Key

Name: Key Date:____________ Period: _____ **Review**

Unit 9 Review – Parametric Equations, Polar Coordinates, and Vector-Valued Functions (WS #1)

Reviews do NOT cover all material from the lessons but will hopefully remind you of key points. To be prepared, you must study all packets from Unit 9.

1. A curve is defined parametrically by $x(t) = t^3 - 3t^2 + 4$ and $y(t) = \sqrt{t^2 + 16}$. What is the equation of the tangent line at the point defined by $t = 3$?

$x(3) = 3^3 - 3(3)^2 + 4 = 4$

$y(3) = \sqrt{3^2+16} = 5$

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \rightarrow \frac{\frac{1}{2}(t^2+16)^{-1/2}(2t)}{3t^2-6t}$

$\left.\frac{dy}{dx}\right|_{t=3} = \frac{y'(3)}{x'(3)} \rightarrow \frac{(3)(3^2+16)^{-1/2}}{3(3)^2-6(3)} \rightarrow \frac{\frac{3}{5}}{9} \rightarrow \frac{3}{5}\cdot\frac{1}{9} = \frac{3}{45} = \frac{1}{15}$

point: $(4, 5)$

slope: $m = 1/15$

$\boxed{y - 5 = \frac{1}{15}(x-4)}$

2. An object moves in the xy-plane so that its position at any time t is given by the parametric equations $x(t) = t^2 + 3$ and $y(t) = t^3 + 5t$. What is the rate of change of y with respect to x when $t = 1$?

$x'(t) = 2t$

$x'(1) = 2$

$y'(t) = 3t^2 + 5$

$y'(1) = 3 + 5 = 8$

$\left.\frac{dy}{dx}\right|_{t=1} = \frac{8}{2} = \boxed{4}$

since $\left.\frac{dy}{dx}\right|_{t=1} = \frac{y'(1)}{x'(1)} = \boxed{4}$

3. A curve in the xy-plane is defined by $(x(t), y(t))$, where $x(t) = 3t$ and $y(t) = t^2 + 1$ for $t \geq 0$. What is $\frac{d^2y}{dx^2}$ in terms of t?

$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{2t}{3}$

$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} \rightarrow \frac{\frac{d}{dx}\left(\frac{2t}{3}\right)}{3} \rightarrow \frac{\frac{2}{3}}{3} \rightarrow \frac{2}{3}\cdot\frac{1}{3} = \boxed{\frac{2}{9}}$

4. If $x(\theta) = \cot\theta$ and $y(\theta) = \csc\theta$, what is $\frac{d^2y}{dx^2}$ in terms of θ?

$\frac{dy}{dx} = \frac{y'(\theta)}{x'(\theta)} = \frac{-\csc\theta\cot\theta}{-\csc^2\theta} \rightarrow \frac{\cot\theta}{\csc\theta} \rightarrow \frac{\cos\theta}{\sin\theta}\cdot\frac{1}{\csc\theta} \rightarrow \frac{\cos\theta}{\sin\theta}\cdot\frac{\sin\theta}{1} = \cos\theta$

$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}(\cos\theta)}{-\csc^2\theta} \rightarrow \frac{-\sin\theta}{-\csc^2\theta} \rightarrow \sin\theta\cdot\sin^2\theta \rightarrow \boxed{\sin^3\theta}$

5. What is the length of the curve defined by the parametric equations $x(t) = 7 + 4t$ and $y(t) = 6 - t$ for the interval $0 \leq t \leq 9$?

$* \text{Length} = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$

$\int_0^9 \sqrt{(4)^2 + (-1)^2}\,dt$

$\int_0^9 \sqrt{17}\,dt$

$\left.\sqrt{17}t\right]_0^9 = 9\sqrt{17} - 0\sqrt{17} = \boxed{9\sqrt{17}}$

6. What is the length of the curve defined by the parametric equations $x(\theta) = 3\cos 2\theta$ and $y(\theta) = 3\sin 2\theta$ for the interval $0 \le \theta \le \frac{\pi}{2}$?

$x'(\theta) = -3\sin(2\theta)\cdot 2$

$y'(\theta) = 3\cos(2\theta)\cdot 2$

$\int_0^{\pi/2} \sqrt{(36\sin 2\theta)^2 + (36\cos 2\theta)^2}\, d\theta$

$\int \sqrt{36(\sin^2 2\theta + \cos^2 2\theta)}\, d\theta$

$\int_0^{\pi/2} \sqrt{36(1)}\, d\theta \rightarrow \int 6\, d\theta$

$6t \Big]_0^{\pi/2} = 6(\pi/2) - 6(0)$

$= \boxed{3\pi}$

7. If f is a vector-valued function defined by $\langle 2t^3 + 3t^2 + 4t + 1,\ t^3 - 4t - 1\rangle$ then $f''(2) =$

$f'(t) = \langle 6t^2 + 6t + 4,\ 3t^2 - 4\rangle$

$f''(t) = \langle 12t + 6,\ 6t\rangle$

$f''(2) = \langle 24 + 6,\ 12\rangle$

$\boxed{f''(2) = \langle 30, 12\rangle}$

8. At time t, $0 \le t \le 2\pi$, the position of a particle moving along a path in the xy-plane is given by the vector-valued function, $f(t) = \langle e^t \sin 3t,\ e^t \cos 3t\rangle$. Find the slope of the path of the particle at time $t = \frac{\pi}{6}$.

$f'(t) = \dfrac{y'(t)}{x'(t)} \rightarrow \dfrac{e^t\cos(3t) - e^t\sin(3t)\cdot 3}{e^t\sin(3t) + e^t\cdot\cos(3t)\cdot 3} \rightarrow \dfrac{\cancel{e^t}(\cos 3t - 3\sin 3t)}{\cancel{e^t}(\sin 3t + 3\cos(3t))}$

$f'(\pi/6) = \dfrac{\cos(3\cdot\pi/6) - 3\sin(3\cdot\pi/6)}{\sin(3\cdot\pi/6) + 3\cos(3\cdot\pi/6)} \rightarrow \dfrac{\cos(\pi/2) - 3\sin(\pi/2)}{\sin(\pi/2) + 3\cos(\pi/2)} \rightarrow \dfrac{0-3}{1+3(0)} \rightarrow \dfrac{-3}{1} \rightarrow \boxed{-3}$

9. Find the vector-valued function $f(t)$ that satisfies the initial conditions $f(0) = \langle -2, 5\rangle$ and $f'(t) = \langle 10t^4,\ 2t\rangle$.

$x(t) = \int 10t^4\, dt = \dfrac{10t^5}{5} + C$

$x(0) = 2(0)^5 + C$

$-2 = C_1$

$x(t) = 2t^5 - 2$

$y(t) = \int 2t\, dt = \dfrac{2t^2}{2} + C$

$y(0) = (0)^2 + C$

$5 = C_2$

$y(t) = t^2 + 5$

$\boxed{f(t) = \langle 2t^5 - 2,\ t^2 + 5\rangle}$

10. **Calculator active** For $t \ge 0$, a particle is moving along a curve so that its position at time t is $(x(t), y(t))$. At time $t = 1$ the particle is at position $(3, 4)$. It is known that $\frac{dx}{dt} = \sin 2t$ and $\frac{dy}{dt} = \frac{\sqrt{t}}{e^{2t}}$. Find the y-coordinate of the particles position at time $t = 3$.

$y(1) = 4$

$y(3) = y(1) + \int_1^3 y'(t)\, dt$

$y(3) = 4 + \int_1^3 \dfrac{\sqrt{t}}{e^{2t}}\, dt$

$y(3) = 4 + 0.0796$

$\boxed{y(3) = 4.0796}$

11. A particle moving in the xy-plane has position given by parametric equations $x(t) = t$ and $y(t) = 4 - t^2$.

A. Find the velocity vector.

$$\boxed{\langle 1, -2t \rangle}$$

B. Find the speed when $t = 1$.

*speed is $\sqrt{[x'(t)]^2+[y'(t)]^2} \rightarrow \sqrt{[x'(1)]^2+[y'(1)]^2} = \sqrt{(1)^2+(-2)^2} = \boxed{\sqrt{5}}$

C. Find the acceleration vector.

$$\boxed{\langle 0, -2 \rangle}$$

12. It is known the acceleration vector for a particle moving in the xy-plane is given by $a(t) = \langle t, \sin t \rangle$. When $t = 0$, the velocity vector $v(0) = \langle 0, -1 \rangle$ and the position vector $p(0) = \langle 0, 0 \rangle$. Find the position vector at time $t = 2$.

$x'(t) = \int t\,dt = \frac{t^2}{2} + c$

$x'(0) = 0$ $\quad 0 = \frac{0^2}{2} + c \quad \underline{c = 0}$

$x'(t) = \frac{1}{2}t^2$

$y'(t) = \int \sin t\,dt = -\cos(t) + c$

$y'(0) = -1$ $\quad -1 = -\cos(0) + c$

$-1 = -1 + c$

$0 = c$

$y'(t) = -\cos(t)$

$x(t) = \int \frac{1}{2}t^2\,dt = \frac{1}{2}\left(\frac{t^3}{3}\right) + c$

$x(0) = 0$ $\quad 0 = \frac{1}{6}(0)^3 + c \quad \underline{c = 0}$

$y(t) = \int -\cos(t)\,dt = -\sin(t) + c$

$y(0) = 0$ $\quad 0 = -\sin(0) + c$

$c = 0$

$p(t) = \langle \frac{1}{6}t^3, -\sin(t) \rangle$

$p(2) = \langle \frac{1}{6}(8), -\sin(2) \rangle$

$\boxed{p(2) = \langle \frac{4}{3}, -\sin(2) \rangle}$

13. Find the slope of the tangent line to the polar curve $r = 2\cos 4\theta$ at the point where $\theta = \frac{\pi}{4}$.

$x(\theta) = r\cos\theta$

$x(\theta) = 2\cos(4\theta)\cdot\cos\theta$

$x'(\theta) = -2\sin 4\theta \cdot 4 \cdot \cos\theta + 2\cos(4\theta)\cdot -\sin\theta$

$x'(\theta) = -8\sin 4\theta\cos\theta - 2\cos 4\theta\sin\theta$

$x'(\pi/4) = -8\sin(\pi)\cos(\pi/4) - 2\cos(\pi)\sin(\pi/4)$

$= 0\left(\frac{\sqrt{2}}{2}\right) - 2(-1)\left(\frac{\sqrt{2}}{2}\right) = \sqrt{2}$

$y(\theta) = r\sin\theta$

$y(\theta) = 2\cos 4\theta \cdot \sin\theta$

$y'(\theta) = -2\sin 4\theta \cdot 4 \cdot \sin\theta + 2\cos 4\theta \cdot \cos\theta$

$y'(\pi/4) = -8\sin(\pi)\sin(\pi/4) + 2\cos(\pi)\cos(\pi/4)$

$= 0 + 2(-1)\left(\frac{\sqrt{2}}{2}\right) = -\sqrt{2}$

slope $= \frac{y'(\pi/4)}{x'(\pi/4)} = \frac{-\sqrt{2}}{\sqrt{2}} = \boxed{-1}$

14. **Calculator active.** For a certain polar curve $r = f(\theta)$, it is known that $\frac{dx}{d\theta} = \cos\theta - \theta\sin\theta$ and $\frac{dy}{d\theta} = \sin\theta + \theta\cos\theta$. What is the value of $\frac{d^2y}{dx^2}$ at $\theta = 6$?

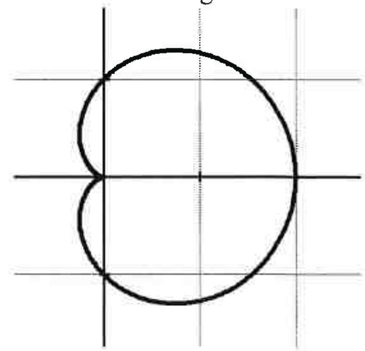
$$\frac{d^2y}{dx^2} = \frac{\frac{d}{d\theta}\left[\frac{dy}{dx}\right]}{dx/d\theta} \rightarrow \frac{\frac{d}{d\theta}\left[\frac{\sin\theta + \theta\cos\theta}{\cos\theta - \theta\sin\theta}\right]}{\cos\theta - \theta\sin\theta} \text{ at } \theta = 6 \rightarrow \frac{5.466085}{2.636663}$$

$$\boxed{2.073}$$

15. **Calculator active.** Find the total area enclosed by the polar curve $r = 1 + \cos\theta$ shown in the figure above.

$$\text{Area} = \int_{\theta_1}^{\theta_2} \frac{1}{2} r^2 \, d\theta$$

$$= \int_0^{2\pi} \frac{1}{2}\left[1+\cos\theta\right]^2 d\theta = \boxed{4.712}$$



16. **Calculator active.** Find the area of the inner loop of the polar curve $r = 3 - 6\sin\theta$.

$0 = 3 - 6\sin\theta$

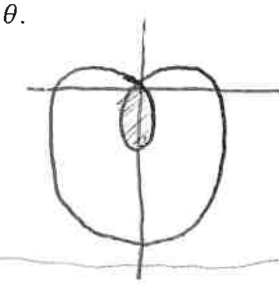
$6\sin\theta = 3$

$\sin\theta = 1/2$

$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

$$\text{Area} = \int_{\pi/6}^{5\pi/6} \frac{1}{2}\left[3-6\sin\theta\right]^2 d\theta$$

$$\boxed{4.8916}$$



17. Find the total area of the common interior of the polar graphs $r = 5 - 3\sin\theta$ and $r = 5 - 3\cos\theta$.

$5 - 3\sin\theta = 5 - 3\cos\theta$

$-3\sin\theta = -3\cos\theta$

$\sin\theta = \cos\theta$

or

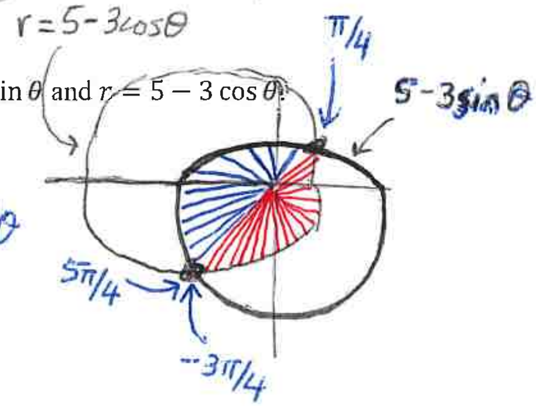
$\tan\theta = 1$

$\theta = \pi/4, 5\pi/4$

(Intersections of both graphs)

$$\int_{-3\pi/4}^{\pi/4} \frac{1}{2}\left[5-3\cos\theta\right]^2 d\theta + \int_{\pi/4}^{5\pi/4} \frac{1}{2}\left[5-3\sin\theta\right]^2 d\theta$$

$$= \boxed{50.2505}$$



18. **Calculator active.** The figure shows the graphs of the polar curves $r = 4\cos 3\theta$ and $r = 4$. What is the sum of the areas of the shaded regions?

*rose curve completes one cycle in $[0, \pi]$

circle − rose curve

$$\int_0^{2\pi} \frac{1}{2}[4]^2 - \int_0^{\pi} \frac{1}{2}\left[4\cos(3\theta)\right]^2 d\theta$$

$$= \boxed{37.699}$$