \sin t)^2}\,dt$

20. $\frac{d}{dt}\langle e^{2t}t^3, e^{2t}(4-2t)\rangle =$

(A) $\langle t^2e^{2t}(2t+3), 2e^{2t}(3-2t)\rangle$

(B) $\langle 6t^2e^{2t}, -4te^{2t}\rangle$

(C) $\langle 5t^2e^{2t}, 2e^{2t}\rangle$

(D) $\langle 5t^2e^{2t}(1+t), 2e^{2t}(3-2t)\rangle$

21. $\int_e^{e^2} \left\langle \frac{3}{t}, 4t \right\rangle dt =$

(A) $\langle 3, 2e^2 - 2e\rangle$ (B) $\langle 3, 2e^4 - e^2\rangle$

(C) $\langle 5, 2e^4 - e^2\rangle$ (D) $\langle 3, 2e^4 - 2e^2\rangle$

22. A particle moves along a plane curve with velocity $\mathbf{v}(t) = \langle 3t+4, \sqrt{t+1}\rangle$. If $\mathbf{r}(0) = \langle 3, 1\rangle$, what is the position of the particle at $t = 3$?

(A) $\left\langle \frac{57}{2}, \frac{17}{3} \right\rangle$ (B) $\left\langle \frac{57}{2}, \frac{19}{3} \right\rangle$

(C) $\left\langle \frac{57}{2}, 6 \right\rangle$ (D) $\langle 171, 34\rangle$

23. The solution to the vector differential equation $\mathbf{r}'(t) = \langle 4t,\ 2t^3 \rangle$, given $\mathbf{r}(0) = \langle 1, 2 \rangle$, is

(A) $\langle 2t^2, t^4 \rangle$

(B) $\left\langle 2t^2 - 1,\ \frac{1}{2}t^4 - 2 \right\rangle$

(C) $\left\langle 2t^2 + 1,\ \frac{1}{2}t^4 + 2 \right\rangle$

(D) $\langle 4t + 1,\ 2t^3 + 2 \rangle$

24. For the vector function $\mathbf{r}(t) = \langle e^t, 2t \rangle$, find the tangent vector to the curve traced out by $\mathbf{r} = \mathbf{r}(t)$ at $t = 0$.

25. The position vector of a particle moving in the xy-plane is

$$\mathbf{r}(t) = \langle 2t^3 + 1, 3\cos(2t - 6) \rangle,\ t \geq 0$$

(a) Find the velocity vector of the particle at $t = 3$.

(b) Find the acceleration vector of the particle at $t = 3$.

(c) Find the speed of the particle at $t = 3$.

(d) Write an integral that represents the total distance the particle travels from $t = 0$ to $t = 3$.

(Unit 9) **Parametric functions**: where x and y coordinates on a graph are given in terms of a third variable "t": x=f(t) and y = g(t) are parametric equations and **t** is called the parameter.
*Eliminate the parameter, and use substitution to write rectangular equation.
*Rectangular equation only shows path of graph
*Parametric Equation tracks more info: Includes the **path, speed, and direction of graph**

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{y'(t)}{x'(t)} \quad \text{and} \quad \frac{d^2y}{dx^2}=\frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

*Horizontal tangent occurs where $\frac{dy}{dt}=0$

*Vertical tangent occurs where $\frac{dx}{dt}=0$

*Beware of $\frac{0}{0}$, which is neither a horizontal nor vertical tangent

Parametric Arc Length: $(s) = \int_{t1}^{t2}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$

Speed of particle: $\left|\vec{v}(t)\right| = \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}$

Distance of particle $\int_{t_1}^{t_2}|v| = \int_{t_1}^{t_2}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$

*Distance of particle IS the parametric arc length
* $\int speed$ is the **total distance traveled** along the curve (arc length)
* $\int velocity$ is the **displacement** (net change in position where positives and negatives cancel)

* **Final Position = Initial Position + Displacement**

Arc Length:

Arc length (s) $= \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$ or $\int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$

Parametric $s = \int_{t1}^{t2}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$ or $\int_{t1}^{t2}\sqrt{(x'(t))^2+(y'(t))^2}\,dt$

$ds^2 = dx^2 + dy^2$

Area of a Surface of Revolution in Parametric Form

Revolution about **x-axis**: $S = 2\pi\int_{t_1}^{t_2} y(t)\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dx$ Revolution about **y-axis**: $S = 2\pi\int_{t_1}^{t_2} x(t)\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dx$

(Ch. 10.4 – 10.5) Polar Equations: Ordered pairs are expressed as (r,θ) with θ as the independent variable.
r = distance from origin θ = directed angle from polar axis
origin is called the pole. x-axis is called the polar axis

Polar to Rectangular
$x = r\cos\theta$
$y = r\sin\theta$

Rectangular to Polar
$r=\sqrt{x^2+y^2}$
$\tan\theta = \frac{y}{x}$

Polar Derivatives:
$$\frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}=\frac{y'(\theta)}{x'(\theta)}$$

Region bounded by a Polar Curve:
Area of a circular sector: $A=\frac{1}{2}r^2\theta$
Polar Area Enclosed Region: $A=\int_a^b \frac{1}{2}r^2\,d\theta$

Arc Length Polar Curve: $s=\int_{\theta_1}^{\theta_2}\sqrt{r^2+[r'(\theta)]^2}\,d\theta$

(Ch. 10.4) Special Polar Graphs

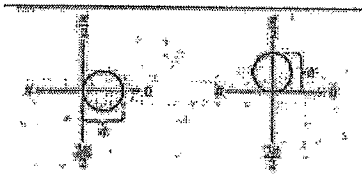
Circles: $r = a\cos\theta$ or $r = a\sin\theta$
*Traces out 1 rotation CCW from $[0,\pi]$
*coefficient a is the length of the diameter
*cosine graph symmetric to x-axis
*sine graph symmetric to y-axis

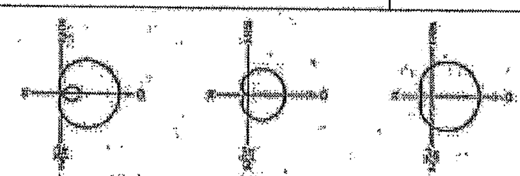
CCW = counter clockwise

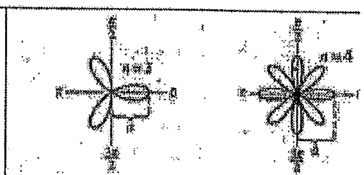
Limacons: r = a ± b cos θ or r = a ± b sin θ (a > 0, b > 0)
* Traces out 1 rotation Clockwise from $[0,2\pi]$
*constant + coefficient = outer radius
*constant – coefficient = inner radius
Graphs going through poles (origin):
Cardioid: **once**
Limacon with inner loop: **twice**
Dimpled Limacon: **none**

Rose Curves $r=a\cos(n\theta)$, $r=a\sin(n\theta)$
*coefficient a is the length of each petal
*If **n** is odd, then there are **n** petals on graph
*If **n** is even, then there are **2n** petals on graph

*If **n** is odd, 1 rotation traces out CCW from $[0,\pi]$
*If **n** is even, 1 rotation traces out CCW from $[0,2\pi]$


$r = a\cos\theta$ Circle $r = a\sin\theta$ Circle


$\frac{a}{b}<1$ Limacon with inner loop $\frac{a}{b}=1$ Cardioid (heart-shaped) $1<\frac{a}{b}<2$ Dimpled limacon


$r = a\cos n\theta$ Rose curve $r = a\cos n\theta$ Rose curve