BC Calculus **Ch. 12.3 Notes** **Velocity and Acceleration**

Now we look to combine our knowledge of parametric equations, curves, vectors, and vector-valued functions to form a model for motion along a curve.

Position vector: r(t) = x(t) **i** + y(t) **j**

Definitions of Velocity and Acceleration

If x and y are twice-differentiable functions of t, and $\mathbf{r}$ is a vector-valued function given by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$, then the velocity vector, acceleration vector, and speed at time t are as follows.

$$\textbf{Velocity} = \mathbf{v}(t) = \mathbf{r}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j}$$

$$\textbf{Acceleration} = \mathbf{a}(t) = \mathbf{r}''(t) = x''(t)\mathbf{i} + y''(t)\mathbf{j}$$

$$\textbf{Speed} = \|\mathbf{v}(t)\| = \|\mathbf{r}'(t)\| = \sqrt{[x'(t)]^2 + [y'(t)]^2}$$

EXAMPLE 1 **Finding Velocity and Acceleration Along a Plane Curve**

Find the velocity vector, speed, and acceleration vector of a particle that moves along the plane curve C described by

$$\mathbf{r}(t) = 2\sin\frac{t}{2}\mathbf{i} + 2\cos\frac{t}{2}\mathbf{j}. \qquad \text{Position vector}$$

<u>Example 2:</u> **Sketching Velocity and Acceleration Vectors in the Plane**

Sketch the path of an object moving along the plane curve given by

$$\mathbf{r}(t) = (t^2 - 4)\mathbf{i} + t\mathbf{j} \qquad \text{Position vector}$$

and find the velocity and acceleration vectors when $t = 0$ and $t = 2$.

Example 3: **Finding a Position Function by Integration**

An object starts from rest at the point $P(1, 2, 0)$ and moves with an acceleration of

$$\mathbf{a}(t) = \mathbf{j} + 2\mathbf{k} \qquad \text{Acceleration vector}$$

where $\|\mathbf{a}(t)\|$ is measured in feet per second per second. Find the location of the object after $t = 2$ seconds.

THEOREM 12.3 Position Function for a Projectile

Neglecting air resistance, the path of a projectile launched from an initial height h with initial speed v_0 and angle of elevation θ is described by the vector function

$$\mathbf{r}(t) = (v_0 \cos \theta)t\mathbf{i} + \left[h + (v_0 \sin \theta)t - \frac{1}{2}gt^2\right]\mathbf{j}$$

where g is the gravitational constant.

Example 4: **Describing the Path of a Baseball**

A baseball is hit 3 feet above ground level at 100 feet per second and at an angle of 45° with respect to the ground, as shown in Figure 12.18. Find the maximum height reached by the baseball. Will it clear a 10-foot-high fence located 300 feet from home plate?

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Key

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Find the velocity vector, speed, and acceleration vector of a particle that moves along the plane curve C described by

$$\mathbf{r}(t) = 2\sin\frac{t}{2}\mathbf{i} + 2\cos\frac{t}{2}\mathbf{j}. \qquad \text{Position vector}$$

a) $v(t) = r'(t) = 2\cos(t/2)\cdot\frac{1}{2}i - 2\sin(t/2)\cdot\frac{1}{2}j = \boxed{\cos(t/2)i - \sin(t/2)j}$

b) $\|r'(t)\| = \sqrt{\cos^2(t/2) + \sin^2(t/2)} = \sqrt{1} = \boxed{1}$

c) $a(t) = r''(t) = -\sin(t/2)\cdot\frac{1}{2}i - \cos(t/2)\cdot\frac{1}{2}j = \boxed{-\frac{1}{2}\sin(t/2)i - \frac{1}{2}\cos(t/2)j}$

Example 2: **Sketching Velocity and Acceleration Vectors in the Plane**

parametric equations

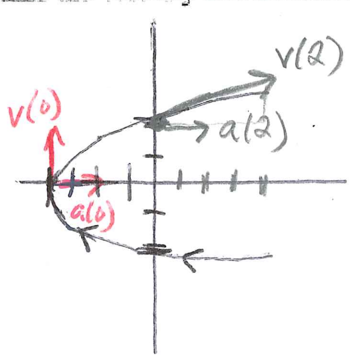
$x = t^2 - 4$ and $y = t$

rectangular equation: $x = y^2 - 4$

Sketch the path of an object moving along the plane curve given by

$$\mathbf{r}(t) = (t^2 - 4)\mathbf{i} + t\mathbf{j} \qquad \text{Position vector}$$

and find the velocity and acceleration vectors when $t = 0$ and $t = 2$.



$v(t) = r'(t) = 2ti + j$

$a(t) = r''(t) = 2i$

$v(0) = 2(0)i + j = j$

$a(0) = 2i$

$v(2) = 2(2)i + j = 4i + j$

$a(2) = 2i$

$r(0) = (-4, 0)$

$r(2) = (0, 2)$

Example 3: **Finding a Position Function by Integration**

$v(0)=0$ $r(0)=1i+2j$

An object starts from rest at the point $P(1, 2, 0)$ and moves with an acceleration of

$$\mathbf{a}(t) = \mathbf{j} + 2\mathbf{k} \qquad \text{Acceleration vector}$$

where $\|\mathbf{a}(t)\|$ is measured in feet per second per second. Find the location of the object after $t = 2$ seconds.

*Integrate twice to find position function.

$v(t) = \int a(t)\,dt = \int j + 2k\,dt = tj + 2tk + C$

$v(0) = C_1 i + C_2 j + C_3 k = 0 \to C_1 = C_2 = C_3 = 0$

$\boxed{v(t) = tj + 2tk}$

$r(t) = \int v(t)\,dt = \int tj + 2tk\,dt = \frac{t^2}{2}j + t^2 k + C$

*$C = C_1 i + C_2 j + C_3 k$

$C = C_4 i + C_5 j + C_6 k$

$r(0) = C_4 i + C_5 j + C_6 k = i + 2j$

$C_4 = 1,\ C_5 = 2,\ C_6 = 0$

$\boxed{r(t) = 1i + \left(\frac{t^2}{2} + 2\right)j + t^2 k}$

$\boxed{r(2) = i + 4j + 4k = \langle 1, 4, 4 \rangle}$

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$h = 3,\ v_0 = 100,\ \theta = 45°$

$g = 32\ ft/s^2$

$r(t) = [100\cos(\pi/4)]ti + [3 + (100\sin\pi/4)t - 16t^2]j$

$= 100\left(\frac{\sqrt{2}}{2}\right)ti + \left[3 + 100\left(\frac{\sqrt{2}}{2}\right)t - 16t^2\right]j = (50\sqrt{2}t)i + [3 + 50\sqrt{2}t - 16t^2]j$

$v(t) = r'(t) = 50\sqrt{2}i + (50\sqrt{2} - 32t)j$

* Max height occurs where $y'(t) = 0$ * $\quad y'(t) = 50\sqrt{2} - 32t = 0 \quad t = \frac{25\sqrt{2}}{16} \approx \boxed{2.21 \text{ secs}}$

Max height: $y = 3 + 50\sqrt{2}[2.21] - 16[2.21]^2 = \frac{649}{8} \approx \boxed{81 \text{ feet}}$

*Find t when $x(t) = 300$ ft.

$50\sqrt{2}t = 300$

$t = 3\sqrt{2} \approx 4.24$ secs.

* Find $y(t)$ when $t = 4.24$ secs

$y = 3 + 50\sqrt{2}(4.24) - 16(4.24)^2 = 15$ ft.

*Therefore, ball clears 10 ft. fence for Home run