BC Calculus Units 6-8 Quiz Review WS 2

Key

Evaluate the below: Show all work!

1) *Integration by Parts (IBP)

The function f has a continuous derivative. The table gives the values of f and its derivatives for $x = 2$ and $x = 7$. If $\int_2^7 f(x)\,dx = 10$, what is the value of $\int_2^7 2xf'(x)\,dx$?

*IBP

	f	g'
+	$2x$	$f'(x)$
−	2	$f(x)$
+	0	$\int f(x)$

x	$f(x)$	$f'(x)$
2	3	5
7	9	−4

$$2xf(x) - 2\int f(x)\,dx$$

$$2xf(x)\Big]_2^7 - 2\underbrace{\int_2^7 f(x)\,dx}_{10}$$

$$2(7)\cdot f(7) - 2(2)\cdot f(2)$$

$$14(9) - 4(3) = 114$$

$$114 - 2(10)$$

$$114 - 20 = \boxed{94}$$

2) $\int 3x \ln x^2\, dx$

IBP * $\int fg' = fg - \int f'g$

$$\int 3x \cdot 2\ln x\,dx$$

$$\int 6x \ln x\,dx$$

$f = \ln x \qquad g' = 6x\,dx$

$f' = \frac{1}{x} \qquad g = \frac{6x^2}{2} = 3x^2$

$$= 3x^2 \ln x - \int \frac{1}{x}\cdot 3x^2\,dx$$

$$= 3x^2 \ln x - \int 3x\,dx$$

$$\boxed{3x^2\ln x - \frac{3x^2}{2} + C \quad \text{or} \quad \frac{3}{2}x^2\ln x^2 - \frac{3}{2}x^2 + C}$$

3) $\int x \cos 4x\, dx$

*Tab Method

	f	g'
+	x	$\cos(4x)$
−	1	$\frac{1}{4}\sin(4x)$
+	0	$\frac{-1}{16}\cos(4x)$

$$\int \cos(4x)\,dx$$

$u = 4x$

$\frac{du}{dx} = 4$

$dx = \frac{du}{4}$

$$\int \cos u \cdot \frac{du}{4}$$

$$\frac{1}{4}\int \cos u\,du$$

$$\frac{1}{4}\sin(4x)$$

$$\boxed{\frac{1}{4}x\sin(4x) + \frac{1}{16}\cos(4x) + C}$$

4) $\int \frac{4x+1}{2x^2-3x-2}\,dx$

*Partial Fraction Decomposition

$(2x+1)(x-2)$

$\frac{4x+1}{(2x+1)(x-2)} = \frac{A}{2x+1} + \frac{B}{x-2}$

$x=-1/2 \quad x=2$

$\frac{4(-1/2)+1}{(-1/2-2)} = \frac{-1}{-5/2} = \frac{2}{5} = A$

$\frac{4(2)+1}{2(2)+1} = \frac{9}{5} = B$

$\int \frac{2/5}{2x+1} + \frac{9/5}{x-2}\,dx$

$u=2x+1$

$\frac{du}{dx}=2$

$dx=\frac{du}{2}$

$\frac{2}{5}\int \frac{1}{u}\cdot\frac{du}{2}$

$\boxed{\frac{1}{5}\ln|2x+1| + \frac{9}{5}\ln|x-2| + C}$

5) $\int_2^\infty \frac{3}{x(x-1)}\,dx$

*Partial Fraction first

$\frac{3}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1}$

$x=0 \quad x=1$

$\frac{3}{0-1} = -3 = A$

$\frac{3}{1} = B$

VA: $x=0, x=1$ (no impact)

$\lim_{b\to\infty} \int_2^b -\frac{3}{x} + \frac{3}{x-1}\,dx$

$-3\ln|x| + 3\ln|x-1|\Big]_2^b$

$3\ln|x-1| - 3\ln|x|\Big]_2^b$

$3\ln\left|\frac{x-1}{x}\right|\Big]_2^b = 3\ln\left|\frac{b-1}{b}\right| - 3\ln\left(\frac{1}{2}\right)$

$\lim_{b\to\infty} 3\ln\left|\frac{b-1}{b}\right| - 3\ln(1/2)$

$3(0) - 3\ln(1/2) = \boxed{-3\ln\left(\frac{1}{2}\right)}$

$\boxed{OR}$ $-3\ln(2)^{-1} = \boxed{3\ln 2}$

6) Non-Calculator (Show your work)

$\int_0^5 \frac{1}{\sqrt{5-x}}\,dx$

$5-x=0$

$x=5$

VA at $x=5$ (Discontinuity at $x=5$)

$\lim_{b\to 5^-} \int_0^b \frac{1}{(5-x)^{1/2}}\,dx$

$u=5-x$

$\frac{du}{dx}=-1$

$dx=-1du$

$\int (u)^{-1/2}(-1du)$

$\frac{-u^{1/2}}{1/2} \to -2u^{1/2}$

$-2(5-x)^{1/2}\Big]_0^b$

$\lim_{b\to 5^-} -2(5-b)^{1/2} - \left(-2(5-0)^{1/2}\right)$

$-2(0)^{1/2} + 2\sqrt{5}$

$= \boxed{2\sqrt{5}}$

7)

Let $h(x) = \int_0^x \sqrt{1+4t^2}\,dt$. Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $h(3)$.

$n=2$

$\Delta x = \frac{b-a}{n} \rightarrow \frac{3-0}{2} = 1.5$

$h'(x) = \frac{d}{dx}\int_0^x \sqrt{1+4t^2}\,dt$

$h'(x) = \sqrt{1+4x^2}$

x	y_0	$h'(x) = \sqrt{1+4x^2}$	$y = y_0 + h'(x)(\Delta x)$
0	0	$h'(0) = \sqrt{1+0} = 1$	$y = 0 + 1(1.5) = 1.5$
1.5	1.5	$h'(1.5) = \sqrt{1+4(1.5)^2} = 3.1622$	$y = 1.5 + 3.1622(1.5) = 6.243$
3	$\boxed{6.243}$		

$\boxed{h(3) \approx 6.243}$

8)

Let $y = f(x)$ be the solution to the differential equation $\frac{dy}{dx} = x + y$ with initial condition $f(0) = 3$. What is the approximation for $f(0.5)$ obtained using Euler's method with 2 steps of equal length, starting at $x = 0$?

$\Delta x = \frac{b-a}{n} = \frac{0.5-0}{2}$

$\Delta x = \frac{1}{4} = 0.25$

x	y_0	$\frac{dy}{dx} = x+y$	$y = y_0 + y'(\Delta x)$
0	3	$y' = 0+3 = 3$	$y = 3 + 3(0.25) = 3.75$
0.25	3.75	$y' = 0.25+3.75 = 4$	$y = 3.75 + 4(0.25) = 4.75$
0.5	$\boxed{4.75}$		

$\boxed{f(0.5) \approx 4.75}$

9)

A populations rate of growth is modeled by the logistic differential equation $\frac{dP}{dt} = \frac{1}{1000}P(600-P)$, where t is in days and $P(0) = 60$. What is the greatest rate of change for this population? * set $P''(t) = 0$

$P' = 0.6P - \frac{1}{1000}P^2$

$P' = 0.6P - 0.001P^2$

$P'' = 0.6 - 0.002P$

$0 = 0.6 - 0.002P$

$0.002P = 0.6$

$P = 300$

$\left.\frac{dP}{dt}\right|_{P=300} = \frac{1}{1000}(300)[600-300]$

$= \boxed{90 \text{ people/day}}$

10)

$L = \_\_?$

Using the logistic differential equation $\frac{dP}{dt} = \frac{1}{5}P - \frac{1}{2000}P^2$, identify the carrying capacity.

$* \frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right)$

$\frac{dP}{dt} = \frac{1}{5}P\left[1 - \frac{P}{400}\right]$

Carrying capacity is 400 since $L = 400$

11)

$L = 1000$

A rate of change $\frac{dP}{dt}$ of a population is modeled by a logistic differential equation. If $\lim_{t\to\infty} P(t) = 1000$ and the rate of change of the population is 100 when the population size is 50, which of the following differential equations describe the situation?

A. $\frac{dP}{dt} = 50P\left(1 - \frac{P}{1000}\right)$

B. $\frac{dP}{dt} = 100P\left(1 - \frac{P}{1000}\right)$

C. $\frac{dP}{dt} = \frac{19}{40}P\left(1 - \frac{P}{1000}\right)$

D. $\frac{dP}{dt} = \frac{40}{19}P\left(1 - \frac{P}{1000}\right)$

$* \frac{dP}{dt} = 100$ when $P = 50$

$*$ test the answer choices to find solution

$\frac{dP}{dt} = \frac{40}{19}P\left[1 - \frac{P}{1000}\right]$

$100 = \frac{40}{19}(50)\left[1 - \frac{50}{1000}\right]$

$100 = 100$ ✓

12)

No Calculator. Suppose $F(x) = \int_0^x \sqrt{3 - 4\sin^2 t}\; dt$. What is the length of the arc along the curve $y = F(x)$ for $0 \le x \le \frac{\pi}{3}$?

$* L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$

$f'(x) = \frac{d}{dx}\int_0^x \sqrt{3 - 4\sin^2 t} = \sqrt{3 - 4\sin^2 x}$

$L = \int_0^{\pi/3} \sqrt{1 + \left[\sqrt{3 - 4\sin^2 x}\right]^2}\, dx = \int_0^{\pi/3} \sqrt{1 + 3 - 4\sin^2 x}$

$\int_0^{\pi/3} \sqrt{4 - 4\sin^2 x}$

$* 4\sin^2 x + 4\cos^2 x = 4$

$4 - 4\sin^2 x = 4\cos^2 x$

$\int_0^{\pi/3} \sqrt{4\cos^2 x}\, dx = \int_0^{\pi/3} 2\cos x\, dx$

$\int_0^{\pi/3} 2\cos x\, dx \rightarrow 2\sin x\Big]_0^{\pi/3}$

$2\sin(\pi/3) - 2\sin(0)$

$= 2 \cdot \frac{\sqrt{3}}{2} - 0$

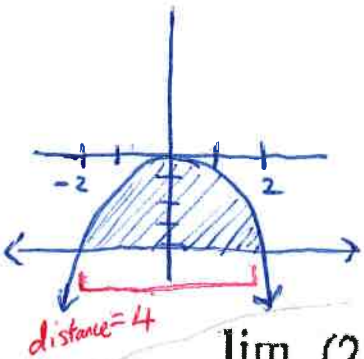
$L = \boxed{\sqrt{3}}$

$\boxed{L = \sqrt{3}}$

13)

Let R be the region bounded by the graphs of $f(x) = -x^2$ and $g(x) = -4$. Write an expression including one or more integrals that gives the perimeter of the region R. **Do Not Evaluate.**

$$*L = \int_a^b \sqrt{1+[f'(x)]^2}\,dx$$



Perimeter of graph $= 4 + \int_{-2}^{2} \sqrt{1+[-2x]^2}\,dx$ $\quad f'(x) = -2x$

$$\boxed{4 + \int_{-2}^{2} \sqrt{1+4x^2}\,dx}$$

14) $\lim\limits_{x\to 0+} (2x)^{3x}$

$y = (2x)^{3x}$

$\ln y = \ln(2x)^{3x}$

$\ln y = 3x \cdot \ln(2x)$

$\lim\limits_{x\to 0^+} \dfrac{\ln(2x)}{\frac{1}{3x}} \to \dfrac{-\infty}{\infty}$ ← Indeterminate Form (Apply L'Hopital's Rule)

$L'H \to \lim\limits_{x\to 0^+} \dfrac{\ln(2x)}{(3x)^{-1}} \to \dfrac{\frac{2}{2x}}{-1(3x)^{-2}(3)} \to \dfrac{\frac{1}{x}}{\frac{-3}{(3x)^2}} \to \dfrac{\frac{1}{x}}{\frac{-1}{3x^2}} \to \dfrac{1}{x} \cdot \dfrac{-3x^2}{1}$

$\lim\limits_{x\to 0^+} \dfrac{-3x}{1} \to 0$

$\ln y = 0$

$e^{\ln y} = e^0$

$y = e^0$

$\boxed{y = 1}$

15) $\lim\limits_{x\to 0+} (\csc x)^{\sin x}$

$y = (\csc x)^{\sin x}$

$\ln y = \ln(\csc x)^{\sin x}$

$\ln y = \sin x \cdot \ln(\csc x)$

$\ln y = \dfrac{\ln(\csc x)}{\csc x}$

$\lim\limits_{x\to 0^+} \dfrac{\ln(\csc x)}{\csc x} \to \dfrac{-\infty}{\infty}$

$\lim\limits_{x\to 0^+} \dfrac{\frac{-\csc x \cot x}{\csc x}}{-\csc x \cot x} \to \dfrac{\cancel{-\csc x \cot x}}{\csc x} \cdot \dfrac{1}{\cancel{-\csc x \cot x}}$

$\lim\limits_{x\to 0^+} \dfrac{1}{\csc x} \to \lim\limits_{x\to 0^+} \sin x = 0$

$\ln y = 0$

$e^{\ln y} = e^0$

$y = e^0$

$\boxed{y = 1}$