BC Calculus Ch. 10.4b Notes Polar Equations and Derivatives

Before we jump into the calculus of polar, we need to develop some proficiency converting back and forth from polar to rectangular coordinates and equations.

Rectangular/Polar Coordinates and Equations

Rectangular coordinates are in the form (x, y), where x is the independent variable.

Polar coordinates are in the form (r, θ), where θ is the independent variable.

For coordinate conversions:

Polar to Rectangular

$x = r\cos\theta$

$y = r\sin\theta$

Rectangular to Polar

$r = \sqrt{x^2 + y^2}$

$\tan\theta = \dfrac{y}{x}$

Example 1:

Convert the following coordinates as specified.

a) Convert $\left(2, \dfrac{5\pi}{6}\right)$ to rectangular coordinates.

b) Convert $(3, -3)$ to polar coordinates.

Example 2:

Convert the following equations from rectangular to polar.

a) $y = 2x + 1$

b) $y = 4$

c) $x^2 + y^2 = 16$

Example 3:

Convert the following equations from polar to rectangular.

a) $r = 2\cos\theta$

b) $\theta = \dfrac{3\pi}{4}$

c) $r = \csc\theta$

Slope of a Polar Equation

To find the slope of a tangent line to a polar graph $r = f(\theta)$, we can use $x = r\cos\theta$ and $y = r\sin\theta$ and the product rule:

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{r\cos\theta + r'\sin\theta}{-r\sin\theta + r'\cos\theta}, \text{ provided that } \frac{dx}{d\theta} \neq 0$$

Or .... create $y(\theta)$ and $x(\theta)$ first, then find $y'(\theta)$ and $x'(\theta)$.

Example 4:

Find $\frac{dy}{dx}$ and the slope of the graph of the polar curve at the given value of θ. $r = 2 + 2\sin\theta$ at $\theta = \frac{\pi}{2}$

Example 5:

Find the **points,** (r,θ) of horizontal and vertical tangency to the graph of $r = 2 - 2\cos\theta$. Beware of $\frac{0}{0}$.

Key

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Before we jump into the calculus of polar, we need to develop some proficiency converting back and forth from polar to rectangular coordinates and equations.

Rectangular/Polar Coordinates and Equations

Rectangular coordinates are in the form (x, y), where x is the independent variable.

Polar coordinates are in the form (r, θ), where θ is the independent variable.

For coordinate conversions:

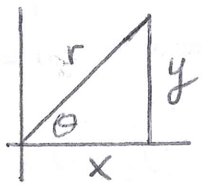
Polar to Rectangular

$x = r\cos\theta$

$y = r\sin\theta$

$\cos\theta = \frac{x}{r}$

$\sin\theta = \frac{y}{r}$

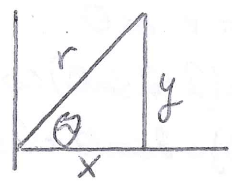


Rectangular to Polar

$r = \sqrt{x^2 + y^2}$ (pythagorean theorem to find radius)

$\tan\theta = \frac{y}{x}$

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$



$r^2 = x^2 + y^2$

$r = \sqrt{x^2+y^2}$

Example 1:

Convert the following coordinates as specified.

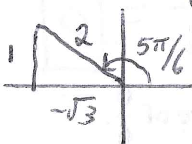
a) Convert $\left(2, \frac{5\pi}{6}\right)$ to rectangular coordinates.

$r = 2$

$\theta = 5\pi/6$

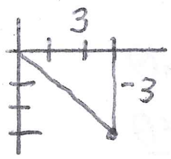
$x = 2\cos\left(\frac{5\pi}{6}\right) = 2\left(\frac{-\sqrt{3}}{2}\right) = -\sqrt{3}$

$y = 2\sin\left(\frac{5\pi}{6}\right) = 2\left(\frac{1}{2}\right) = 1$



$\boxed{(x, y) = (-\sqrt{3}, 1)}$

b) Convert $(3, -3)$ to polar coordinates. → many ways to represent these.



$r = \sqrt{3^2 + (-3)^2} = \sqrt{18} = 3\sqrt{2}$

$\tan\theta = \frac{-3}{3}$ $\quad \theta = \tan^{-1}(-1)$

$\theta = 7\pi/4$

$\boxed{(r, \theta) = (3\sqrt{2}, 7\pi/4) \text{ or } (3\sqrt{2}, -\pi/4)}$

Example 2:

Convert the following equations from rectangular to polar.

a) $y = 2x + 1$ $\quad$ * $y = r\sin\theta$, $x = r\cos\theta$

$r\sin\theta = 2(r\cos\theta) + 1$

$r\sin\theta - 2r\cos\theta = 1$

$r(\sin\theta - 2\cos\theta) = 1$

$r = \frac{1}{\sin\theta - 2\cos\theta}$

*Remove x, y, get in terms of r, θ

b) $y = 4$

$r\sin\theta = 4$

$r = \frac{4}{\sin\theta}$

c) $x^2 + y^2 = 16$

$(r\cos\theta)^2 + (r\sin\theta)^2 = 16$

$r^2\cos^2\theta + r^2\sin^2\theta = 16$

$r^2(\cos^2\theta + \sin^2\theta) = 16$

$r^2(1) = 16$

$\boxed{r = 4}$

*circle: for all θ values, each has radius of 4.

Example 3:

Convert the following equations from polar to rectangular.

a) $r = 2\cos\theta$ $\quad$ *replace trig portion first.

$r = 2\left(\frac{x}{r}\right)$

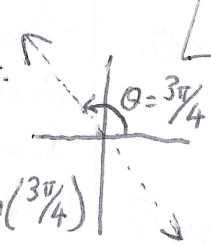
$r^2 = 2x$

$x^2 + y^2 = 2x$

$x^2 - 2x + y^2 = 0$

$(x-1)^2 + y^2 = 1$ or $y^2 = 1 - (x-1)^2$

b) $\theta = \frac{3\pi}{4}$



$\tan\theta = \tan(3\pi/4)$

$\frac{y}{x} = -1$

$\boxed{y = -x}$

c) $r = \csc\theta$

$r = \frac{r}{y}$

$ry = r$

$\boxed{y = 1}$

Slope of a Polar Equation

To find the slope of a tangent line to a polar graph $r = f(\theta)$, we can use $x = r\cos\theta$ and $y = r\sin\theta$ and the product rule:

$$\frac{dy}{dx} = \boxed{\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}} = \frac{r\cos\theta + r'\sin\theta}{-r\sin\theta + r'\cos\theta}, \text{ provided that } \frac{dx}{d\theta} \neq 0$$

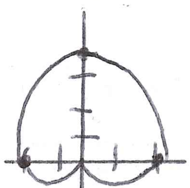
Or . . . . create $\underline{y(\theta)}$ and $\underline{x(\theta)}$ first, then find $\boxed{y'(\theta) \text{ and } x'(\theta).}$

Example 4:

Find $\frac{dy}{dx}$ and the slope of the graph of the polar curve at the given value of θ. $r = 2 + 2\sin\theta$ at $\theta = \frac{\pi}{2}$

* cardioid

* Find $x(\theta)$ and $y(\theta)$ first.



$[0, 2\pi]$

$x = r\cos\theta$

$x = (2 + 2\sin\theta)\cos\theta$

$x(\theta) = 2\cos\theta + 2\sin\theta\cos\theta$

$\underline{x(\theta) = 2\cos\theta + \sin 2\theta}$

$y = r\sin\theta$

$y = (2 + 2\sin\theta)\sin\theta$

$\underline{y(\theta)} = 2\sin\theta + 2\sin^2\theta$

$x'(\theta) = -2\sin\theta + \cos 2\theta(2)$

$y'(\theta) = 2\cos\theta + 4\sin\theta \cdot \cos\theta$

$x'(\pi/2) = -2\sin(\pi/2) + 2\cos(\pi) = -2 - 2 = -4$

$y'(\pi/2) = 2\cos(\pi/2) + 4\sin(\pi/2)\cos(\pi/2) = 0 + 0 = 0$

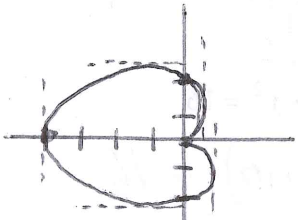
$\frac{dy}{dx} = \frac{y'(\theta)}{x'(\theta)} = \frac{y'(\pi/2)}{x'(\pi/2)} = \frac{0}{-4} = \boxed{0}$

Example 5:

Find the **points**, (r, θ) of horizontal and vertical tangency to the graph of $r = 2 - 2\cos\theta$. Beware of $\frac{0}{0}$.

horizontal tangent: $\frac{dy}{dx} = 0$

vertical tangent: $\frac{dy}{dx} =$ undefined



$x(\theta) = (2 - 2\cos\theta)\cos\theta$

$\underline{x(\theta) = 2\cos\theta - 2\cos^2\theta}$

$y(\theta) = (2 - 2\cos\theta)\sin\theta$

$= 2\sin\theta - 2\sin\theta\cos\theta$

$\underline{y(\theta) = 2\sin\theta - \sin 2\theta}$

$\underline{x'(\theta) = -2\sin\theta + 4\cos\theta\sin\theta}$

$\underline{y'(\theta) = 2\cos\theta - 2\cos 2\theta}$

<u>horizontal tangent:</u>

set $y'(\theta) = 0$

$2\cos\theta - 2\cos 2\theta = 0$

$2\cos\theta - \cancel{2}(2\cos^2\theta - 1) = 0$

$-4\cos^2\theta + 2\cos\theta + 2 = 0$

$4\cos^2\theta - 2\cos\theta - 2 = 0$

$2\cos^2\theta - \cos\theta - 1 = 0$

$(2\cos\theta + 1)(\cos\theta - 1) = 0$

$\cos\theta = -1/2 \quad \cos\theta = 1$

$\theta = \frac{2\pi}{3}, \frac{4\pi}{3} \quad \theta = \cancel{0}, \cancel{2\pi}$

<u>vertical tangent</u>

$-2\sin\theta + 4\cos\theta\sin\theta = 0$

$-2\sin\theta(1 - 2\cos\theta) = 0$

$\sin\theta = 0 \quad | \quad \cos\theta = \frac{1}{2}$

$\theta = \cancel{0}, \pi \quad | \quad \theta = \frac{\pi}{3}, \frac{5\pi}{3}$

* Beware of $\frac{0}{0}$

horiz. tangent pts:

$(\underline{3}, 2\pi/3), (\underline{3}, \frac{4\pi}{3})$

vert. tangent pts:

$(4, \pi), (1, \pi/3)$

$(1, 5\pi/3)$