BC Calculus   Ch. 11 Notes and Review

11.1 Vector is a quantity that involve both magnitude and direction (Scalar quantity involve just magnitude)

The directed line segment $\overrightarrow{PQ}$ has initial point P and terminal point Q, and its length (magnitude) is denoted by $\|\overrightarrow{PQ}\|$

$\|\overrightarrow{PQ}\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ Vector $\vec{v} = \overrightarrow{PQ}$

Definition of Component Form of a Vector in the Plane

If $\mathbf{v}$ is a vector in the plane whose initial point is the origin and whose terminal point is (v_1, v_2), then the **component form of v** is given by

$$\mathbf{v} = \langle v_1, v_2 \rangle.$$

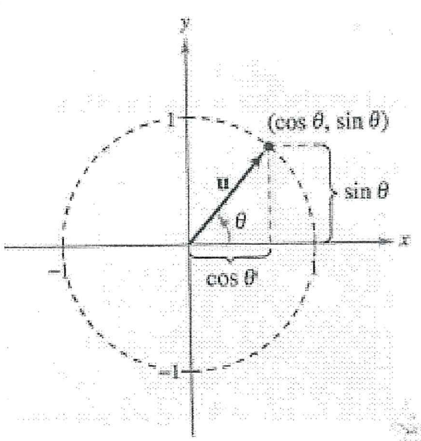
The coordinates v_1 and v_2 are called the **components of v.** If both the initial point and the terminal point lie at the origin, then $\mathbf{v}$ is called the **zero vector** and is denoted by $\mathbf{0} = \langle 0, 0 \rangle$.

Standard Unit Vectors

The unit vectors <1,0> and <0,1> are called standard unit vectors and are denoted by **i** = <1, 0> and **j** = <0, 1>

EXAMPLE 1 Finding the Component Form and Length of a Vector

Find the component form and length of the vector $\mathbf{v}$ that has initial point $(3, -7)$ and terminal point $(-2, 5)$.



If $\mathbf{u}$ is a unit vector and θ is the angle (measured counterclockwise) from the positive x-axis to $\mathbf{u}$, then the terminal point of $\mathbf{u}$ lies on the unit circle, and you have

$$\mathbf{u} = \langle \cos\theta, \sin\theta \rangle = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j} \qquad \text{Unit vector}$$

as shown in Figure 11.11. Moreover, it follows that any other nonzero vector $\mathbf{v}$ making an angle θ with the positive x-axis has the same direction as $\mathbf{u}$, and you can write

$$\mathbf{v} = \|\mathbf{v}\|\langle \cos\theta, \sin\theta \rangle = \|\mathbf{v}\|\cos\theta\,\mathbf{i} + \|\mathbf{v}\|\sin\theta\,\mathbf{j}.$$

EXAMPLE 2 Writing a Vector of Given Magnitude and Direction

The vector $\mathbf{v}$ has a magnitude of 3 and makes an angle of $30° = \pi/6$ with the positive x-axis. Write $\mathbf{v}$ as a linear combination of the unit vectors $\mathbf{i}$ and $\mathbf{j}$.

11.2 Coordinates in Space and Vectors in Space

Distance Formula	$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$
Equation of a Sphere	$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$
Midpoint Rule	$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$.
Length:	$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$

Definition of Parallel Vectors

Two nonzero vectors **u** and **v** are **parallel** if there is some scalar c such that $\mathbf{u} = c\mathbf{v}$.

Definition of Dot Product

The **dot product** of $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ is

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2.$$

*The Dot Product returns a scalar and not a vector
*Vectors are orthogonal (normal, perpendicular) if
u * v = 0

THEOREM 11.5 Angle Between Two Vectors

If θ is the angle between two nonzero vectors **u** and **v**, then

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|}.$$

Alternative form of the dot product

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos\theta$$

Definition of Work:
The work W done by a constant force F as its point of application moves along the vector $\overrightarrow{PQ}$ is given by either of the following:

1. $W = \|\text{proj}_{\overrightarrow{PQ}} \mathbf{F}\| \|\overrightarrow{PQ}\|$ Projection form
2. $W = \mathbf{F} \cdot \overrightarrow{PQ}$ Dot product form

Definition of Cross Product of Two Vectors in Space

Let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ be vectors in space. The **cross product** of **u** and **v** is the vector

$$\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2)\mathbf{i} - (u_1 v_3 - u_3 v_1)\mathbf{j} + (u_1 v_2 - u_2 v_1)\mathbf{k}.$$

*The cross product yields a vector , also called the vector product

Example 3: Finding the cross product Given $\mathbf{u} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$, find $\mathbf{u} \times \mathbf{v}$

THEOREM 11.8 Geometric Properties of the Cross Product

Let $\mathbf{u}$ and $\mathbf{v}$ be nonzero vectors in space, and let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$.

1. $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.
2. $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \, \|\mathbf{v}\| \sin \theta$
3. $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are scalar multiples of each other.
4. $\|\mathbf{u} \times \mathbf{v}\| =$ area of parallelogram having $\mathbf{u}$ and $\mathbf{v}$ as adjacent sides.

THEOREM 11.9 The Triple Scalar Product

For $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$, $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, and $\mathbf{w} = w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k}$, the triple scalar product is given by

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

THEOREM 11.10 Geometric Property of Triple Scalar Product

The volume V of a parallelepiped with vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ as adjacent edges is given by

$$V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|.$$

EXAMPLE 4 Volume by the Triple Scalar Product

Find the volume of the parallelepiped shown in Figure 11.42 having $\mathbf{u} = 3\mathbf{i} - 5\mathbf{j} + \mathbf{k}$, $\mathbf{v} = 2\mathbf{j} - 2\mathbf{k}$, and $\mathbf{w} = 3\mathbf{i} + \mathbf{j} + \mathbf{k}$ as adjacent edges.

11.5 Lines in Space

THEOREM 11.11 Parametric Equations of a Line in Space

A line L parallel to the vector $\mathbf{v} = \langle a, b, c \rangle$ and passing through the point $P(x_1, y_1, z_1)$ is represented by the **parametric equations**

$$x = x_1 + at, \quad y = y_1 + bt, \quad \text{and} \quad z = z_1 + ct.$$

Symmetric Equations

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

Example 5: Find a set of parametric equations of the line that passes through the points $(-2, 1, 0)$ and $(1, 3, 5)$.

THEOREM 11.13 Distance Between a Point and a Plane

The distance between a plane and a point Q (not in the plane) is

$$D = \|\text{proj}_{\mathbf{n}}\overrightarrow{PQ}\| = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|}$$

where P is a point in the plane and $\mathbf{n}$ is normal to the plane.

THEOREM 11.12 Standard Equation of a Plane in Space

The plane containing the point (x_1, y_1, z_1) and having a normal vector $\mathbf{n} = \langle a, b, c\rangle$ can be represented, in **standard form**, by the equation

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0.$$

Example 6: Find the distance between the point $Q(1, 5, -4)$ and the plane given by

$$3x - y + 2z = 6.$$

THEOREM 11.14 Distance Between a Point and a Line in Space

The distance between a point Q and a line in space is given by

$$D = \frac{\|\overrightarrow{PQ} \times \mathbf{u}\|}{\|\mathbf{u}\|}$$

where $\mathbf{u}$ is a direction vector for the line and P is a point on the line.

Example 7: Find the distance between the point $Q(3, -1, 4)$ and the line given by

$$x = -2 + 3t, \quad y = -2t, \quad \text{and} \quad z = 1 + 4t.$$

Key

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Find the component form and length of the vector **v** that has initial point $(3, -7)$ and terminal point $(-2, 5)$.

$V = \langle V_1, V_2 \rangle$

$V_1 = -2 - 3 = -5$

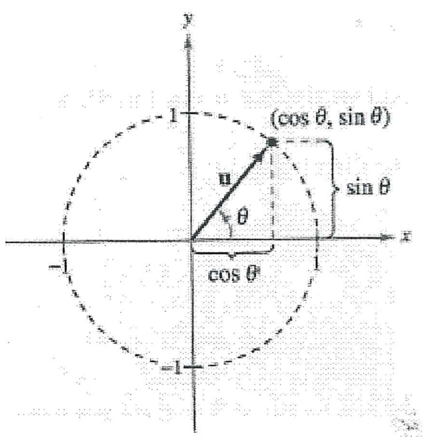
$V_2 = 5 - -7 = 12$

$V = \langle -5, 12 \rangle$

$\|v\| = \sqrt{(-5)^2 + 12^2}$

$= \sqrt{169}$

$\boxed{\|v\| = 13}$



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The vector **v** has a magnitude of 3 and makes an angle of $30° = \pi/6$ with the positive x-axis. Write **v** as a linear combination of the unit vectors **i** and **j**.

$\|v\| = 3$

$V = 3\cos(\pi/6)\,i + 3\sin(\pi/6)\,j = \boxed{\frac{3\sqrt{3}}{2} i + \frac{3}{2} j}$

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$*$ determinant $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

$\boxed{\text{Ex. 31}}$

Example: Finding the cross product

Given $\mathbf{u} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$, find $\mathbf{u} \times \mathbf{v}$

$$u \times v = \begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{vmatrix} i & j & k \\ 1 & -2 & 1 \\ 3 & 1 & -2 \end{vmatrix} \end{array} = \begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} i - \begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix} j + \begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} k$$

$$(4-1)i - (-2-3)j + (1+6)k$$

$$\boxed{3i + 5j + 7k}$$

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EXAMPLE 4 Volume by the Triple Scalar Product

Find the volume of the parallelepiped shown in Figure 11.42 having $\mathbf{u} = 3\mathbf{i} - 5\mathbf{j} + \mathbf{k}$, $\mathbf{v} = 2\mathbf{j} - 2\mathbf{k}$, and $\mathbf{w} = 3\mathbf{i} + \mathbf{j} + \mathbf{k}$ as adjacent edges.

$$V = |u \cdot (v \times w)| = \begin{vmatrix} 3 & -5 & 1 \\ 0 & 2 & -2 \\ 3 & 1 & 1 \end{vmatrix} = 3\begin{vmatrix} 2 & -2 \\ 1 & 1 \end{vmatrix} - (-5)\begin{vmatrix} 0 & -2 \\ 3 & 1 \end{vmatrix} + 1\begin{vmatrix} 0 & 2 \\ 3 & 1 \end{vmatrix} = 12 + 30 - 6 = \boxed{36}$$

11.5 Lines in Space

THEOREM 11.11 Parametric Equations of a Line in Space

A line L parallel to the vector $\mathbf{v} = \langle a, b, c\rangle$ and passing through the point $P(x_1, y_1, z_1)$ is represented by the **parametric equations**

Ex. 5 $x = x_1 + at,\quad y = y_1 + bt,\quad \text{and}\quad z = z_1 + ct.$

Symmetric Equations

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

Find a set of parametric equations of the line passing through points $(-2,1,0)$ and $(1,3,5)$

$$v = \overrightarrow{PQ} = \langle 1 - -2,\ 3-1,\ 5-0 \rangle = \langle 3, 2, 5 \rangle$$

$$\boxed{\begin{aligned} x &= -2 + 3t \\ y &= 1 + 2t \\ z &= 5t \end{aligned}}$$

THEOREM 11.13 Distance Between a Point and a Plane

The distance between a plane and a point Q (not in the plane) is

$$D = \|\text{proj}_{\mathbf{n}}\overrightarrow{PQ}\| = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|}$$

where P is a point in the plane and $\mathbf{n}$ is normal to the plane.

* plane in standard form: $a(x-x_1)+b(y-y_1)+c(z-z_1)=0$ has normal vector $n=\langle a,b,c\rangle$

Ex. 6 Find distance between point $Q(1,5,-4)$ and the plane given by $3x-y+2z=6$

$n=\langle 3,-1,2\rangle$ point $P=(2,0,0)$ $\overrightarrow{PQ}=\langle -1,5,-4\rangle$

$$D=\frac{|\langle -1,5,-4\rangle\cdot\langle 3,-1,2\rangle|}{\sqrt{9+1+4}}=\frac{|-3-5-8|}{\sqrt{14}}=\boxed{\frac{16}{\sqrt{14}}}$$

THEOREM 11.14 Distance Between a Point and a Line in Space

The distance between a point Q and a line in space is given by

$$D = \frac{\|\overrightarrow{PQ} \times \mathbf{u}\|}{\|\mathbf{u}\|}$$

where $\mathbf{u}$ is a direction vector for the line and P is a point on the line.

Ex. 7 Find distance between point $Q(3,-1,4)$ and line given by

$x=-2+3t$ $y=-2t$ $z=1+4t$ *direction values $\langle 3,-2,4\rangle$

$u=\langle 3,-2,4\rangle$ $P=$ (let $t=0$) $P=(-2,0,1)$

$\overrightarrow{PQ}=\langle 3--2,-1-0,4-1\rangle=\langle 5,-1,3\rangle$

$$\overrightarrow{PQ}\times u=\begin{vmatrix} \overset{+}{i} & \overset{-}{j} & \overset{+}{k}\\ 5 & -1 & 3\\ 3 & -2 & 4\end{vmatrix}=\begin{vmatrix}-1 & 3\\ -2 & 4\end{vmatrix}i-\begin{vmatrix}5 & 3\\ 3 & 4\end{vmatrix}j+\begin{vmatrix}5 & -1\\ 3 & -2\end{vmatrix}k=2i-11j-7k$$

$$\frac{\|\overrightarrow{PQ}\times u\|}{\|u\|}=\frac{\sqrt{2^2+11^2+7^2}}{\sqrt{3^2+2^2+4^2}}=\boxed{\frac{\sqrt{174}}{\sqrt{29}}=\sqrt{6}\approx 2.45}$$