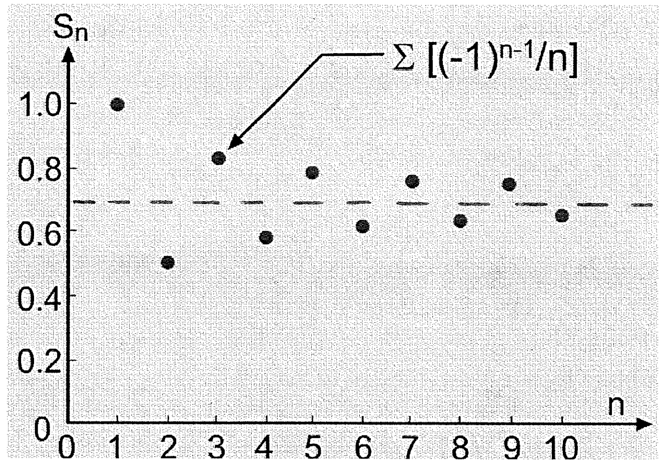
AP Calculus BC 9.7B Notes Lagrange Error Bound

Recall the **Alternating Series Remainder**:

Suppose an alternating series satisfies the convergent conditions of the Alternating Series Test. If the Series has a Sum S, then $|R_n| = |S - S_n| \leq a_{n+1}$, meaning that the maximum error for the nth term partial Sum S_n is no greater than the absolute value of the first unused term, a_{n+1}, which means that $S \in [S_n - R_n, S_n + R_n]$



<u>Lagrange Form of the Remainder (also called Lagrange Error Bound or Taylor's Theorem Remainder)</u>

When a Taylor polynomial is used to approximate a function, we need a way to see how accurately the polynomial approximates the function.

$$f(x) = P_n(x) + R_n(x) \text{ so } R_n(x) = f(x) - P_n(x)$$

Written in words:

Function = Polynomial Approximation + Remainder,

so

Remainder = Function − Polynomial Approximation

or

$$R_n(x) = |T_n(x) - f(x)| = \left| \frac{\left|f^{(n+1)}(z)\right|}{(n+1)!}(x-c)^{n+1} \right|$$

<u>Taylor's Theorem</u>: If a function f is differentiable through order $n+1$ in an interval containing c, then for each x in the interval, there exists a number z between x and c such that

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + R_n(x)$$

where the remainder $R_n(x)$ (or error) is given by $R_n(x) = \left| \frac{\left|f^{(n+1)}(z)\right|}{(n+1)!}(x-c)^{n+1} \right|$, the **Lagrange Remainder (or Error Bound).**

LaGrange Error Bound : $$R_n(x) = \left| \frac{\left|f^{(n+1)}(z)\right|}{(n+1)!}(x-c)^{n+1} \right|$$

When applying Taylor's Formula, we would not expected to be able to find the exact value of z. (If we could do this, then an approximation would not be necessary). Rather, we are merely interested in a safe upper bound (maximum value) for the $|f^{(n+1)}(z)|$ between x and c, from which we will be able to tell how large the remainder $R_n(x)$ is.

- We want to maximize the (n+1)st derivative on the interval from [x , c] or [c,x]. The maximum error bound is the worst case scenario for the interval in which our actual approximation can live

Example 1:
Calculator Permitted

Let f be a function with 5 derivatives on the interval $[2,3]$. Assume that $\left|f^{(5)}(x)\right| < 0.2$ for all x in the interval $[2,3]$ and that a fourth-degree Taylor polynomial for f at $c = 2$ is used to estimate $f(3)$

(a) How accurate is this approximation? Give three decimal places.

(b) Suppose that $P_4(3) = 1.763$. Use your answer to (a) to find an interval in which $f(3)$ must reside.

(c) Could $f(3)$ equal 1.778?

(d) Could $f(3)$ equal 1.764?

Example 2:
Calculator Permitted

(a) Find the fifth-degree Maclaurin polynomial for $\sin x$. Then use your polynomial to approximate $\sin 1$, and use Taylor's Theorem to find the maximum error for your approximation. Give three decimal places.

(b) Use your answer to (a) to find an interval $[a,b]$ such that $a \le \sin 1 \le b$

(c) Could $\sin 1$ equal 0.9? Why or why not?

Example 3:
No Calculator

(a) Write the fourth-degree Maclaurin polynomial for $f(x)=e^x$. Then use your polynomial to approximate *e*, and find a Lagrange error bound for the maximum error when $|x|\le 1$.

(b) Use your answer to (a) to find an interval $[a,b]$ such that $a\le e\le b$.

Example 4:
Calculator Permitted

The function *f* has derivatives of all orders for all real numbers *x*. Assume that $f(2)=6$, $f'(2)=4$, $f''(2)=-7$, $f'''(2)=8$.

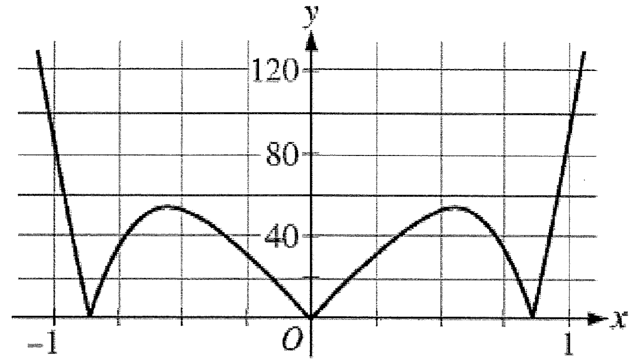
(a) Write the third-degree Taylor polynomial for *f* about $x=2$, and use it to approximate $f(2.3)$. Give three decimal places.

(b) The fourth derivative of *f* satisfies the inequality $\left|f^{(4)}(x)\right|\le 9$ for all *x* in the closed interval $[2,2.3]$. Use the Lagrange error bound on the approximation of $f(2.3)$ found in part (a) to find an interval $[a,b]$ such that $a\le f(2.3)\le b$. Give three decimal places.

(c) Based on the information above, could $f(2.3)$ equal 6.992? Explain why or why not.

Question 6

Let $f(x) = \sin(x^2) + \cos x$. The graph of $y = \left|f^{(5)}(x)\right|$ is shown above.



Graph of $y = \left|f^{(5)}(x)\right|$

(a) Write the first four nonzero terms of the Taylor series for $\sin x$ about $x = 0$, and write the first four nonzero terms of the Taylor series for $\sin(x^2)$ about $x = 0$.

(b) Write the first four nonzero terms of the Taylor series for $\cos x$ about $x = 0$. Use this series and the series for $\sin(x^2)$, found in part (a), to write the first four nonzero terms of the Taylor series for f about $x = 0$.

(c) Find the value of $f^{(6)}(0)$.

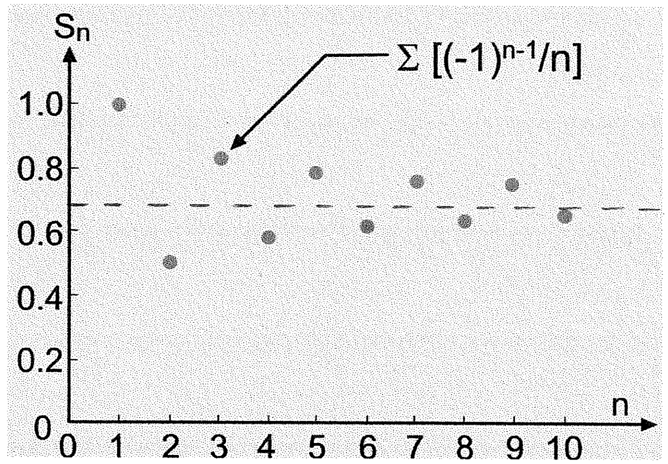
(d) Let $P_4(x)$ be the fourth-degree Taylor polynomial for f about $x = 0$. Using information from the graph of $y = \left|f^{(5)}(x)\right|$ shown above, show that $\left|P_4\left(\frac{1}{4}\right) - f\left(\frac{1}{4}\right)\right| < \frac{1}{3000}$.

Key

AP Calculus BC **9.7B Notes** **LaGrange Error Bound**

Recall the **Alternating Series Remainder**:

Suppose an alternating series satisfies the convergent conditions of the Alternating Series Test. If the Series has a Sum S, then $|R_n| = |S - S_n| \le a_{n+1}$, meaning that the maximum error for the nth term partial Sum S_n is no greater than the absolute value of the first unused term, a_{n+1}, which means that $S \in [S_n - R_n, S_n + R_n]$



Lagrange Form of the Remainder (also called Lagrange Error Bound or Taylor's Theorem Remainder)

When a Taylor polynomial is used to approximate a function, we need a way to see how accurately the polynomial approximates the function.

$$f(x) = \underbrace{P_n(x)}_{\text{polynomial approximation}} + \underbrace{R_n(x)}_{\text{remainder}} \text{ so } R_n(x) = f(x) - P_n(x)$$

Written in words:

Function = Polynomial Approximation + Remainder,

so

Remainder = Function − Polynomial Approximation

or

$$R_n(x) = |T_n(x) - f(x)| = \left|\frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}\right|$$

Taylor's Theorem: If a function f is differentiable through order $n+1$ in an interval containing c, then for each x in the interval, there exists a number z between x and c such that

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \boxed{R_n(x)}$$

Lagrange Error Bound or (Taylor Error)

where the remainder $R_n(x)$ (or error) is given by $R_n(x) = \left|\frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}\right|$, **the Lagrange Remainder (or Error Bound).**

*notice subscript is the same as the degree here (n)

LaGrange Error Bound :

$$R_n(x) = \left| \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1} \right|$$

When applying Taylor's Formula, we would not expected to be able to find the exact value of z. (If we could do this, then an approximation would not be necessary). Rather, we are merely interested in a safe upper bound (maximum value) for the $|f^{(n+1)}(z)|$ between x and c, from which we will be able to tell how large the remainder $R_n(x)$ is.

- We want to maximize the (n+1)st derivative on the interval from [x , c] or [c,x]. The maximum error bound is the worst case scenario for the interval in which our actual approximation can live

Example 1:
Calculator Permitted

Abs value of 5th derivative evaluated at x.
*magnitude of 5th derivative always less than 0.2.

Let f be a function with 5 derivatives on the interval $[2,3]$. Assume that $|f^{(5)}(x)| < 0.2$ for all x in the interval $[2,3]$ and that a fourth-degree Taylor polynomial for f at $c = 2$ is used to estimate $f(3)$

$\boxed{x=3 \quad c=2}$

(a) How accurate is this approximation? Give three decimal places.

max value

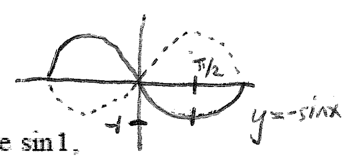
$$R_4(3) = \left| \frac{f^{(5)}(z)}{5!}(3-2)^5 \right| \leq \left| \frac{0.2}{5!}(1)^5 \right| = 0.000166$$

(b) Suppose that $P_4(3) = 1.763$. Use your answer to (a) to find an interval in which $f(3)$ must reside.

$1.763 - 0.000166 \leq f(3) \leq 1.763 + 0.00166$ $\qquad 1.7613 \leq f(3) \leq 1.76466$

(c) Could $f(3)$ equal 1.778? No, since 1.778 is not in interval

(d) Could $f(3)$ equal 1.764? yes, since 1.764 is in interval



Example 2:
Calculator Permitted

(a) Find the fifth-degree Maclaurin polynomial for $\sin x$. Then use your polynomial to approximate $\sin 1$, and use Taylor's Theorem to find the maximum error for your approximation. Give three decimal places.

$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!}$ $\quad \sin 1 \approx 1 - \frac{1}{6} + \frac{1}{120} = \frac{120-20+1}{120} = \frac{101}{120}$ $\quad P_{(5)}(1) = \frac{101}{120} = 0.8416$

$f'(x) = \cos x$
$f''(x) = -\sin x$
$f''' = -\cos x$
$f^4 = \sin x$
$f^5 = \cos x$
$f^6 = -\sin x$

$R_{(5)}(1) = \left| \frac{f^{(6)}(z)}{6!}(1-0)^6 \right| \leq \left| \frac{1}{6!}(1)^6 \right| = 0.00138$ $\qquad [0.8416 - 0.00138, 0.8416 + 0.00138]$

Remainder for this 5th degree polynomial

$x = 1 \rightarrow -\sin 0$
$c = 0 \rightarrow |-\sin 1| \approx 1$ (max value)

(b) Use your answer to (a) to find an interval $[a, b]$ such that $a \leq \sin 1 \leq b$ $\qquad 0.84027 \leq \sin 1 \leq 0.84305$

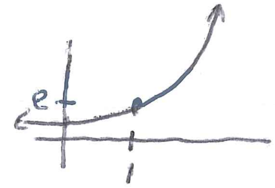
(c) Could $\sin 1$ equal 0.9? Why or why not? No, 0.9 is outside the interval $[0.84027, 0.84305]$

$$R_n(x) = \left| \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1} \right|$$

$x < z < c$

or

$c < z < x$



Example 3:
No Calculator

(a) Write the fourth-degree Maclaurin polynomial for $f(x) = e^x$. Then use your polynomial to approximate e, and find a Lagrange error bound for the maximum error when $|x| \le 1$.

$e^x \approx P_4(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$ $\quad e^1 \approx P_4(1) = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} = \frac{65}{24}$

$R_4(1) = \left| \frac{f^{(5)}(z)}{5!}(1-0)^5 \right| \le \left| \frac{3}{5!} \right| = \frac{1}{40} = 0.025$

$x = 1$

$c = 0 \quad e^0 = 1, \; e^1 \approx 2.718 \to 3$ (max value)

(b) Use your answer to (a) to find an interval $[a,b]$ such that $a \le e \le b$.

$\frac{65}{24} - 0.025 \le e \le \frac{65}{24} + 0.025$

Example 4:
Calculator Permitted

infinitely many terms

The function f has derivatives of all orders for all real numbers x. Assume that $f(2) = 6$, $f'(2) = 4$, $f''(2) = -7$, $f'''(2) = 8$.

$\boxed{c = 2}$

(a) Write the third-degree Taylor polynomial for f about $x = 2$, and use it to approximate $f(2.3)$. Give three decimal places.

$f(x) \approx P_3(x) = 6 + 4(x-2) + \frac{-7}{2!}(x-2)^2 + \frac{8}{3!}(x-2)^3$

$f(2.3) \approx P_3(2.3) = 6 + 4(2.3-2) - \frac{7}{2}(2.3-2)^2 + \frac{8}{3!}(2.3-2)^3$

$= 6 + 4(0.3) - \frac{7}{2}(0.3)^2 + \frac{4}{3}(0.3)^3 = \boxed{6.921}$

(b) The fourth derivative of f satisfies the inequality $|f^{(4)}(x)| \le 9$ for all x in the closed interval $[2, 2.3]$. Use the Lagrange error bound on the approximation of $f(2.3)$ found in part (a) to find an interval $[a,b]$ such that $a \le f(2.3) \le b$. Give three decimal places.

max value

$R_3(2.3) = \left| \frac{f^{(4)}(z)}{4!}(2.3-2)^4 \right| \le \left| \frac{9}{4!}(0.3)^4 \right| = 0.00303$

$x = 2.3$ $\quad x - c$

$c = 2$

$6.921 - 0.00303 \le f(2.3) \le 6.921 + 0.00303$

$6.917 \le f(2.3) \le 6.924$

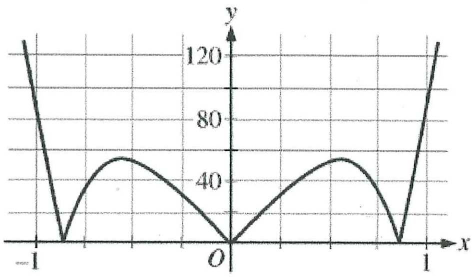
(c) Based on the information above, could $f(2.3)$ equal 6.992? Explain why or why not.

No, since 6.992 is outside the interval $[6.917, 6.924]$

AP® CALCULUS BC
2011 SCORING GUIDELINES

Question 6

Let $f(x) = \sin(x^2) + \cos x$. The graph of $y = \left|f^{(5)}(x)\right|$ is shown above.



Graph of $y = \left|f^{(5)}(x)\right|$

(a) Write the first four nonzero terms of the Taylor series for $\sin x$ about $x = 0$, and write the first four nonzero terms of the Taylor series for $\sin(x^2)$ about $x = 0$.

(b) Write the first four nonzero terms of the Taylor series for $\cos x$ about $x = 0$. Use this series and the series for $\sin(x^2)$, found in part (a), to write the first four nonzero terms of the Taylor series for f about $x = 0$.

(c) Find the value of $f^{(6)}(0)$.

(d) Let $P_4(x)$ be the fourth-degree Taylor polynomial for f about $x = 0$. Using information from the graph of $y = \left|f^{(5)}(x)\right|$ shown above, show that $\left|P_4\left(\frac{1}{4}\right) - f\left(\frac{1}{4}\right)\right| < \frac{1}{3000}$.

$$* \; R_n(x) = \left|\frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}\right|$$

a) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$

$\sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \dots$

b) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$

$f(x) = 1 + x^2 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{3!} - \frac{x^6}{6!}$

$f(x) = 1 + \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6} - \frac{x^6}{720}$

$1 + \frac{x^2}{2} + \frac{x^4}{4!} - \frac{120x^6}{720} - \frac{1x^6}{720}$

$f(x) = 1 + \frac{x^2}{2} + \frac{x^4}{4!} - \frac{121}{6!}x^6$

c) Since $\frac{f^{(6)}(0)}{6!}x^6 = \frac{-121}{6!}x^6$,

$\boxed{f^{(6)}(0) = -121}$

d) $R_4(x) = \left|P_4(x) - f(x)\right| = \left|P_4\left(\frac{1}{4}\right) - f\left(\frac{1}{4}\right)\right| = \left|\frac{f^{(5)}(z)}{5!}(x-c)^5\right| \le \left|\frac{40}{5!}\left(\frac{1}{4}-0\right)^5\right|$

$x = 1/4$

$C = 0$

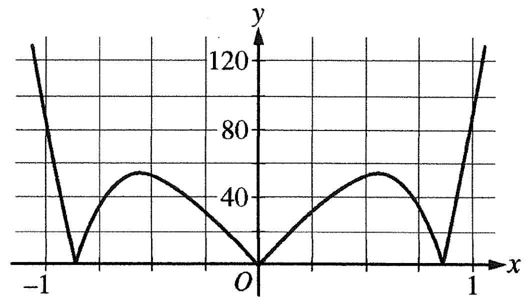
max y-value in interval $0 < x < 1/4$

$= \left|\frac{40}{5!}\left(\frac{1}{4}\right)^5\right| = \frac{40}{120 \cdot 4^5} = \boxed{\frac{1}{3072} < \frac{1}{3000}}$

$= \frac{1}{3 \cdot 4^5} = \frac{1}{3 \cdot 1024}$

Question 6

Let $f(x) = \sin\left(x^2\right) + \cos x$. The graph of $y = \left|f^{(5)}(x)\right|$ is shown above.



Graph of $y = \left|f^{(5)}(x)\right|$

(a) Write the first four nonzero terms of the Taylor series for $\sin x$ about $x = 0$, and write the first four nonzero terms of the Taylor series for $\sin\left(x^2\right)$ about $x = 0$.

(b) Write the first four nonzero terms of the Taylor series for $\cos x$ about $x = 0$. Use this series and the series for $\sin\left(x^2\right)$, found in part (a), to write the first four nonzero terms of the Taylor series for f about $x = 0$.

(c) Find the value of $f^{(6)}(0)$.

(d) Let $P_4(x)$ be the fourth-degree Taylor polynomial for f about $x = 0$. Using information from the graph of $y = \left|f^{(5)}(x)\right|$ shown above, show that $\left|P_4\left(\frac{1}{4}\right) - f\left(\frac{1}{4}\right)\right| < \frac{1}{3000}$.

(a) $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots$

$\sin\left(x^2\right) = x^2 - \dfrac{x^6}{3!} + \dfrac{x^{10}}{5!} - \dfrac{x^{14}}{7!} + \cdots$

$3 : \begin{cases} 1 : \text{series for } \sin x \\ 2 : \text{series for } \sin\left(x^2\right) \end{cases}$

(b) $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \cdots$

$f(x) = 1 + \dfrac{x^2}{2} + \dfrac{x^4}{4!} - \dfrac{121x^6}{6!} + \cdots$

$3 : \begin{cases} 1 : \text{series for } \cos x \\ 2 : \text{series for } f(x) \end{cases}$

(c) $\dfrac{f^{(6)}(0)}{6!}$ is the coefficient of x^6 in the Taylor series for f about $x = 0$. Therefore $f^{(6)}(0) = -121$.

1 : answer

(d) The graph of $y = \left|f^{(5)}(x)\right|$ indicates that $\max\limits_{0 \le x \le \frac{1}{4}} \left|f^{(5)}(x)\right| < 40$.

Therefore

$$\left|P_4\left(\frac{1}{4}\right) - f\left(\frac{1}{4}\right)\right| \le \frac{\max\limits_{0 \le x \le \frac{1}{4}} \left|f^{(5)}(x)\right|}{5!} \cdot \left(\frac{1}{4}\right)^5 < \frac{40}{120 \cdot 4^5} = \frac{1}{3072} < \frac{1}{3000}.$$

$2 : \begin{cases} 1 : \text{form of the error bound} \\ 1 : \text{analysis} \end{cases}$