BC Calculus Units 6-8 Quiz Review WS 3

Key

Evaluate the below: Show all work!

1) Integration by Parts (IBP)

The function f has a continuous derivative. The table gives the values of f and its derivatives for $x = 1$ and $x = 6$. If $\int_1^6 f(x)\,dx = 9$, what is the value of $\int_1^6 3xf'(x)\,dx$?

*IBP (Tab Method)

x	$f(x)$	$f'(x)$
1	2	4
6	8	-3

	f	g'
+	$3x$	$f'(x)$
−	3	$f(x)$
+	0	$\int f(x)$

$$\int_1^6 3xf'(x)dx = \left[3xf(x) - \int 3f(x)dx\right]_1^6$$

$$3xf(x)\Big]_1^6 - 3\int_1^6 f(x)dx \to 9$$

$$3(6)\cdot f(6) - 3(1)\cdot f(1)$$

$$= 18(8) - 3(2) - 3(9) = 144 - 6 - 27 = \boxed{111}$$

2) $\int 4xe^{3x+2}dx$

*IBP (Tab Method)

	f	g'
+	$4x$	e^{3x+2}
−	4	$\frac{1}{3}e^{3x+2}$
+	0	$\frac{1}{9}e^{3x+2}$

$\int e^{3x+2}dx$ | $u = 3x+2$, $\frac{du}{dx} = 3$, $dx = \frac{du}{3}$ | $\int e^u \cdot \frac{du}{3}$, $\frac{1}{3}\int e^u du$ | $\frac{1}{3}e^u \to \frac{1}{3}e^{3x+2}$

$$\boxed{\frac{4}{3}xe^{3x+2} - \frac{4}{9}e^{3x+2} + C}$$

3) $\int \frac{2}{(x-1)(x+2)(x-4)}dx$

$$\frac{2}{(x-1)(x+2)(x-4)} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{x-4}$$

$x=1$ $\quad x=-2$ $\quad x=4$

$$\frac{2}{(1+2)(1-4)} \to \frac{2}{3(-3)} = \frac{2}{-9} = A$$

$$\frac{2}{(-2-1)(-2-4)} \to \frac{2}{(-3)(-6)} \to \frac{1}{9} = B$$

$$\frac{2}{(4-1)(4+2)} \to \frac{2}{3(6)} \to \frac{2}{18} = \frac{1}{9} = C$$

$$\int \frac{-2/9}{x-1} + \frac{1/9}{x+2} + \frac{1/9}{x-4}dx$$

$$\boxed{-\frac{2}{9}\ln|x-1| + \frac{1}{9}\ln|x+2| + \frac{1}{9}\ln|x-4| + C}$$

4) $\int_2^{\infty} x^{-3}\, dx$

$\lim_{b\to\infty} \int_2^b x^{-3}\,dx \qquad \frac{x^{-2}}{-2} \rightarrow -\frac{1}{2x^2}\Big]_2^b$

$\lim_{b\to\infty} \frac{-1}{2b^2} - \left(\frac{-1}{2(2)^2}\right)$

$0 + \frac{1}{8} = \boxed{\frac{1}{8}}$

5) $\int_0^3 \frac{1}{\sqrt{9-x^2}}\, dx$

$9-x^2=0$
$x=3$

VA at $x=3$ (Discontinuity at $x=3$)

$\lim_{b\to 3^-} \int_0^b \frac{1}{\sqrt{a^2-u^2}}\,dx$

$\arcsin\left(\frac{x}{3}\right)\Big]_0^b = \arcsin\left(\frac{b}{3}\right) - \arcsin\left(\frac{0}{3}\right)$

$\lim_{b\to 3^-} \arcsin\left(\frac{b}{3}\right) - \arcsin 0$

* $\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\left(\frac{u}{a}\right)$

$= \arcsin(1) - 0$

$\int \frac{1}{\sqrt{3^2-x^2}} \rightarrow \arcsin\left(\frac{x}{3}\right)$

$= \boxed{\frac{\pi}{2}}$

6)

Let $h(x) = \int_1^x \frac{1}{t^2}\,dt$. Use Euler's method, starting at $x = 1$ with two steps of equal size, to approximate $h(3)$.

$\Delta x = \frac{b-a}{n} \rightarrow \frac{3-1}{2} = \frac{2}{2}$

$\Delta x = 1$

$h'(x) = \frac{d}{dx}\int_1^x \frac{1}{t^2}\,dt$

$h'(x) = \frac{1}{x^2}$

x	y_0	$h'(x) = \frac{1}{x^2}$	$y = y_0 + h'(x)(\Delta x)$
1	$h(1) = \int_1^1 \frac{1}{t^2} = 0$	$h'(1) = \frac{1}{1^2} = 1$	$y = 0 + 1(1) = 1$
2	1	$h'(2) = \frac{1}{2^2} = \frac{1}{4} = 0.25$	$y = 1 + 0.25(1) = 1.25$
3	$\boxed{1.25}$		

$\boxed{h(3) \approx 1.25}$

7)

The table below gives the values of f', the derivative of f. If $f(2) = 1$, what is the approximation to $f(2.3)$ obtained by using Euler's method with 3 steps of equal size?

x	2	2.1	2.2	2.3
$f'(x)$	-0.1	-0.15	-0.3	-0.5

$\Delta x = \frac{b-a}{n} \to \frac{2.3-2}{3}$

$\Delta x = 0.1$

x	y_0	$f'(x)$	$y = y_0 + f'(x)(\Delta x)$
2	1	$f'(2) = -0.1$	$y = 1 + (-0.1)(0.1) = 0.99$
2.1	0.99	$f'(2.1) = -0.15$	$y = 0.99 + (-0.15)(0.1) = 0.975$
2.2	0.975	$f'(2.2) = -0.3$	$y = 0.975 + (-0.3)(0.1) = 0.945$
2.3	$\boxed{0.945}$		

$\boxed{f(2.3) \approx 0.945}$

8)

A populations rate of growth is modeled by the logistic differential equation $\frac{dP}{dt} = \frac{1}{400}P(100 - P)$, where t is in days and $P(0) = 10$. What is the greatest rate of change for this population? * set $P'' = 0$

$\frac{dP}{dt} = \frac{100}{400}P - \frac{1}{400}P^2$

$P' = 0.25P - 0.0025P^2$

$P'' = 0.25 - 0.005P$

$0 = 0.25 - 0.005P$

$0.005P = 0.25$

$P = 50$

$\left.\frac{dP}{dt}\right|_{P=50} = \frac{1}{400}(50)[100-50]$

$= \boxed{\frac{25}{4} \text{ people per day}}$

9)

Using the logistic differential equation $\frac{dP}{dt} = \frac{1}{3}P - \frac{1}{120}P^2$, what is $\lim_{t\to\infty} P(t)$?

* $\frac{dy}{dt} = ky\left[1 - \frac{y}{L}\right]$

* Limiting value is 40

$\frac{dP}{dt} = \frac{1}{3}P\left[1 - \frac{P}{40}\right]$

$\boxed{\lim_{t\to\infty} P(t) = 40}$

10)

$L = 100$

A rate of change $\frac{dP}{dt}$ of a population is modeled by a logistic differential equation. $\boxed{\text{If } \lim_{t\to\infty} P(t) = 100}$ and the rate of change of the population is 5 when the population size is 20, which of the following differential equations describe the situation?

$* \frac{dP}{dt} = 5, \; P = 20$

A. $\frac{dP}{dt} = 5P\left(1 - \frac{P}{20}\right)$

B. $\frac{dP}{dt} = 20P\left(1 - \frac{P}{100}\right)$

C. $\boxed{\frac{dP}{dt} = \frac{5}{16}P\left(1 - \frac{P}{100}\right)}$ $\rightarrow 5 = \frac{5}{16}(20)\left[1 - \frac{20}{100}\right]$

$5 = 5$ ✓

D. $\frac{dP}{dt} = \frac{16}{5}P\left(1 - \frac{P}{100}\right)$

11)

A rate of change for a population is modeled by the differential equation $\frac{dP}{dt} = 0.3P(66 - P)$. What is the population when the rate of change is the greatest? $* \text{ set } P'' = 0$

$P' = 19.8P - 0.3P^2$

$P'' = 19.8 - 0.6P$

$0 = 19.8 - 0.6P$

$0.6P = 19.8$

$P = 33$

$\boxed{\text{Population when rate of change is greatest is 33}}$

12) $* \; L = \int_a^b \sqrt{1 + [F'(x)]^2}\,dx$

No Calculator. Suppose $F(x) = \int_0^x \sqrt{3 - 4\cos^2 t}\; dt$. What is the length of the arc along the curve $y = F(x)$ for $0 \le x \le \frac{\pi}{3}$?

$f'(x) = \frac{d}{dx}\int_0^x \sqrt{3 - 4\cos^2 t}\,dt$

$f'(x) = \sqrt{3 - 4\cos^2 x}$

$L = \int_0^{\pi/3} \sqrt{1 + \left[\sqrt{3 - 4\cos^2 x}\right]^2}\,dx$

$L = \int_0^{\pi/3} \sqrt{1 + 3 - 4\cos^2 x}\,dx$

$* \; 4\sin^2 x + 4\cos^2 x = 4$

$4\sin^2 x = 4 - 4\cos^2 x$

$L = \int_0^{\pi/3} \sqrt{4 - 4\cos^2 x}\,dx$

$L = \int_0^{\pi/3} \sqrt{4\sin^2 x}\,dx$

$L = \int_0^{\pi/3} 2\sin x\,dx$

$-2\cos x \Big]_0^{\pi/3}$

$-2\cos(\pi/3) - (-2\cos(0))$

$-2(1/2) + 2$

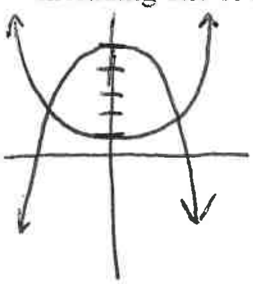
$-1 + 2 = \boxed{1}$

13)

$x^2+1=-x^2+5$ | $x^2=2$

$2x^2=4$ | $x=\pm\sqrt{2}$ $\quad f'(x)=2x \quad g'(x)=-2x$

Let R be the region bounded by the graphs of $f(x) = x^2 + 1$ and $g(x) = -x^2 + 5$. Write an expression including one or more integrals that gives the length of the region R. **Do Not Evaluate.**



$$L_1 = \int_{-\sqrt{2}}^{\sqrt{2}} \sqrt{1+[f'(x)]^2}\,dx \to \int_{-\sqrt{2}}^{\sqrt{2}} \sqrt{1+[2x]^2}\,dx \to \int_{-\sqrt{2}}^{\sqrt{2}} \sqrt{1+4x^2}\,dx$$

$$L_2 = \int_{-\sqrt{2}}^{\sqrt{2}} \sqrt{1+[g'(x)]^2}\,dx \to \int_{-\sqrt{2}}^{\sqrt{2}} \sqrt{1+[2x]^2}\,dx \to \int_{-\sqrt{2}}^{\sqrt{2}} \sqrt{1+4x^2}\,dx$$

$$\boxed{L_1+L_2 = 2\int_{-\sqrt{2}}^{\sqrt{2}} \sqrt{1+4x^2}\,dx}$$

14) $\lim\limits_{x\to\infty} (x+1)^{e^{-x}}$

$y=(x+1)^{e^{-x}}$

$\ln y = \ln(x+1)^{e^{-x}}$

$\ln y = e^{-x}\cdot \ln(x+1)$

$\ln y = \dfrac{\ln(x+1)}{e^x}$

$\lim\limits_{x\to\infty} \dfrac{\ln(x+1)}{e^x} \to \dfrac{\infty}{\infty}$

$L'H \to \lim\limits_{x\to\infty} \dfrac{\frac{1}{x+1}}{e^x}$

$\lim\limits_{x\to\infty} \dfrac{1}{e^x(x+1)} \to 0$

$\ln y = 0$

$e^{\ln y} = e^0$

$y=e^0$

$\boxed{y=1}$

15) $\lim\limits_{x\to\pi/2^-} (\sin x)^{\tan x}$

$y=\sin x^{\tan x}$

$\ln y = \ln(\sin x)^{\tan x}$

$\ln y = \tan x \cdot \ln(\sin x)$

$\ln y = \dfrac{\ln(\sin x)}{\cot x}$

$\lim\limits_{x\to\pi/2^-} \dfrac{\ln(\sin x)}{\cot x} \to L'H \to \lim\limits_{x\to\pi/2^-} \dfrac{\frac{\cos x}{\sin x}}{-\csc^2 x}$

$\dfrac{\cos x}{\sin x}\cdot\dfrac{1}{-\csc^2 x} \to \dfrac{\cos x}{\sin x}\cdot\dfrac{-\sin^2 x}{1} \to -\cos x \sin x$

$\lim\limits_{x\to\pi/2^-} -\cos x\sin x = -\cos(\pi/2)\sin(\pi/2) \to -0(1) = 0$

$\ln y = 0$

$e^{\ln y} = e^0$

$y=e^0$

$\boxed{y=1}$