BC Calculus Ch. 6.3 Notes Partial Fractions and Logistic Differential Equation

Partial Fraction Decomposition: Cover up Method for Non – repeating Linear Factors in the denominator

Steps:

1) Factor Denominator
2) Solve for x for each factor
3) Cover up a factor in the denominator of the problem
4) Plug in the x-value from #2 and solve for x.
5) Pair the value found with the factor from the cover up. $\left(\frac{value}{factor}\right)$
6) Repeat steps 2 – 3 for the other factors.

Example:

Evaluate $\int \frac{x+5}{x^2+x-2}dx$

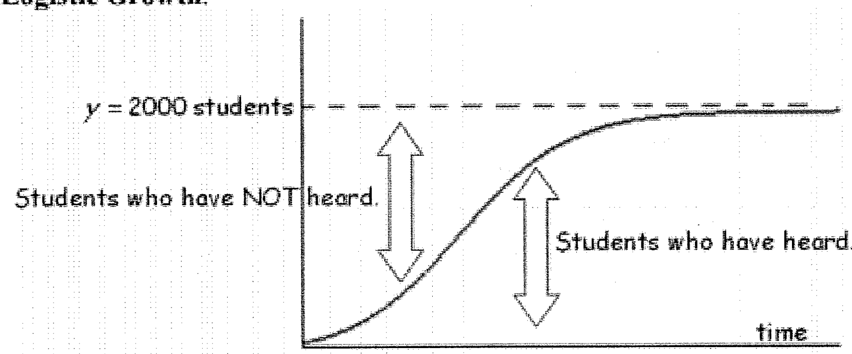
*In order for partial fraction decomposition to work, the degree of the denominator must be greater than that of the numerator. When it is not, we look to test out long division method first.

Logistic Growth: Population that grow exponentially cannot continue to do so indefinitely. After a while, the population begin to compete for resources, such as food and water. The growth begins to taper off as it approaches some carrying capacity of the system. The **Carrying Capacity** is the maximum number of individuals of a given species that an area's resources can sustain indefinitely without significantly depleting or degrading those resources

In this case, the growth rate is not only proportional to the current value, but also how far the current value is from the carrying capacity.

Imagine a rumor spreading throughout a school of 2000 students. The rate at which the rumor spreads is directly proportional to BOTH the students who have heard the rumor AND the students who have yet to hear the rumor as the number of people hearing the rumor approaches 2000.

The curve for the spread of the rumor might look like something shown below. This type of curve and growth is called **Logistic Growth**.



For quantities, y, that grow logistically with a carrying capacity of $y = L$, we can state the relation mathematically the following way:

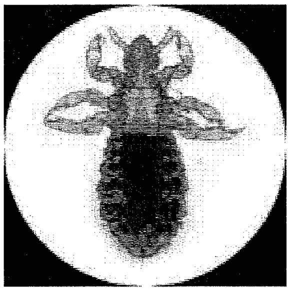
$$\frac{dy}{dt} = ky(L-y)$$

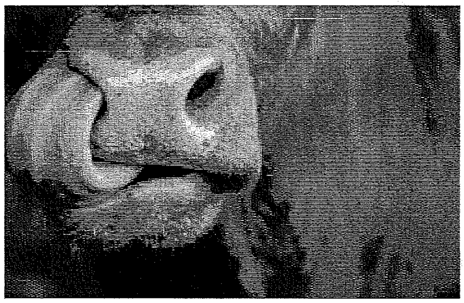
Memorize the solution:

Logistic Growth

$$\text{If } \frac{dy}{dt} = ky(L-y), \text{ then } y = \frac{L}{1+Ce^{-Lkt}}$$

Think "Lice Minus Licked." It's gross, but it works.





******Notice the prominence of our carrying capacity value L in each form of the equation. This is very important, especially when asked for the* ***limit at infinity*** *OR when asked to find the y-value when the y-values are* ***increasing most rapidly*** *(i.e. the inflection value: $y = \frac{L}{2}$).*

Example 2:

(Calculator) The population of Alaska since from 1900 to 2000 can be modeled by the following logistic equation.

$$P(t) = \frac{895598}{1+71.57e^{-0.0516t}}$$

where P is the population and t years after 1900, with $t = 0$ corresponding to 1900.

a) What is the predicted population of Alaska in 2020?
b) How fast was the population of Alaska changing in 1920? In 1940? In 1999?
c) When was Alaska growing the fastest, and what was the population then?
d) What information does the equation tell us about the population of Alaska in the long run?

The AP exam loves to ask questions that require you to recognize the parameters of logistic growth for either the equation or the differential equation written in a DIFFERERNT FORMAT. This requires you to manipulate the equation to fit one of the two standard forms below:

$$\frac{dy}{dt} = ky(L-y) \rightarrow y = \frac{L}{1+Ce^{-Lkt}}$$

Example 3:

The growth rate of a population P of bears in a newly established wildlife preserve is modeled by the differential equation $\frac{dP}{dt} = 0.008P(100-P)$, where t measured in years.

a) What is the carrying capacity for bears in this wildlife preserve?
b) What is the bear population when the population is growing the fastest?
c) What is the rate of change of population when it is growing the fastest?

$$\frac{dy}{dt} = ky(L-y) \rightarrow y = \frac{L}{1+Ce^{-Lkt}}$$

Example 4:

Suppose that a population develops according to the logistic differential equation $\frac{dP}{dt} = 0.2P - 0.002P^2$, where t is measured in weeks, $t \geq 0$.

a) If $P(0) = 5$, what is $\lim_{t\to\infty} P(t)$?
b) If $P(0) = 60$, what is $\lim_{t\to\infty} P(t)$?
c) If $P(0) = 120$, what is $\lim_{t\to\infty} P(t)$?
d) Sketch the solution curves for a), b), and c). Which one has an inflection point?

Example 5:

The rate at which the flu spreads through a community is modeled by the logistic differential equation $\frac{dP}{dt} = 0.001P(3000 - P)$, where t is measured in days, $t \geq 0$.

a) If $P(0) = 50$, solve for P as a function of t.
b) Use your solution to a) to find the size of the population when $t = 2$ days.
c) Use your solution to a) to find the number of days that have occurred when the flu is spreading the fastest.

Key

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Steps:

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5) Pair the value found with the factor from the cover up. $\left(\frac{value}{factor}\right)$
6) Repeat steps 2 – 3 for the other factors.

Example:

Evaluate $\int \frac{x+5}{x^2+x-2}dx$

$\int \frac{x+5}{(x+2)(x-1)}dx$

$\int \frac{\quad}{x+2} + \frac{\quad}{x-1}dx$

$x+2=0,\ x=-2,\quad x-1=0,\ x=1$

$\int \frac{-1}{(x+2)} + \frac{2}{(x-1)}dx$

$\boxed{-\ln|x+2| + 2\ln|x-1| + C}$

condense

$\ln|x-1|^2 - \ln|x+2| + C$

$\boxed{\ln\left|\frac{(x-1)^2}{x+2}\right| + C}$

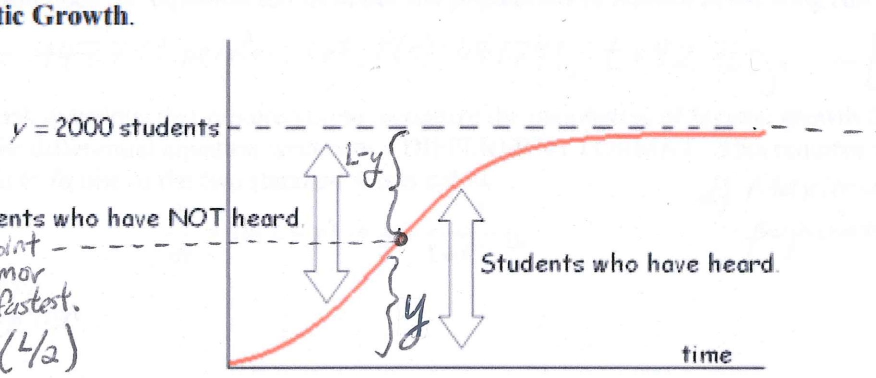
*In order for partial fraction decomposition to work, the degree of the denominator must be greater than that of the numerator. When it is not, we look to test out long division method first.

Logistic Growth: Population that grow exponentially cannot continue to do so indefinitely. After a while, the population begin to compete for resources, such as food and water. The growth begins to taper off as it approaches some carrying capacity of the system. The **Carrying Capacity** is the maximum number of individuals of a given species that an area's resources can sustain indefinitely without significantly depleting or degrading those resources

In this case, the growth rate is not only proportional to the current value, but also how far the current value is from the carrying capacity.

Imagine a rumor spreading throughout a school of 2000 students. The rate at which the rumor spreads is directly proportional to BOTH the students who have heard the rumor AND the students who have yet to hear the rumor as the number of people hearing the rumor approaches 2000.

The curve for the spread of the rumor might look like something shown below. This type of curve and growth is called **Logistic Growth**.



Limits of growth

inflection point rate when rumor is spreading fastest. (L/2)

For quantities, y, that grow logistically with a carrying capacity of $y = L$, we can state the relation mathematically the following way:

$$\frac{dy}{dt} = ky(L-y)$$

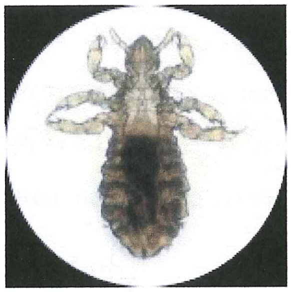
*rumor is spreading at a rate that is jointly proportional to the # of students who have heard and the # of students who haven't heard it.

Memorize the solution:

Logistic Growth

$$\text{If } \frac{dy}{dt} = ky(L-y), \text{ then } y = \frac{L}{1+Ce^{-Lkt}}$$

Think "Lice Minus Licked." It's gross, but it works.



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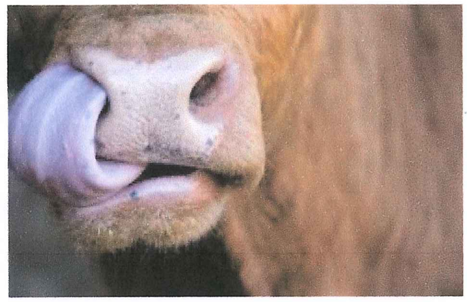


******Notice the prominence of our carrying capacity value L in each form of the equation. This is very important, especially when asked for the **limit at infinity** OR when asked to find the y-value when the y-values are **increasing most rapidly** (i.e. the inflection value: $y = \frac{L}{2}$).*

b) $P'(20) = 1678.078$ people/yr.
$P'(40) = 4127.794$ people/yr.
$P'(99) = 9741.748$ people/yr.

Example 2:

(Calculator) The population of Alaska since from 1900 to 2000 can be modeled by the following logistic equation.

$$P(t) = \frac{895598}{1+71.57e^{-0.0516t}}$$

where P is the population and t years after 1900, with $t = 0$ corresponding to 1900.

a) What is the predicted population of Alaska in 2020? $(t=120) \rightarrow P(120) = 781217.6$ people
b) How fast was the population of Alaska changing in 1920? In 1940? In 1999? $P'(20)$, $P'(40)$, $P'(99)$
c) When was Alaska growing the fastest, and what was the population then?
d) What information does the equation tell us about the population of Alaska in the long run?

c) $\frac{L}{2} = \frac{895598}{2} = 447799$ people, set $P(t) = 447799$, $t = 82.765$ yrs $\sim \boxed{1982}$

The AP exam loves to ask questions that require you to recognize the parameters of logistic growth for either the equation or the differential equation written in a DIFFERERNT FORMAT. This requires you to manipulate the equation to fit one of the two standard forms below:

$$\frac{dy}{dt} = ky(L-y) \rightarrow y = \frac{L}{1+Ce^{-Lkt}}$$

d) Maximum sustainable population is 895598 people

*C is not always initial population..

Example 3:

The growth rate of a population P of bears in a newly established wildlife preserve is modeled by the differential equation $\frac{dP}{dt} = \overset{k}{0.008}\overset{y}{P}(\overset{L-y}{100-P})$, where t measured in years.

a) What is the carrying capacity for bears in this wildlife preserve? $L = 100$
b) What is the bear population when the population is growing the fastest? $L/2 = \frac{100}{2} = 50$
c) What is the rate of change of population when it is growing the fastest?

$$\left.\frac{dP}{dt}\right|_{P=50} = 0.008(50)(100-50) = \boxed{20 \text{ bears/year}}$$

$$\frac{dy}{dt} = ky(L-y) \rightarrow y = \frac{L}{1+Ce^{-Lkt}}$$

$$\frac{dP}{dt} = 0.002P(100-P)$$

K L

Example 4:

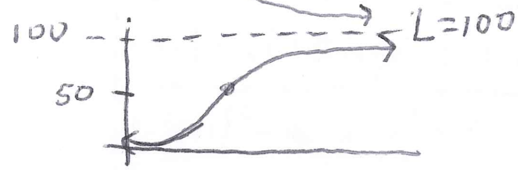
Suppose that a population develops according to the logistic differential equation $\frac{dP}{dt} = 0.2P - 0.002P^2$, where t is measured in weeks, $t \ge 0$.

a) If $P(0) = 5$, what is $\lim_{t\to\infty} P(t)$? 100

b) If $P(0) = 60$, what is $\lim_{t\to\infty} P(t)$? 100

c) If $P(0) = 120$, what is $\lim_{t\to\infty} P(t)$? 100

d) Sketch the solution curves for a), b), and c). Which one has an inflection point?



A only

Example 5: K L

The rate at which the flu spreads through a community is modeled by the logistic differential equation $\frac{dP}{dt} = 0.001P(3000-P)$, where t is measured in days, $t \ge 0$.

a) If $P(0) = 50$, solve for P as a function of t.

b) Use your solution to a) to find the size of the population when $t = 2$ days.

c) Use your solution to a) to find the number of days that have occurred when the flu is spreading the fastest.

a) $P(t) = \frac{L}{1+Ce^{-Lkt}} = \frac{3000}{1+Ce^{-0.001(3000)t}} = \frac{3000}{1+Ce^{-3t}}$

$P(0) = \frac{3000}{1+Ce^{-3(0)}}$

$50 = \frac{3000}{1+Ce^{0}}$

$50 = \frac{3000}{1+C}$

$50 + 50C = 3000$

$50C = 2950$

$\boxed{C = 59}$

$\boxed{P(t) = \frac{3000}{1+59e^{-3t}}}$

b) $P(2) = \frac{3000}{1+59e^{-3(2)}} = 2617.238$ people

c) flu spread fastest at $L/2 = \frac{3000}{2} = 1500$

set $1500 = \frac{3000}{1+59e^{-3t}}$

$1+59e^{-3t} = \frac{3000}{1500}$

$1+59e^{-3t} = 2$

$59e^{-3t} = 1$

$e^{-3t} = 1/59$

$\ln e^{-3t} = \ln(1/59)$

$-3t = \ln(1/59)$

$t = -\frac{1}{3}\ln(1/59)$

$\boxed{t = 1.359 \text{ days}}$