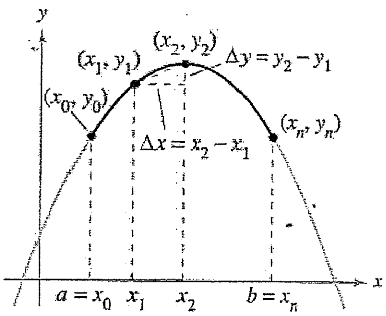
Calculus BC   Arc Length Notes (Chapter 7.4) Notes



Distance:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Arc length (s)

$$s \approx \sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}$$

$$= \sum_{i=1}^{n} \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$$

$$= \sum_{i=1}^{n} \sqrt{(\Delta x_i)^2 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2 (\Delta x_i)^2}$$

$$= \sum_{i=1}^{n} \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\,(\Delta x_i).$$

$$s = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\,(\Delta x_i)$$

$$s = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \sqrt{1 + [f'(c_i)]^2}\,(\Delta x_i)$$

$$= \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

Definition of Arc Length

Let the function given by $y = f(x)$ represent a smooth curve on the interval $[a, b]$. The **arc length** of f between a and b is

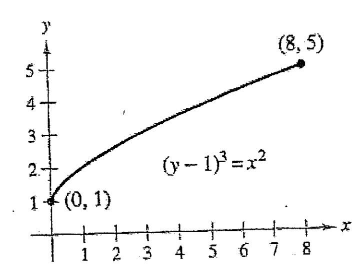
$$s = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx.$$

Ex. 1: Find the arc length of the graph $y = \frac{x^3}{6} + \frac{1}{2x}$

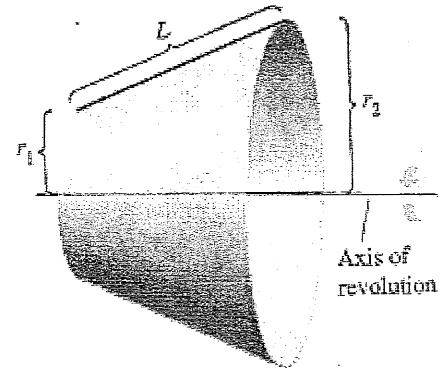
on the interval $\left[\frac{1}{2}, 2\right]$

Ex. 2

Find the arc length of the graph $(y - 1)^3 = x^2$ on the interval [0, 8]



Definition of Surface of Revolution:
If the graph of a continuous function is revolved about a line, the resulting surface is a surface of revolution.



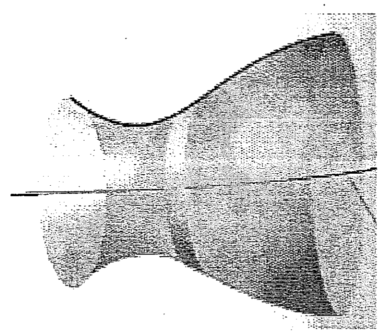
The area of a surface of revolution is derived from the formula for the lateral surface area of the frustum (region between 2 parallel planes in the solid) of a right circular cone.

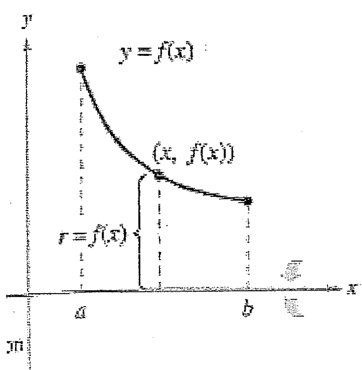
S = $2\pi r L$

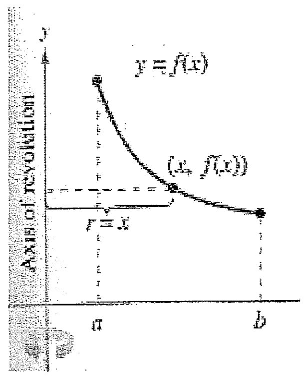
Area of a Surface of Revolution:

$$S = 2\pi \int_a^b r(x)\sqrt{1 + [f'(x)]^2}\, dx$$

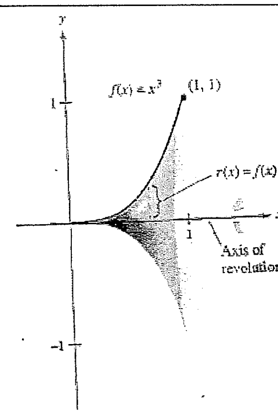
y is a function of x.



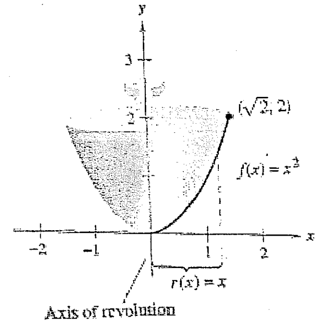




Ex. 3 Find the area of the surface formed by revolving the graph of $f(x) = x^3$ on the interval [0, 1] about the x-axis

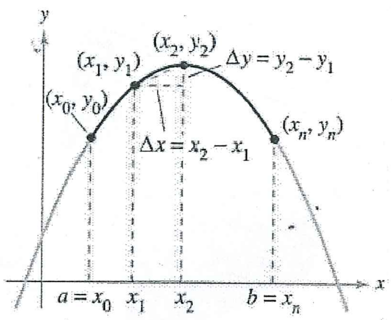


Ex. 4 Find the area of the surface formed by revolving the graph of $f(x) = x^2$ on the interval $[0, \sqrt{2}]$ about the y-axis



Calculus BC **Arc Length Notes (Chapter 7.4) Notes**

$\boxed{\text{Key}}$



Distance:

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Arc length (s)

$$s \approx \sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}$$

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$$s = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2} (\Delta x_i)$$

$$s = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \sqrt{1 + [f'(c_i)]^2} (\Delta x_i)$$

$$= \int_a^b \sqrt{1 + [f'(x)]^2}\, dx \quad \text{or} \quad \int_a^b \sqrt{1 + [g'(y)]^2}\, dy$$

Definition of Arc Length

Let the function given by $y = f(x)$ represent a smooth curve on the interval $[a, b]$. The **arc length** of f between a and b is

$$s = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx.$$

Ex. 1: Find the arc length of the graph $y = \frac{x^3}{6} + \frac{1}{2x}$ $= \frac{1}{6}x^3 + \frac{1}{2}x^{-1}$

on the interval $\left[\frac{1}{2}, 2\right]$

$f'(x) = \frac{1}{6} \cdot 3x^2 + \frac{1}{2}(-1)x^{-2}$

$f'(x) = \frac{x^2}{2} - \frac{1}{2x^2}$

$s = \int_{1/2}^{2} \sqrt{1 + \left(\frac{x^2}{2} - \frac{1}{2x^2}\right)^2}\, dx$

$\sqrt{1 + \frac{x^4}{4} - \frac{1}{4} - \frac{1}{4} + \frac{1}{4x^4}}$

$\sqrt{\frac{x^4}{4} + \frac{1}{2} + \frac{1}{4x^4}}$

$\sqrt{\frac{1}{4}\left(x^4 + 2 + \frac{1}{x^4}\right)}$

$\sqrt{\frac{1}{4}\left(x^2 + \frac{1}{x^2}\right)^2}$

$S = \int_{1/2}^{2} \frac{1}{2}\left(x^2 + \frac{1}{x^2}\right) dx = \int_{1/2}^{2} \frac{1}{2}x^2 + \frac{1}{2}x^{-2}\, dx$

$\frac{1}{2}\left(\frac{x^3}{3}\right) + \frac{1}{2}\left(\frac{x^{-1}}{-1}\right)$

$= \left.\frac{x^3}{6} - \frac{1}{2x}\right]_{1/2}^{2}$

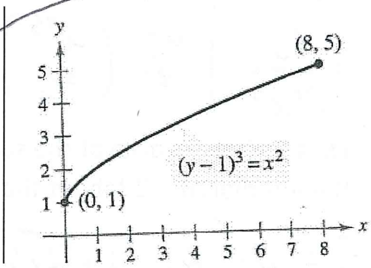
$= \frac{8}{6} - \frac{1}{4} - \left(\frac{1}{48} - 1\right)$

$= \frac{4}{3} - \frac{1}{4} - \frac{1}{48} + 1$

$S = \boxed{\frac{33}{16}}$

Ex. 2

Find the arc length of the graph $(y - 1)^3 = x^2$ on the interval $[0, 8]$



$x = \sqrt{(y-1)^3}$

$x = (y-1)^{3/2}$

$\frac{dx}{dy} = \frac{3}{2}(y-1)^{1/2}(1)$

$\frac{dx}{dy} = \frac{3}{2}(y-1)^{1/2}$

$S = \int_1^5 \sqrt{1 + \left[\frac{3}{2}(y-1)^{1/2}\right]^2}\, dy$

$\sqrt{1 + \frac{9}{4}(y-1)}$

$\sqrt{1 + \frac{9}{4}y - \frac{9}{4}}$

$\sqrt{-\frac{5}{4} + \frac{9}{4}y}$

$\sqrt{\frac{9y-5}{4}}$

$\frac{1}{2}\sqrt{9y-5}$

$s = \frac{1}{2}\int (9y-5)^{1/2}\, dy$

$u = 9y - 5$

$\frac{du}{dy} = 9$

$dy = \frac{du}{9}$

$\frac{1}{2}\int u^{1/2} \cdot \frac{du}{9} = \frac{1}{18}\frac{u^{3/2}}{3/2}$

$\frac{1}{18} \cdot \frac{2}{3} u^{3/2}$

$\left.\frac{1}{27}(9y-5)^{3/2}\right]_1^5$

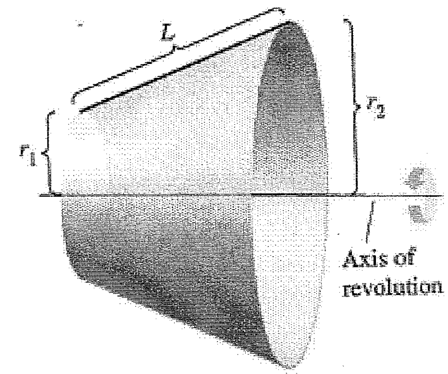
$\frac{1}{27}\left(40^{3/2} - 4^{3/2}\right) \approx \boxed{9.073}$

Calculus BC Area of Surface of Revolution Notes (Chapter 7.4b) Notes

Definition of Surface of Revolution:
If the graph of a continuous function is revolved about a line, the resulting surface is a surface of revolution.

The area of a surface of revolution is derived from the formula for the lateral surface area of the frustum (region between 2 parallel planes in the solid) of a right circular cone.

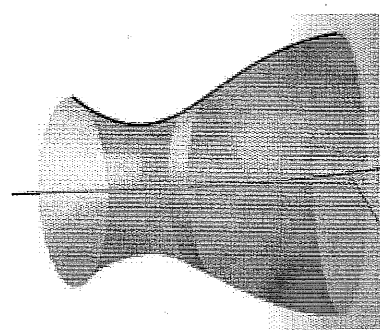
S = $2\pi rL$

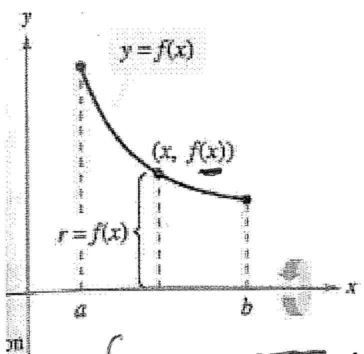


Area of a Surface of Revolution:

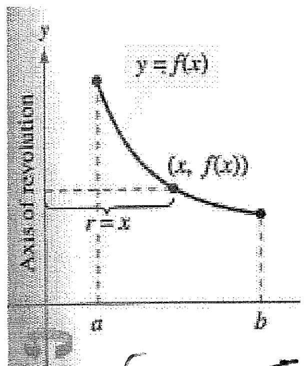
$$S = 2\pi\int_a^b r(x)\sqrt{1+[f'(x)]^2}\,dx$$

y is a function of x.





$$S = 2\pi\int f(x)\sqrt{1+f'(x)^2}\,dx$$



$$S = 2\pi\int x\sqrt{1+f'(x)^2}\,dx$$

Ex. 3 Find the area of the surface formed by revolving the graph of $f(x)= x^3$ on the interval [0, 1] about the x-axis

$S = 2\pi\int_0^1 x^3\sqrt{1+(3x^2)^2}) = \int x^3(1+9x^4)^{1/2}dx$

$u = 1+9x^4$

$\frac{du}{dx} = 36x^3$

$dx = \frac{du}{36x^3}$

$= 2\pi\int \cancel{x^3}\cdot u^{1/2}\cdot\frac{du}{36\cancel{x^3}}$

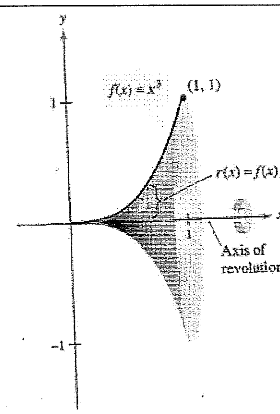
$\frac{2\pi}{36}\left(\frac{u^{3/2}}{3/2}\right)$

$\frac{\pi}{18}\cdot\frac{2}{3}(1+9x^4)^{3/2}$

$\frac{\pi}{27}(1+9x^4)^{3/2}\Big]_0^1$

$\frac{\pi}{27}(10^{3/2}-1)$

$\approx \boxed{3.563}$



Ex. 4 Find the area of the surface formed by revolving the graph of $f(x) = x^2$ on the interval $[0, \sqrt{2}]$ about the y-axis

$S = 2\pi\int x(\sqrt{1+(2x)^2})dx$

$= 2\pi\int x(1+4x^2)^{1/2}dx$

$u = 1+4x^2$

$\frac{du}{dx} = 8x$

$dx = \frac{du}{8x}$

$2\pi\int x\cdot u^{1/2}\cdot\frac{du}{8x}$

$\frac{2\pi}{8}\int u^{1/2}du$

$\frac{\pi}{4}\left(\frac{u^{3/2}}{3/2}\right) = \frac{\pi}{4}\cdot\frac{2}{3}(1+4x^2)^{3/2}$

$= \frac{\pi}{6}(1+4x^2)^{3/2}\Big]_0^{\sqrt{2}}$

$= \frac{\pi}{6}[9^{3/2}-1] = \frac{\pi}{6}(26) = \frac{26\pi}{6} = \frac{13\pi}{3} \approx \boxed{13.614}$