BC Calculus Ch. 12.2 Notes — Differentiation and Integration

The **derivative of a vector-valued function r** is defined by

$$\mathbf{r}'(t) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}$$

THEOREM 12.1 Differentiation of Vector-Valued Functions

1. If $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$, where f and g are differentiable functions of t, then
$$\mathbf{r}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j}. \quad \text{Plane}$$
2. If $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, where f, g, and h are differentiable functions of t, then
$$\mathbf{r}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}. \quad \text{Space}$$

Finding Intervals on which a curve is smooth

r(t) = f(t)I + g(t)j + h(t)k is smooth on an open interval if f', g', and h' are continuous on I AND r'(t) $\neq$ 0

Example 1: Find the intervals on which the epicycloid C given by

$$\mathbf{r}(t) = (5\cos t - \cos 5t)\mathbf{i} + (5\sin t - \sin 5t)\mathbf{j}, \quad 0 \le t \le 2\pi$$

Properties of the Derivatives: Let **u** and **r** be differentiable vector-valued functions of t, let f be a differentiable real-valued function of t, and let c be a scalar.

1. $D_t[c\mathbf{r}(t)] = c\mathbf{r}'(t)$
2. $D_t[\mathbf{r}(t) \pm \mathbf{u}(t)] = \mathbf{r}'(t) \pm \mathbf{u}'(t)$
3. $D_t[f(t)\mathbf{r}(t)] = f(t)\mathbf{r}'(t) + f'(t)\mathbf{r}(t)$
4. $D_t[\mathbf{r}(t) \cdot \mathbf{u}(t)] = \mathbf{r}(t) \cdot \mathbf{u}'(t) + \mathbf{r}'(t) \cdot \mathbf{u}(t)$
5. $D_t[\mathbf{r}(t) \times \mathbf{u}(t)] = \mathbf{r}(t) \times \mathbf{u}'(t) + \mathbf{r}'(t) \times \mathbf{u}(t)$
6. $D_t[\mathbf{r}(f(t))] = \mathbf{r}'(f(t))f'(t)$
7. If $\mathbf{r}(t) \cdot \mathbf{r}(t) = c$, then $\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$.

Ex. 2: Using Properties of the Derivative

For the vector-valued functions given by

$$\mathbf{r}(t) = \frac{1}{t}\mathbf{i} - \mathbf{j} + \ln t\mathbf{k} \quad \text{and} \quad \mathbf{u}(t) = t^2\mathbf{i} - 2t\mathbf{j} + \mathbf{k}$$

find

a. $D_t[\mathbf{r}(t) \cdot \mathbf{u}(t)]$ and **b.** $D_t[\mathbf{u}(t) \times \mathbf{u}'(t)]$.

II. Integration of Vector-Valued Functions

Definition of Integration of Vector-Valued Functions

1. If $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$, where f and g are continuous on $[a, b]$, indefinite integral (antiderivative) of $\mathbf{r}$ is

$$\int \mathbf{r}(t)\,dt = \left[\int f(t)\,dt\right]\mathbf{i} + \left[\int g(t)\,dt\right]\mathbf{j}$$

and its definite integral over the interval $a \le t \le b$ is

$$\int_a^b \mathbf{r}(t)\,dt = \left[\int_a^b f(t)\,dt\right]\mathbf{i} + \left[\int_a^b g(t)\,dt\right]\mathbf{j}.$$

Example 3:

Find the indefinite integral

$$\int (t\,\mathbf{i} + 3\mathbf{j})\,dt.$$

Example 4: Definite Integral of Vector-Valued Functions

Evaluate the integral

$$\int_0^1 \mathbf{r}(t)\,dt = \int_0^1 \left(\sqrt[3]{t}\,\mathbf{i} + \frac{1}{t+1}\mathbf{j} + e^{-t}\mathbf{k}\right)dt.$$

Example 5: The Antiderivative of Vector -Valued Functions

Find the antiderivative of

$$\mathbf{r}'(t) = \cos 2t\mathbf{i} - 2\sin t\mathbf{j} + \frac{1}{1+t^2}\mathbf{k}$$

that satisfies the initial condition $\mathbf{r}(0) = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$.

Key

BC Calculus Ch. 12.2 Notes Differentiation and Integration

The **derivative of a vector-valued function r** is defined by

$$\mathbf{r}'(t) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}$$

THEOREM 12.1 Differentiation of Vector-Valued Functions

1. If $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$, where f and g are differentiable functions of t, then
$\mathbf{r}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j}$. Plane
2. If $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, where f, g, and h are differentiable functions of t, then
$\mathbf{r}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}$. Space

Finding Intervals on which a curve is smooth

r(t) = f(t)I + g(t)j + h(t)k is smooth on an open interval if f', g', and h' are continuous on I AND r'(t) $\neq$ 0

Example 1: Find the intervals on which the epicycloid C given by

$$\mathbf{r}(t) = (5\cos t - \cos 5t)\mathbf{i} + (5\sin t - \sin 5t)\mathbf{j}, \quad 0 \le t \le 2\pi$$ is a smooth curve

* set $r'(t) = 0$

$r'(t) = (-5\sin t + 5\sin 5t)i + (5\cos t - 5\cos 5t)j$

$0 = (-5\sin t + 5\sin 5t)i + (5\cos t - 5\cos 5t)j$

$t = 0, \pi/2, \frac{3\pi}{2}, 2\pi, \pi$

C is smooth in intervals $(0, \pi/2), (\pi/2, \pi), (\pi, 3\pi/2), (\frac{3\pi}{2}, 2\pi)$

Properties of the Derivatives: Let **u** and **r** be differentiable vector-valued functions of t, let f be a differentiable real-valued function of t, and let c be a scalar.

1. $D_t[c\mathbf{r}(t)] = c\mathbf{r}'(t)$
2. $D_t[\mathbf{r}(t) \pm \mathbf{u}(t)] = \mathbf{r}'(t) \pm \mathbf{u}'(t)$
3. $D_t[f(t)\mathbf{r}(t)] = f(t)\mathbf{r}'(t) + f'(t)\mathbf{r}(t)$
4. $D_t[\mathbf{r}(t) \cdot \mathbf{u}(t)] = \mathbf{r}(t) \cdot \mathbf{u}'(t) + \mathbf{r}'(t) \cdot \mathbf{u}(t)$
5. $D_t[\mathbf{r}(t) \times \mathbf{u}(t)] = \mathbf{r}(t) \times \mathbf{u}'(t) + \mathbf{r}'(t) \times \mathbf{u}(t)$
6. $D_t[\mathbf{r}(f(t))] = \mathbf{r}'(f(t))f'(t)$
7. If $\mathbf{r}(t) \cdot \mathbf{r}(t) = c$, then $\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$.

Ex. 2: Using Properties of the Derivative

For the vector-valued functions given by

$$\mathbf{r}(t) = \frac{1}{t}\mathbf{i} - \mathbf{j} + \ln t\mathbf{k} \quad \text{and} \quad \mathbf{u}(t) = t^2\mathbf{i} - 2t\mathbf{j} + \mathbf{k}$$

find $r'(t) = \frac{-1}{t^2}i - 0 + \frac{1}{t}k$ $\quad u'(t) = 2ti - 2j + 0$

a. $D_t[\mathbf{r}(t) \cdot \mathbf{u}(t)]$ and **b.** $D_t[\mathbf{u}(t) \times \mathbf{u}'(t)]$.

$u''(t) = 2i - 0 + 0$

a) $(\frac{1}{t}i - j + \ln t k) \cdot (2ti - 2j) + (\frac{-1}{t^2}i + \frac{1}{t}k)(t^2 i - 2tj + k)$

$2 + 2 + 0 \quad - 1 + \frac{1}{t} = \boxed{3 + \frac{1}{t}}$

b) $[u(t) \times u''(t)] + [u'(t) \times u'(t)]$

$$\begin{array}{ccc} + & - & + \end{array}$$
$$\begin{vmatrix} i & j & k \\ t^2 & -2t & 1 \\ 2 & 0 & 0 \end{vmatrix} = i\begin{vmatrix} -2t & 1 \\ 0 & 0 \end{vmatrix} - j\begin{vmatrix} t^2 & 1 \\ 2 & 0 \end{vmatrix} + k\begin{vmatrix} t^2 & -2t \\ 2 & 0 \end{vmatrix}$$

$= i(0) - j(0-2) + k(0+4t)$

$= \boxed{2j + 4tk}$

II. Integration of Vector-Valued Functions

Definition of Integration of Vector-Valued Functions

1. If $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$, where f and g are continuous on $[a, b]$, **indefinite integral (antiderivative)** of $\mathbf{r}$ is

$$\int \mathbf{r}(t)\,dt = \left[\int f(t)\,dt\right]\mathbf{i} + \left[\int g(t)\,dt\right]\mathbf{j}$$

and its **definite integral** over the interval $a \le t \le b$ is

$$\int_a^b \mathbf{r}(t)\,dt = \left[\int_a^b f(t)\,dt\right]\mathbf{i} + \left[\int_a^b g(t)\,dt\right]\mathbf{j}.$$

Example 3:

Find the indefinite integral

$$\int (t\,\mathbf{i} + 3\mathbf{j})\,dt.$$

$$\boxed{\frac{t^2}{2}i + 3tj + C}$$

Example 4: Definite Integral of Vector-Valued Functions

Evaluate the integral

$$\int_0^1 \mathbf{r}(t)\,dt = \int_0^1 \left(\sqrt[3]{t}\,\mathbf{i} + \frac{1}{t+1}\mathbf{j} + e^{-t}\mathbf{k}\right) dt.$$

$$= \frac{3}{4}t^{4/3}\Big]_0^1 i + \ln|t+1|\Big]_0^1 j + -e^{-t}\Big]_0^1 k$$

$$= \frac{3}{4}i + (\ln 2)j + (-e^{-1} - (-e^0))k$$

$$\boxed{\frac{3}{4}i + (\ln 2)j + \left(1 - \frac{1}{e}\right)k}$$

Example 5: The Antiderivative of Vector-Valued Functions

Find the antiderivative of

$$\int \mathbf{r}'(t) = \int \cos 2t\mathbf{i} - 2\sin t\mathbf{j} + \frac{1}{1+t^2}\mathbf{k}$$

that satisfies the initial condition $\mathbf{r}(0) = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$.

$$\int \frac{1}{a^2+u^2} = \arctan\left(\frac{u}{a}\right) + C$$

$$\left(\frac{1}{2}\sin 2t + C_1\right)i + (2\cos t + C_2)j + (\arctan(t) + C_3)k$$

$$r(0) = \left(\frac{1}{2}(0) + C_1\right)i + (2(1) + C_2)j + (0 + C_3)k$$

$$r(0) = 3i - 2j + k$$

$$C_1 = 3, \quad C_2 = -4 \quad C_3 = 1$$

$$\boxed{r(t) = \left(\frac{1}{2}\sin 2t + 3\right)i + (2\cos t - 4)j + (\arctan t + 1)k}$$