BC Calculus Ch. 10.3 Notes Parametric Equations and Calculus

Parametric Equations & Formulas for Calculus

If a smooth curve C is given by the equations $x = f(t)$ and $y = g(t)$, then the slope of C at the point (x, y) is given by $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$ where $\frac{dx}{dt} \neq 0$, and the second derivative is given by

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] =$$

Example 1:

Without a calculator, given $x = 2\sqrt{t}$, $y = 3t^2 - 2t$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ and evaluate at $t = 1$.

Example 2:

Without a calculator, given $x = 4\cos t$, $y = 3\sin t$, write an equation of the tangent line to the curve at the point where $t = \frac{3\pi}{4}$.

Example 3:

Without a calculator, find all points of horizontal and vertical tangency given $x = t^2 + t$, $y = t^2 - 3t + 5$.

Parametric Arc Length

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$ is the length of the arc from $t = a$ to $t = b$

Example 7:

Without a calculator, find the arc length of the given curve if $x = t^2$, $y = 4t^3 - 1$, $0 \le t \le 1$.

It's time to revisit particle motion.

Horizontal and Vertical Velocity Component VECTORS

- $x'(t)=\dfrac{dx}{dt}$ is the rate at which the x-coordinate is changing with respect to t or the velocity of a particle in the horizontal direction.

- $y'(t)=\dfrac{dy}{dt}$ is the rate at which the y-coordinate is changing with respect to t or the velocity of a particle in the vertical direction.

- $\bar{s}=\langle x(t),y(t)\rangle=(x(t),y(t))$ is the position at any time t.

- $\bar{v}=\langle x'(t),y'(t)\rangle=(x'(t),y'(t))$ is the velocity vector at any time t.

- $\bar{a}=\langle x''(t),y''(t)\rangle=(x''(t),y''(t))$ is the acceleration vector at any time t.

  *note: the vectors may or may not be contained within the chevrons $\langle\ \rangle$.

- $\dfrac{dy}{dx}$ is the rate of change of y with respect to x or the slope of the tangent line to the curve or the slope of the position vector.

- $\dfrac{d^2y}{dx^2}$ is the rate of change of the slope of the curve with respect to x.

- $|\bar{v}(t)|=\|\bar{v}(t)\|=\sqrt{\left(\dfrac{dx}{dt}\right)^2+\left(\dfrac{dy}{dt}\right)^2}$ is the **<u>speed of a particle</u>** or the **<u>magnitude</u>** or the **<u>length</u>** or the **<u>norm</u>** of the velocity vector.

- $\int_a^b\sqrt{\left(\dfrac{dx}{dt}\right)^2+\left(\dfrac{dy}{dt}\right)^2}\,dt$ is the **length of the arc** from $t=a$ to $t=b$ or the **distance traveled** by a particle from $t=a$ to $t=b$.

*Remember that $\int_a^b |v(t)|\,dt$ is the total distance traveled, whether it be along a straight line or curve.

Example 5:
(No Calculator) A particle moves in the $xy-$plane so that at any time t, $t \geq 0$, the position of the particle is given by $x(t) = t^3 + 4t^2$, $y(t) = t^4 - t^3$.

(a) Find the velocity vector at $t = 1$, (b) the speed of the particle at $t = 1$, and (c) the acceleration vector at $t = 1$.

Example 6:
(No Calculator) A particle moves in the $xy-$plane so that $x = \sqrt{3} - 4\cos t$ and $y = 1 - 2\sin t$, where $0 \leq t \leq 2\pi$

(a) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

(b) The path of the particle intersects the x-axis twice. Write an expression that represents the distance traveled by the particle between the two x-intercepts. Do not evaluate.

BC Calculus Ch. 10.3 Notes Parametric Equations and Calculus

Key

Parametric Equations & Formulas for Calculus

If a smooth curve C is given by the equations $x = f(t)$ and $y = g(t)$, then the slope of C at the point (x, y) is given by $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{y'(t)}{x'(t)}$ where $\frac{dx}{dt} \neq 0$, and the second derivative is given by

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d}{dx}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right) = \boxed{\frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}}$$

Example 1:

Without a calculator, given $x = 2\sqrt{t}$, $y = 3t^2 - 2t$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ and evaluate at $t = 1$.

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{6t-2}{t^{-1/2}} = \sqrt{t}(6t-2) = 6t^{3/2} - 2t^{1/2}$$

$x = 2t^{1/2}$

$x'(t) = t^{-1/2}$

$y'(t) = 6t - 2$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(6t^{3/2} - 2t^{1/2}\right)}{t^{-1/2}} = \frac{9t^{1/2} - t^{-1/2}}{t^{-1/2}}$$

$$\frac{d^2y}{dx^2} = 9t - 1$$

$$\left.\frac{dy}{dx}\right|_{t=1} = 6 - 2 = \boxed{4}$$

particle is moving at velocity of 4 thru a curved path (up 4 times faster than moving right)

$$\left.\frac{d^2y}{dx^2}\right|_{t=1} = 9(1) - 1 = \boxed{8}$$

accelerating thru curved path at 8

*confirm $\left.\frac{dy}{dx}\right|_{x=1}$ with calculator.

Example 2:

Without a calculator, given $x = 4\cos t$, $y = 3\sin t$, write an equation of the tangent line to the curve at the point where $t = \frac{3\pi}{4}$. find point and slope.

$$x\left(\frac{3\pi}{4}\right) = 4\cos\left(\frac{3\pi}{4}\right) = 4\left(-\frac{\sqrt{2}}{2}\right) = -2\sqrt{2}$$

$$y\left(\frac{3\pi}{4}\right) = 3\sin\left(\frac{3\pi}{4}\right) = 3\left(\frac{\sqrt{2}}{2}\right) = \frac{3\sqrt{2}}{2}$$

point: $\left(-2\sqrt{2}, \frac{3\sqrt{2}}{2}\right)$

Tangent line: $y - \frac{3\sqrt{2}}{2} = \frac{3}{4}\left(x + \frac{2\sqrt{2}}{2}\right)$

slope: $$\left.\frac{dy}{dx}\right|_{t=3\pi/4} = \frac{y'(t)}{x'(t)} = \frac{3\cos t}{-4\sin t} = \frac{3\cos(3\pi/4)}{-4\sin(3\pi/4)} = \left(\frac{-3}{4}\right)\left(\frac{-\sqrt{2}/2}{\sqrt{2}/2}\right) = \frac{3}{4}$$

$m = 3/4$

Example 3:

Without a calculator, find all points of horizontal and vertical tangency given $x = t^2 + t$, $y = t^2 - 3t + 5$.

Horizontal tangent is where $\frac{dy}{dx} = \frac{0}{\text{nonzero}}$ find where $\frac{dy}{dt} = 0$

Vertical tangent is where $\frac{dy}{dx} = \frac{\text{nonzero}}{0}$, find where $\frac{dx}{dt} = 0$

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{2t-3}{2t+1}$$

Horizontal tangent:

$2t - 3 = 0$

$t = 3/2$

point: $\left(\left(\frac{3}{2}\right)^2 + \frac{3}{2}, \left(\frac{3}{2}\right)^2 - 3\left(\frac{3}{2}\right) + 5\right)$

vertical tangent

$2t + 1 = 0$

$t = -1/2$

point: $\left(\left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right), \left(-\frac{1}{2}\right)^2 - 3\left(-\frac{1}{2}\right) + 5\right)$

Parametric Arc Length

$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$ is the length of the arc from $t = a$ to $t = b$

$d = \sqrt{(\Delta x)^2 + (\Delta y)^2}$

* Recall Rectangular Arc Length:

$$s = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

Example 4:

Without a calculator, find the arc length of the given curve if $x = t^2$, $y = 4t^3 - 1$, $0 \le t \le 1$.

$x'(t) = 2t \quad y'(t) = 12t^2$

$$L = \int_0^1 \sqrt{(2t)^2 + (12t^2)^2}\, dt$$

$$= \int \sqrt{4t^2 + 144t^4}\, dt$$

$$= \int \sqrt{4t^2(1 + 36t^2)}\, dt$$

$$\int 2t\sqrt{1 + 36t^2}\, dt$$

$u = 1 + 36t^2 \quad dt = \frac{du}{72t}$

$\frac{du}{dt} = 72t$

$$\int 2t \cdot u^{1/2} \cdot \frac{du}{72t}$$

$$\frac{1}{36}\int u^{1/2}\, du$$

$$\frac{1}{36} \cdot \frac{u^{3/2}}{3/2}$$

$$\frac{2}{3} \cdot \frac{1}{36}\left(1 + 36t^2\right)^{3/2}\Big]_0^1$$

$$\frac{1}{54}\left(37^{3/2} - 1^{3/2}\right)$$

$$= \boxed{\frac{1}{54}\left(37^{3/2} - 1\right)}$$

It's time to revisit particle motion.

Horizontal and Vertical Velocity Component VECTORS

- $x'(t) = \frac{dx}{dt}$ is the rate at which the x-coordinate is changing with respect to t or the velocity of a particle in the horizontal direction. (horizontal velocity)

arrow notation denotes vectors

- $y'(t) = \frac{dy}{dt}$ is the rate at which the y-coordinate is changing with respect to t or the velocity of a particle in the vertical direction. (vertical velocity)

- $\vec{s} = \langle x(t), y(t) \rangle = (x(t), y(t))$ is the position at any time t.

- $\vec{v} = \langle x'(t), y'(t) \rangle = (x'(t), y'(t))$ is the velocity vector at any time t.

- $\vec{a} = \langle x''(t), y''(t) \rangle = (x''(t), y''(t))$ is the acceleration vector at any time t.

  *note: the vectors may or may not be contained within the chevrons $\langle\ \rangle$. (angle brackets)

- $\frac{dy}{dx}$ is the rate of change of y with respect to x or the slope of the tangent line to the curve or the slope of the position vector.

- $\frac{d^2y}{dx^2}$ is the rate of change of the slope of the curve with respect to x.

Abs value of velocity is speed. $\|\vec{v}(t)\|$ is the resultant vector.

- $|\vec{v}(t)| = \|\vec{v}(t)\| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$ is the <u>**speed of a particle**</u> or the <u>**magnitude**</u> or the <u>**length**</u> or the <u>**norm**</u> of the velocity vector.

- $\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$ is the **length of the arc** from $t = a$ to $t = b$ or the **distance traveled** by a particle from $t = a$ to $t = b$. .

*Remember that $\int_a^b |v(t)|\, dt$ is the total distance traveled, whether it be along a straight line or curve.

* $\int$ speed is the total distance traveled along the curve (arc length)

* $\int$ velocity is the displacement (net change in position where positive and negatives cancel)

Example 5:

(No Calculator) A particle moves in the xy – plane so that at any time t, $t \geq 0$, the position of the particle is given by $x(t) = t^3 + 4t^2$, $y(t) = t^4 - t^3$.

position vector $\vec{s}(t) = \langle t^3+4t^2, t^4-t^3 \rangle$

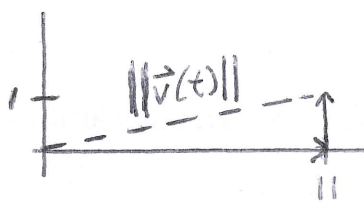
(a) Find the velocity vector at $t = 1$, (b) the speed of the particle at $t = 1$, and (c) the acceleration vector at $t = 1$.

$= \langle 5, 0 \rangle$

a) $\vec{v}(t) = \langle 3t^2+8t, 4t^3-3t^2 \rangle$

$\vec{v}(1) = \langle 11, 1 \rangle$

b)



speed $=$

$\|\vec{v}(1)\| = \sqrt{11^2+1^2}$

$= \boxed{\sqrt{122}}$

c) $\vec{a}(t) = \langle 6t+8, 12t^2-6t \rangle$

$\vec{a}(1) = \langle 14, 6 \rangle$

*particle picking up speed faster horizontally than vertically

Example 6:

(No Calculator) A particle moves in the xy – plane so that $x = \sqrt{3} - 4\cos t$ and $y = 1 - 2\sin t$, where $0 \leq t \leq 2\pi$

not linear motion (vector/parametric)

(a) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{-2\cos t}{4\sin t} = \boxed{-\frac{1}{2}\cot t}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{-\left(-\frac{1}{2}\csc^2 t\right)}{4\sin t} = \boxed{\frac{1}{8}\csc^3 t}$$

(b) The path of the particle intersects the x-axis twice. Write an expression that represents the distance traveled by the particle between the two x-intercepts. Do not evaluate.

*particle intersects x-axis when $y=0$.

$y = 1 - 2\sin t$

$1 - 2\sin t = 0$

$\sin t = \frac{1}{2}$

$t = \frac{\pi}{6}, \frac{5\pi}{6}$

Distance traveled is parametric arc length.

$$L = \int_{\pi/6}^{5\pi/6} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$\boxed{L = \int_{\pi/6}^{5\pi/6} \sqrt{(4\sin t)^2 + (-2\cos t)^2}\, dt}$$