BC Calculus Notes Euler's Method

For differential equations that cannot be solved, a slope field can be a graphical solution to that differential equation. The problem with this approach is that a slope field is only really good for getting general ***trends*** in solutions and for long-term behavior of solutions. There are times when we will need something more. Maybe we need to determine how a specific solution behaves, including some values that the solution will take.

In these cases we must resort to numerical methods that will allow us to approximate solutions to differential equations. There are many different methods that can be used to approximate solutions to a differential equation. We have already looked at tangent line approximations, which are not bad approximations of a solution near the point of tangency, but that idea can be used repeatedly to get increasingly better approximations. We are going to look at one of the oldest and easiest to use here. This method was originally devised by Euler and is called, oddly enough, Euler's Method.

Euler's method basically involves "***walking out along a tightrope***" from an initial point along it's tangent line. Instead of walking along the same line the whole time (as in a tangent line approximation), we change tangent lines with each step (of length Δx). This involves recalculating the point and slope after each step. This will produce a much more accurate approximation that simply using the original tangent line. The process it self is pretty easy and repetitive, and it is easier demonstrated with an example rather than a complicated formula.

Here's the "machinery" you will need to make it work:

- We first must designate the number of equal steps we would like to take. Call this number n.
- Next, if $x = a$ is our initial x-value, and $x = b$ is our desired x-value to find our approximation at, we calculate Δx the conventional way:

$$\Delta x = \frac{b-a}{n}$$

- Recall slope: $m = \frac{\Delta y}{\Delta x}$, solving for Δy, we get $\Delta y = m(\Delta x)$
- We can now proceed. The following chart will make things easier.

(note: the first x and y used are the initial condition. $\frac{dy}{dx}$ will be given)

MEMORIZE THIS CHART. MEMORIZE THIS CHART.

x	y	$m = \left.\frac{dy}{dx}\right\|_{(x,y)}$	$\Delta y = m(\Delta x)$	$y_{new} = y + \Delta y$
a	$y(a)$			

Example 1:

Given the differential equation $\frac{dy}{dx} = x - 2$ and $y(0) = 5$.

a) Find an approximation for $y(0.8)$ by using Euler's method with two equal steps. Sketch your solution.

b) Solve the differential equation $\frac{dy}{dx} = x - 2$ with the initial condition $y(0) = 5$, and use your solution to find $y(0.8)$.

Example 2:

If $\frac{dy}{dx} = 2x - y$ and if $y = 3$ when $x = 2$, use Euler's method with five equal steps to approximate y when $x = 1.5$.

Example 3:

Assume that f and f' have the values given in the table. Use Euler's method with two equal steps to approximate the value of $f(2.6)$.

x	3	2.8	2.6
$f'(x)$	0.4	0.7	0.9
$f(x)$	2		

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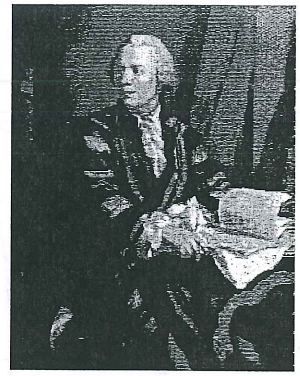
Key



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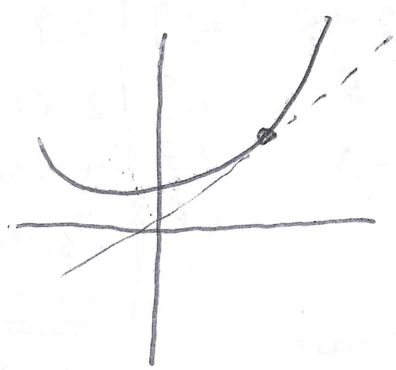

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MEMORIZE THIS CHART. MEMORIZE THIS CHART.

x	y	$m = \left.\frac{dy}{dx}\right\rvert_{(x,y)}$	$\Delta y = m(\Delta x)$	$y_{new} = y + \Delta y$
a	$y(a)$	⟶		

Example 1:

$\Delta x = \frac{b-a}{n} = \frac{0.8-0}{2} = 0.4$

Given the differential equation $\frac{dy}{dx} = x - 2$ and $y(0) = 5$.

a) Find an approximation for $y(0.8)$ by using Euler's method with two equal steps. Sketch your solution.

b) Solve the differential equation $\frac{dy}{dx} = x - 2$ with the initial condition $y(0) = 5$, and use your solution to find $y(0.8)$.

$m = \frac{\Delta y}{\Delta y}$

$\Delta x = 0.4$

x	y	$m = \frac{dy}{dx}$	$\Delta y = m\Delta x$	$y_{new} = y + \Delta y$
0	5	-2	$\Delta y = -2(0.4) = -0.8$	$y_{new} = 5 - 0.8 = 4.2$
0.4	4.2	-1.6	-0.64	$4.2 - 0.64 = 3.56$
0.8	3.56			

So $y(0.8) \approx 3.56$

Example 2:

If $\frac{dy}{dx} = 2x - y$ and if $y = 3$ when $x = 2$, use Euler's method with five equal steps to approximate y when $x = 1.5$.

$\Delta x = \frac{b-a}{n}$

$\Delta x = \frac{2 - 1.5}{5}$

$= -0.1$

x	y	$m = \frac{dy}{dx}$	Δy	y_{new}
2	3	1	-0.1	2.9
+1.9	2.9	0.9	-0.09	2.81
1.8	2.81	0.79	-0.079	2.731
1.7	2.731	0.669	-0.0669	2.6641
1.6	2.6241	-0.5359	-0.05354	2.61051
1.5	2.61051			

So $y(1.5) \approx 2.61051$

Example 3:

Assume that f and f' have the values given in the table. Use Euler's method with two equal steps to approximate the value of $f(2.6)$.

$\Delta x = -0.2$

x	3	2.8	2.6
$f'(x)$	0.4	0.7	0.9
$f(x)$	2		

x	y	$\frac{dy}{dx}$	Δy	y_{new}
3	2	0.4	-0.08	$y = 2 - 0.08 = 1.92$
2.8	1.92	0.7	-0.14	1.78
2.6	1.78			

Therefore, $f(2.6) \approx 1.78$