9.6 Convergence Test Review WS #2 (Stewart's Calculus pg. 784) Determine if the following series converge or diverge. Name the test to justify convergence/divergence. Find the sum of the series if possible.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2-1}{n^2+n}$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{n^2+n}$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+n}$

4. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n-1}{n^2+n}$

5. $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^{n+1}}{2^{3n}}$

6. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3n}{1+8n}\right)^n$

7. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln(n)}}$

8. $\displaystyle\sum_{k=1}^{\infty} \frac{2^k k!}{(k+2)!}$

9. $\displaystyle\sum_{k=1}^{\infty} k^2 e^{-k}$

10. $\sum_{n=1}^{\infty} n^2 e^{-n^3}$

11. $\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n \ln(n)}$

12. $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2 + 25}$

13. $\sum_{n=1}^{\infty} \frac{3^n n^2}{n!}$

14. $\sum_{n=1}^{\infty} \sin(n)$

15. $\sum_{n=0}^{\infty} \frac{n!}{2 \cdot 5 \cdot 8 \cdot \cdots \cdot (3n+2)}$

16. $\sum_{n=1}^{\infty} \frac{n^2 + 1}{n^3 + 1}$

17. $\sum_{n=1}^{\infty} (-1)^n 2^{1/n}$

18. $\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n} - 1}$

19. $\sum_{n=1}^{\infty} (-1)^n \frac{\ln(n)}{\sqrt{n}}$

20. $\displaystyle\sum_{k=1}^{\infty} \frac{k+5}{5^k}$

21. $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^{2n}}{n^n}$

22. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n^2-1}}{n^3+2n^2+5}$

23. $\displaystyle\sum_{n=1}^{\infty} \tan(1/n)$

24. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n/2)}{n^2+4n}$

25. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{e^{n^2}}$

26. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+1}{5^n}$

27. $\displaystyle\sum_{k=1}^{\infty} \frac{k\ln(k)}{(k+1)^3}$

28. $\displaystyle\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}$

29. $\displaystyle\sum_{n=1}^{\infty} \frac{\tan^{-1}(n)}{n\sqrt{n}}$

30. $\displaystyle\sum_{j=1}^{\infty}(-1)^j \frac{\sqrt{j}}{j+5}$

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31. $\displaystyle\sum_{k=1}^{\infty} \frac{5^k}{3^k + 4^k}$

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32. $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)^n}{n^{2n}}$

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33. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(1/n)}{\sqrt{n}}$

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34. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n + n\cos^2(n)}$

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35. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$

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36. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(\ln(n))^{\ln(n)}}$

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37. $\displaystyle\sum_{n=1}^{\infty} (\sqrt[n]{2} - 1)^n$

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38. $\displaystyle\sum_{n=1}^{\infty} (\sqrt[n]{2} - 1)$

Solution Key

9.6 Convergence Test Review WS #2 (Stewart's Calculus pg. 784) Determine if the following series ~~converge~~ converge or diverge. Name the test to justify convergence/divergence. Find the sum of the series if possible.

Note: More than 1 Test could be used to justify convergence for below problems

1. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2-1}{n^2+n}$

$\lim_{n\to\infty} \frac{n^2-1}{n^2+n} = 1 \neq 0$ series diverge by n^{th} term test.

2. $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{n^2+n}$

n^{th} term test inconclusive

Compare with $\frac{1}{n}$ (diverges by p-series)

Limit Comparison Test

$$\lim_{n\to\infty}\left|\frac{\frac{n-1}{n^2+n}}{\frac{1}{n}}\right| = \lim_{n\to\infty}\left|\frac{n-1}{n^2+n}\cdot\frac{n}{1}\right| = \lim_{n\to\infty}\left|\frac{n^2-n}{n^2+n}\right| = 1$$

By LCT, series diverge.

3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+n}$

n^{th} term test inconclusive

compare with $\frac{1}{n^2}$ (converge by p-series $p=2$)

since $\frac{1}{n^2} > \frac{1}{n^2+n}$, By DCT, series converge.

4. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1}\frac{n-1}{n^2+n}$

Since $a_n = \frac{n-1}{n^2+n}$, $\lim_{n\to\infty} \frac{n-1}{n^2+n} = 0$, and $a_{n+1} \le a_n$ for all n, series converge by Alt. Series Test (conditional convergence)

5. $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^{n+1}}{2^{3n}}$

$$= \frac{(-1\cdot 3)^{n+1}}{8^n} = \sum_{n=1}^{\infty}(-1)^{n+1}\cdot 3\left(\frac{3}{8}\right)^n$$

since $|r| = \frac{3}{8} < 1$, series converge by GST. $S = \frac{9/8}{1+3/8} = \frac{\frac{9}{8}}{11/8} = \boxed{\frac{9}{11}}$

6. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3n}{1+8n}\right)^n$

n^{th} term test inconclusive

$$\lim_{n\to\infty} \sqrt[n]{\left(\frac{3n}{1+8n}\right)^n} = \lim_{n\to\infty}\left|\frac{3n}{1+8n}\right| = \frac{3}{8} < 1$$

converges by Root Test.

7. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln(n)}}$

n^{th} term test inconclusive $f(x)$ is pos, dec, continuous

$$\int \frac{1}{x(\ln x)^{1/2}}dx \quad u=\ln x,\ \frac{du}{dx}=\frac{1}{x},\ dx = x\,du \quad \int \frac{1}{x\cdot u^{1/2}}\cdot x\,du$$

$$\int u^{-1/2}du = \frac{u^{1/2}}{1/2} \quad \lim_{b\to\infty} \left[2\sqrt{\ln x}\right]_2^b \quad \lim_{b\to\infty} 2\sqrt{\ln b} - 2\sqrt{\ln 2} = \infty$$

Series Diverges by Integral Test

8. $\displaystyle\sum_{k=1}^{\infty} \frac{2^k k!}{(k+2)!}$

n^{th} term test show divergence

$$\lim_{k\to\infty}\left|\frac{2^{k+1}(k+1)!}{(k+3)!}\cdot\frac{(k+2)!}{2^k\cdot k!}\right| = \lim_{k\to\infty}\left|\frac{2\cdot(k+1)}{k+3}\right| = 2 > 1$$

Diverges by Ratio Test.

9. $\displaystyle\sum_{k=1}^{\infty} k^2 e^{-k}$

n^{th} term test inconclusive $\displaystyle\sum_{k=1}^{\infty}\frac{k^2}{e^k}$

$$\lim_{k\to\infty}\left|\frac{(k+1)^2}{e^{k+1}}\cdot\frac{e^k}{k^2}\right| = \lim_{k\to\infty}\left|\frac{k^2+2k+1}{e\cdot k^2}\right| = \frac{1}{e} < 1$$

converges by Ratio Test.

10. $\sum_{n=1}^{\infty} n^2 e^{-n^3}$ $= \int \frac{x^2}{e^{x^3}} dx$ $u = x^3$ $\frac{du}{dx} = 3x^2$ $dx = \frac{du}{3x^2}$ $\int \frac{x^2}{e^u} \frac{du}{3x^2} = \frac{1}{3}\int e^{-u} du = -\frac{1}{3}\left(\frac{1}{e^{x^3}}\right)$ converges by Integral Test.

$\lim_{b \to \infty} -\frac{1}{3e^{x^3}}\Big]_1^b = \frac{-1}{3e^{b^3}} + \frac{1}{3e} = \frac{1}{3e}$

n^{th} term test inconclusive $f(x)$ is pos, dec, continuous

11. $\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n \ln(n)}$ Since $\lim_{n \to \infty} \frac{1}{n \ln n} = 0$ and $a_{n+1} < a_n$, series converge by Alternating Series Test.

12. $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2 + 25}$ Since $\lim_{n \to \infty} \frac{n}{n^2+25} = 0$ and $a_{n+1} \geq a_n$ for all $n \geq 5$, series converge by Alt. Series Test.

13. $\sum_{n=1}^{\infty} \frac{3^n n^2}{n!}$ $\lim_{n \to \infty} \left| \frac{3^{n+1}(n+1)^2}{(n+1)!} \cdot \frac{n!}{3^n n^2} \right| = \lim_{n \to \infty} \left| \frac{3(n^2+2n+1)}{n^2(n+1)} \right| = 0 < 1$ By Ratio Test, series converge.

n^{th} term test inconclusive

14. $\sum_{n=1}^{\infty} \sin(n)$ Diverges by n^{th} term test

$\lim_{n \to \infty} \sin(n) \neq 0.$

15. $\sum_{n=0}^{\infty} \frac{n!}{2 \cdot 5 \cdot 8 \cdot \dots \cdot (3n+2)}$ $\lim_{n \to \infty} \left| \frac{(n+1)!}{2 \cdot 5 \cdot 8 \cdots 3(n+1)+2} \cdot \frac{2 \cdot 5 \cdot 8 \cdots 3n+2}{n!} \right| = \lim_{n \to \infty} \left| \frac{n+1}{3n+5} \right| = \frac{1}{3} < 1$

Series converge by Ratio Test.

16. $\sum_{n=1}^{\infty} \frac{n^2 + 1}{n^3 + 1}$ $\lim_{n \to \infty} \left| \frac{\frac{n^2+1}{n^3+1}}{\frac{1}{n}} \right| = \lim_{n \to \infty} \left| \frac{n^2+1}{n^3+1} \cdot n \right| = 1$ By Limit Comparison Test, since $\frac{1}{n}$ diverges, series also diverge.

17. $\sum_{n=1}^{\infty} (-1)^n 2^{1/n}$ let $a_n = 2^{\frac{1}{n}}$

since $\lim_{n \to \infty} 2^{\frac{1}{n}} \neq 0$, series diverge by n^{th} term test

18. $\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n} - 1}$ $\lim_{n \to \infty} \frac{1}{\sqrt{n}-1} = 0$ and $\frac{1}{\sqrt{n}-1}$ decreases for all $n \geq 2$, series converge by Alt. Series Test.

19. $\sum_{n=1}^{\infty} (-1)^n \frac{\ln(n)}{\sqrt{n}}$ $\lim_{n \to \infty} \frac{\ln(n)}{\sqrt{n}} = 0$ and $\frac{\ln(n)}{\sqrt{n}}$ decrease for all n.

series converge by Alt. Series Test.

20. $\sum_{k=1}^{\infty} \frac{k+5}{5^k}$ $\lim_{k\to\infty}\left|\frac{k+6}{5^{k+1}}\cdot\frac{5^k}{k+5}\right| = \lim_{k\to\infty}\left|\frac{k+6}{5(k+5)}\right| = \frac{1}{5} < 1$ Series converge by Ratio Test.

21. $\sum_{n=1}^{\infty} \frac{(-2)^{2n}}{n^n}$ $\sqrt[n]{\left(\frac{2}{n}\right)^{2n}}$ $\lim_{n\to\infty}\left|\frac{2^2}{n}\right| = 0 < 1$. By Root test, series converge.

22. $\sum_{n=1}^{\infty} \frac{\sqrt{n^2-1}}{n^3+2n^2+5}$ $\lim_{n\to\infty}\left|\frac{\sqrt{n^2-1}}{n^3+2n^2+5}\cdot\frac{n^2}{1}\right| = \lim_{n\to\infty}\left|\frac{n^2\sqrt{n^2-1}}{n^3+2n^2+5}\right| = 1$ By Limit Comparison Test, since $\frac{1}{n^2}$ converge, series converge as well.

23. $\sum_{n=1}^{\infty} \tan(1/n)$ $\lim_{n\to\infty}\left|\frac{\tan\left(\frac{1}{n}\right)}{\frac{1}{n}}\right|$ $\lim_{t\to 0}\frac{\tan t}{t} = \lim_{t\to 0}\left|\frac{\sin t}{t}\cdot\frac{1}{\cos t}\right| = 1$ By Limit Comparison Test, since $\frac{1}{n}$ diverge, series diverge as well.

compare with $\frac{1}{n}$

24. $\sum_{n=1}^{\infty} \frac{\cos(n/2)}{n^2+4n}$ Compare with $\frac{1}{n^2}$ (converges by p-series)

Since $\frac{1}{n^2} \geq \frac{1}{n^2+4n}$ and $\left|\cos\left(\frac{n}{2}\right)\right| \leq 1$, series converge by Direct Comparison Test.

n^{th} term test inconclusive

25. $\sum_{n=1}^{\infty} \frac{n!}{e^{n^2}}$ $\lim_{n\to\infty}\left|\frac{(n+1)!}{e^{(n+1)^2}}\cdot\frac{e^{n^2}}{n!}\right| = \lim_{n\to\infty}\left|\frac{(n+1)\cdot \cancel{e^{n^2}}}{\cancel{e^{n^2}}^{+2n+1}}\right| = \lim_{n\to\infty}\left|\frac{(n+1)}{e^{2n+1}}\right| = 0 < 1$ Series converge by Ratio Test.

26. $\sum_{n=1}^{\infty} \frac{n^2+1}{5^n}$ $\lim_{n\to\infty}\left|\frac{(n+1)^2+1}{5^{n+1}}\cdot\frac{5^n}{n^2+1}\right| = \lim_{n\to\infty}\left|\frac{n^2+2n+2}{5(n^2+1)}\right| = \frac{1}{5} < 1$ Series converge by Ratio Test

27. $\sum_{k=1}^{\infty} \frac{k\ln(k)}{(k+1)^3}$ Compare with $\frac{\sqrt{k}}{k^2} = \frac{1}{k^{3/2}}$ Since $\frac{1}{k^{3/2}}$ converge by p-series,

$\lim_{k\to\infty}\left|\frac{k\ln(k)}{(k+1)^3}\cdot\frac{k^{3/2}}{1}\right| < 1$, series converge by Limit Comparison Test.

28. $\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}$ $\int\frac{e^{\frac{1}{x}}}{x^2}dx$ $u = x^{-1}$ $\frac{du}{dx} = \frac{-1}{x^2} = -\int e^u du$ $\lim_{b\to\infty} e^{\frac{1}{x}}\Big]_1^b = e^{\frac{1}{b}} - e^1 = 1-e$, series converge by Integral Test.

f(x) is pos; continuous, decreasing

29. $\sum_{n=1}^{\infty} \frac{\tan^{-1}(n)}{n\sqrt{n}}$ Since $0 < \frac{\arctan(n)}{n^{3/2}} < \frac{\pi/2}{n^{3/2}}$, By Direct Comparison Test, since $\frac{\pi/2}{n^{3/2}}$ converges, series also converge.

Since $\lim_{j\to\infty} \frac{\sqrt{j}}{j+5} = 0$, and $a_{j+1} \le a_j$ for all j,

30. $\sum_{j=1}^{\infty} (-1)^j \frac{\sqrt{j}}{j+5}$

series converge by Alt. Series Test.

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31. $\sum_{k=1}^{\infty} \frac{5^k}{3^k + 4^k} > \frac{5^k}{4^k+4^k} = \frac{5^k}{2 \cdot 4^k} = \frac{1}{2}\left(\frac{5}{4}\right)^k$, Since $\frac{1}{2}\left(\frac{5}{4}\right)^k$ diverges by Geometric series test, by Direct Comparison Test, series diverges.

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32. $\sum_{n=1}^{\infty} \frac{(2n)^n}{n^{2n}}$ $\lim_{n\to\infty} \sqrt[n]{\left(\frac{2n}{n^2}\right)^n} = \lim_{n\to\infty} \left|\frac{2n}{n^2}\right| = 0 < 1$, series converge by Root Test.

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33. $\sum_{n=1}^{\infty} \frac{\sin(1/n)}{\sqrt{n}}$ Compare with $\frac{1}{n\sqrt{n}} = \frac{1}{n^{3/2}}$

$\sin\left(\frac{1}{n}\right) \approx \frac{1}{n}$

$\lim_{n\to\infty} \left| \frac{\sin\left(\frac{1}{n}\right)}{\sqrt{n}} \cdot \frac{n\sqrt{n}}{1} \right| = \lim_{n\to\infty} \left(n \cdot \sin\left(\frac{1}{n}\right) \right)$

$= \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = \lim_{t\to 0} \frac{\sin t}{t} = 1$

Since $\frac{1}{n^{3/2}}$ converge by p-series, then by Limit Comparison test, series converge.

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34. $\sum_{n=1}^{\infty} \frac{1}{n + n\cos^2(n)}$

Since $\cos^2(n) < 1$, and $n + n\cos^2(n) < n + n$

then $\frac{1}{n+n\cos^2(n)} \ge \frac{1}{2n}$

By Direct Comparison Test, since $\frac{1}{2n}$ diverges, then series diverge as well.

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35. $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$

$\lim_{n\to\infty} \left| \sqrt[n]{\left(\frac{n}{n+1}\right)^{n^2}} \right| = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty} \frac{1}{\left(\frac{n+1}{n}\right)^n}$

$= \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \lim_{n\to\infty} \left(\frac{1}{e}\right) < 1$

By Root test, series converge.

OMIT ~~36.~~ $\sum_{n=2}^{\infty} \frac{1}{(\ln(n))^{\ln(n)}}$

converges by Direct comparison Test, since $\frac{1}{n^2} > \frac{1}{(\ln n)^{\ln n}}$

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37. $\sum_{n=1}^{\infty} (\sqrt[n]{2} - 1)^n$ $\lim_{n\to\infty} \left| \sqrt[n]{\left(\sqrt[n]{2}-1\right)^n} \right|$

$= \lim_{n\to\infty} \sqrt[n]{2} - 1 = \lim_{n\to\infty} 2^{\frac{1}{n}} - 1 = 1 - 1 = 0 < 1$

so series converge by Root Test.

OMIT ~~38.~~ $\sum_{n=1}^{\infty} (\sqrt[n]{2} - 1)$

let $a_n = \sqrt[n]{2} - 1$, $b_n = \frac{1}{n}$

Since $\lim_{n\to\infty} \left| \frac{\sqrt[n]{2}-1}{\frac{1}{n}} \right| = c$, series diverge by Limit Comparison Test.