BC Taylor Series FRQ 2009 #6

Question 6

The Maclaurin series for e^x is $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots + \frac{x^n}{n!} + \cdots$. The continuous function f is defined by $f(x) = \frac{e^{(x-1)^2} - 1}{(x-1)^2}$ for $x \neq 1$ and $f(1) = 1$. The function f has derivatives of all orders at $x = 1$.

(a) Write the first four nonzero terms and the general term of the Taylor series for $e^{(x-1)^2}$ about $x = 1$.

(b) Use the Taylor series found in part (a) to write the first four nonzero terms and the general term of the Taylor series for f about $x = 1$.

(c) Use the ratio test to find the interval of convergence for the Taylor series found in part (b).

(d) Use the Taylor series for f about $x = 1$ to determine whether the graph of f has any points of inflection.

1) **BC Taylor Series FRQ** 2009 #6

Key

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(d) Use the Taylor series for f about $x = 1$ to determine whether the graph of f has any points of inflection.

a) $e^{(x-1)^2} = 1 + (x-1)^2 + \frac{[(x-1)^2]^2}{2} + \frac{[(x-1)^2]^3}{6} + \ldots \frac{[(x-1)^2]^n}{n!}$

$= 1 + (x-1)^2 + \frac{(x-1)^4}{2} + \frac{(x-1)^6}{6} + \ldots + \frac{(x-1)^{2n}}{n!} + \ldots$

b) $f(x) = \dfrac{1 + (x-1)^2 + \frac{(x-1)^4}{2} + \frac{(x-1)^6}{6} + \ldots + \frac{(x-1)^{2n}}{n!} - 1}{(x-1)^2} = 1 + \frac{(x-1)^2}{2} + \frac{(x-1)^4}{6} + \frac{(x-1)^6}{24} + \ldots \boxed{\frac{(x-1)^{2n}}{(n+1)!}}$

$2!$ $3!$ $4!$

c) *Ratio Test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$

$\lim_{n\to\infty} \left| \frac{(x-1)^{2(n+1)}}{(n+1+1)!} \cdot \frac{(n+1)!}{(x-1)^{2n}} \right| = \lim_{n\to\infty} \left| \frac{(x-1)^{2n+2} \cdot (n+1)!}{(n+2)! \cdot (x-1)^{2n}} \right| = \lim_{n\to\infty} \left| \frac{\cancel{(x-1)^{2n}} \cdot (x-1)^2 \cdot \cancel{(n+1)!}}{(n+2)\cancel{(n+1)!}\,\cancel{(x-1)^{2n}}} \right|$

$\lim_{n\to\infty} \left| \frac{(x-1)^2}{(n+1)!} \right| = 0 < 1$. The Interval of Convergence is therefore $(-\infty, \infty)$.

d) $f'(x) = \frac{2(x-1)^1}{2} + \frac{4(x-1)^3}{6} + \frac{6(x-1)^5}{24} + \ldots \frac{2n(x-1)^{2n-1}}{(n+1)!}$

$f''(x) = 1 + \frac{4 \cdot 3(x-1)^2}{6} + \frac{6 \cdot 5(x-1)^4}{24} + \ldots \frac{2n \cdot (2n-1)(x-1)^{2n-2}}{(n+1)!}$

Since $f''(x) > 0$ for all x. No points of inflection for graph of f.

2009 SCORING GUIDELINES

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(d) Use the Taylor series for f about $x = 1$ to determine whether the graph of f has any points of inflection.

(a) $1 + (x-1)^2 + \frac{(x-1)^4}{2} + \frac{(x-1)^6}{6} + \cdots + \frac{(x-1)^{2n}}{n!} + \cdots$	$2: \begin{cases} 1: \text{first four terms} \\ 1: \text{general term} \end{cases}$
(b) $1 + \frac{(x-1)^2}{2} + \frac{(x-1)^4}{6} + \frac{(x-1)^6}{24} + \cdots + \frac{(x-1)^{2n}}{(n+1)!} + \cdots$	$2: \begin{cases} 1: \text{first four terms} \\ 1: \text{general term} \end{cases}$
(c) $\lim_{n\to\infty} \left\lvert \frac{\frac{(x-1)^{2n+2}}{(n+2)!}}{\frac{(x-1)^{2n}}{(n+1)!}} \right\rvert = \lim_{n\to\infty} \frac{(n+1)!}{(n+2)!}(x-1)^2 = \lim_{n\to\infty} \frac{(x-1)^2}{n+2} = 0$ Therefore, the interval of convergence is $(-\infty, \infty)$.	$3: \begin{cases} 1: \text{sets up ratio} \\ 1: \text{computes limit of ratio} \\ 1: \text{answer} \end{cases}$
(d) $f''(x) = 1 + \frac{4\cdot 3}{6}(x-1)^2 + \frac{6\cdot 5}{24}(x-1)^4 + \cdots + \frac{2n(2n-1)}{(n+1)!}(x-1)^{2n-2} + \cdots$ Since every term of this series is nonnegative, $f''(x) \geq 0$ for all x. Therefore, the graph of f has no points of inflection.	$2: \begin{cases} 1: f''(x) \\ 1: \text{answer} \end{cases}$

BC Taylor Series FRQ 2009 B #6

Question 6

The function f is defined by the power series

$$f(x) = 1 + (x+1) + (x+1)^2 + \cdots + (x+1)^n + \cdots = \sum_{n=0}^{\infty} (x+1)^n$$

for all real numbers x for which the series converges.

(a) Find the interval of convergence of the power series for f. Justify your answer.

(b) The power series above is the Taylor series for f about $x = -1$. Find the sum of the series for f.

(c) Let g be the function defined by $g(x) = \int_{-1}^{x} f(t)\,dt$. Find the value of $g\left(-\frac{1}{2}\right)$, if it exists, or explain why $g\left(-\frac{1}{2}\right)$ cannot be determined.

(d) Let h be the function defined by $h(x) = f\left(x^2 - 1\right)$. Find the first three nonzero terms and the general term of the Taylor series for h about $x = 0$, and find the value of $h\left(\frac{1}{2}\right)$.

2) **BC Taylor Series FRQ 2009 B #6**

Key

Question 6

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a) The ratio is $(x+1)$. For convergence, $|r| < 1$

$|x+1| < 1 \rightarrow -1 < x+1 < 1$

I.O.C. is $\boxed{-2 < x < 0}$

Test endpts: Since series is geometric, the endpoints will diverge

b) $\text{Sum} = \dfrac{a_1}{1-r} = \dfrac{1}{1-(x+1)} = \dfrac{1}{-x}$ for $-2 < x < 0$

c) $\displaystyle\int_{-1}^{x} f(t)\,dt = \int_{-1}^{x} -\frac{1}{x}\,dx = -\ln|x|\Big]_{1}^{x} = -\ln|x|\Big]_{1}^{-1/2} = -\ln\left|\tfrac{-1}{2}\right| - (-\ln(1))$

$= -\ln|2^{-1}| = \boxed{\ln 2}$

d) $f(x) = 1 + (x+1) + (x+1)^2 + \dots + (x+1)^n$

$h(x) = f(x^2-1) = 1 + [x^2-1+1] + [x^2-1+1]^2 + \dots + [x^2-1+1]^n + \dots$

$= \boxed{1 + x^2 + x^4 + \dots + x^{2n} + \dots}$

$h\left(\frac{1}{2}\right) = f\left(\left(\frac{1}{2}\right)^2 - 1\right) = f\left(\frac{-3}{4}\right) = \dfrac{-1}{(-3/4)} = \boxed{\dfrac{4}{3}}$

$f(x) = \dfrac{-1}{x}$

2009 SCORING GUIDELINES (Form B)

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for all real numbers x for which the series converges.

(a) Find the interval of convergence of the power series for f. Justify your answer.

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(d) Let h be the function defined by $h(x) = f\left(x^2 - 1\right)$. Find the first three nonzero terms and the general term of the Taylor series for h about $x = 0$, and find the value of $h\left(\frac{1}{2}\right)$.

(a) The power series is geometric with ratio $(x+1)$. The series converges if and only if $\lvert x+1 \rvert < 1$. Therefore, the interval of convergence is $-2 < x < 0$.	$3: \begin{cases} 1: \text{identifies as geometric} \\ 1: \lvert x+1 \rvert < 1 \\ 1: \text{interval of convergence} \end{cases}$
OR	OR
$\lim\limits_{n \to \infty} \left\lvert \frac{(x+1)^{n+1}}{(x+1)^n} \right\rvert = \lvert x+1 \rvert < 1$ when $-2 < x < 0$. At $x = -2$, the series is $\sum\limits_{n=0}^{\infty} (-1)^n$, which diverges since the terms do not converge to 0. At $x = 0$, the series is $\sum\limits_{n=0}^{\infty} 1$, which similarly diverges. Therefore, the interval of convergence is $-2 < x < 0$.	$3: \begin{cases} 1: \text{sets up limit of ratio} \\ 1: \text{radius of convergence} \\ 1: \text{interval of convergence} \end{cases}$
(b) Since the series is geometric, $f(x) = \sum\limits_{n=0}^{\infty} (x+1)^n = \frac{1}{1-(x+1)} = -\frac{1}{x}$ for $-2 < x < 0$.	1 : answer
(c) $g\left(-\frac{1}{2}\right) = \int_{-1}^{-\frac{1}{2}} -\frac{1}{x}\,dx = -\ln\lvert x \rvert \Big\rvert_{x=-1}^{x=-\frac{1}{2}} = \ln 2$	$2: \begin{cases} 1: \text{antiderivative} \\ 1: \text{value} \end{cases}$
(d) $h(x) = f\left(x^2 - 1\right) = 1 + x^2 + x^4 + \cdots + x^{2n} + \cdots$ $h\left(\frac{1}{2}\right) = f\left(-\frac{3}{4}\right) = \frac{4}{3}$	$3: \begin{cases} 1: \text{first three terms} \\ 1: \text{general term} \\ 1: \text{value of } h\left(\frac{1}{2}\right) \end{cases}$