AP Calculus BC 9.8a Notes Power Series: Taylor & Maclaurin Series

If x is a variable, then an infinite series of the form $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots a_n x^n + \cdots$ is called a **power series.**

$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots a_n(x-c)^n + \cdots$ is a power series **centered at** c, where c is a constant.

The equal sign above means that the left side equals the right side for all values in the domain. This means the above is an IDENTITY. But for WHAT values of x does the identity hold? We have to find them. For all such x values, we say the series CONVERGES. For the values of x for which the identity is NOT true, we say the series diverges.

For a power series centered at c, precisely one of the following is true:

1) The series converges only at c (ALL power series converge at their center!!)
2) THE series converges for all x.
3) There exists an $R > 0$ such that the series converges for $|x-c| < R$ and diverges for $|x-c| > R$.

R is called the **radius of convergence** of the power series.

In part 1) the radius is 0.
In part 2), the radius is ∞
In part 3) The corresponding domain, $[(c-R, c+R)]$, is called the **interval of convergence** or the **domain** of the power series.

Note: to determine if the endpoints are included or not, we must test each endpoint independently.
Note2: We typically use the RATIO TEST to determine the radius of convergence.

Example 1:

Find the nth term for the power series $f(x) = e^x$, then find the radius and interval of convergence for the representative power series.

Example 2:

Find the radius of convergence and the interval of convergence. Be sure to check the endpoints.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-5)^n}{n2^n}$

(b) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{x}{3}\right)^n$

(c) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

(d) $\sum_{n=0}^{\infty} n!(x-3)^n$

We will now look at a special family of power series for which you're almost already acquainted: **Taylor and Maclaurin Series.**

> **Taylor Series centered at $x = c$:**
>
> $$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$
>
> Once again, if $c = 0$, the series is called a **Maclaurin series.**

Notice we now use an equal sign instead of an approximation sign. Do you know why???

Example 3:

Find a Taylor series for $f(x)=e^{5x}$ centered at $c=2$. Give the first four nonzero terms and the general term.

There are three special Maclaurin series you must know. These are the series for e^x, $\sin x$, and $\cos x$. They converge for all x.

Example 4:

If $f(x)=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\frac{x^n}{n!}+\cdots$, Find $f'(x)$ and $f'(0)$. Do you recognize this familiar function?

Example 5:

If $f(x)=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\frac{x^n}{n!}+\cdots$, and $F(x)=\int f(x)dx$ and $F(0)=1$, find $F(x)$. Do you recognize this familiar function?

Example 6:

Derive a Maclaurin series for $\sin x$, then take its derivative to derive a Maclaurin series for $\cos x$. Be sure to include the general term. You may have to adjust your index for $\cos x$ to make it look "pretty."

Once we have these series memorized these series (and perhaps those for $\frac{1}{1-x}$ centered at $c = 0$ and $\ln x$ centered at $c = 1$), we can conveniently manipulate them to suit other similar transcendental functions..

You can manipulate these three special series (or any series we are given) to find other series by using the following techniques. Note: the radius of convergence may change, though)

1) Substitute into a series for x
2) Multiply or divide the series by a constant and/or a variable
3) Add or subtract two series
4) Differentiate or integrate a series (may change the interval, but not the radius of convergence)
5) Recognize the series as the sum of a geometric power series

Example 7:

Find a Maclaurin series for $f(x) = \sin\left(x^2\right)$. Find the first four nonzero terms and the general term.

Example 8:

Find a Maclaurin series for $g(x) = x\cos x$. Find the first four nonzero terms and the general term.

Example 9:

Find a Maclaurin series for $h(x) = \dfrac{e^x + e^{-x}}{2}$. Find the first four nonzero terms and the general term.

Key

*polynomial of n^{th} degree has finite number of terms (only approximates) → powers of x^n

If x is a variable, then an infinite series of the form $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots a_n x^n + \cdots$ is called a **power series**.

Series is a polynomial with infinitely many terms

→ a_n is rule of sequence that generates the coefficients

$$\sum_{n=0}^{\infty} a_n (x-c)^n \boxed{=} a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots a_n(x-c)^n + \cdots$$ is a power series **centered at** c, where c is a constant.

Taylor and Maclaurin Series are special cases of power series.

The equal sign above means that the left side equals the right side for all values in the domain. This means the above is an IDENTITY. But for WHAT values of x does the identity hold? We have to find them. For all such x values, we say the series CONVERGES. For the values of x for which the identity is NOT true, we say the series diverges.

→ this is where the function and series are exactly the same (share the same value)

→ that represents a function $f(x)$

For a power series centered at c, precisely one of the following is true:

1) The series converges only at c (ALL power series converge at their center!!)
2) THE series converges for all x. (where function and the infinite series will have exact same values everywhere)
3) There exists an $R > 0$ such that the series converges for $|x-c| < R$ and diverges for $|x-c| > R$.

(converges not everywhere but within a certain Radius, within a certain set of values)

R is called the **radius of convergence** of the power series.

In part 1) the radius is 0.
In part 2), the radius is ∞

→ radius is a value that is a certain distance from both sides of center.

In part 3) The corresponding domain, $[(c-R, c+R)]$, is called the **interval of convergence** or the **domain** of the power series.

lower bound, upper bound *we have to decide whether to include the endpoints by testing them.

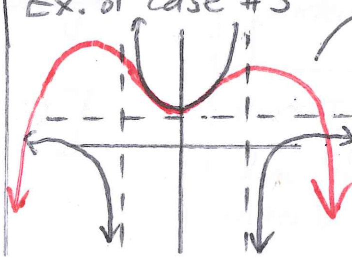
Note: to determine if the endpoints are included or not, we must test each endpoint independently.
Note2: We typically use the RATIO TEST to determine the radius of convergence.

Example of case #2:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

This polynomial will match the sin x graph exactly. In theory, these are one and the same

Ex. of case #3



→ Interval of convergence $(-1, 1)$

radius: 1

center: 0

Example 1:
Find the nth term for the power series $f(x) = e^x$, then find the radius and interval of convergence for the representative power series.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \text{ or } \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!}$$

(choose the simpler form of series)

Use Ratio Test: (*Important to have n^{th} term) to find Radius of Convergence

Treat 'x' as a constant

$$\lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \lim_{n\to\infty} \left| \frac{x}{n+1} \right| = 0 < 1 \text{ for all } x \text{ in the set of Real Numbers}$$

Radius $= \infty$ Interval: $(-\infty, \infty)$

centered at $x = 0$

*This shows that e^x has an infinite polynomial series that represents it perfectly.

Example 2: alternator only contributes to the sign, not the magnitude.

Find the radius of convergence and the interval of convergence. Be sure to check the endpoints.

$f(x)=$ (a) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-5)^n}{n2^n}$

treat x as a constant

$$\lim_{n\to\infty}\left|\frac{(x-5)^{n+1}}{(n+1)2^{n+1}}\cdot\frac{n2^n}{(x-5)^n}\right| = \lim_{n\to\infty}\left|\frac{(x-5)n}{2(n+1)}\right|$$

$$\frac{|x-5|}{2}\lim_{n\to\infty}\left|\frac{n}{n+1}\right| = 1 = \frac{|x-5|}{2} < 1$$

$|x-5|<2$ matches $|x-c|<r$ — center, within, radius

*will converge within r units of the center

Radius $=2$, center $c=5$ $\quad [(5-2, 5+2)]$

$[(3, 7)]$

Test $x=7$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(7-5)^n}{n2^n}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(2)^n}{n\cdot 2^n}$$ converges by AST.

Test $x=3$ $\quad \sum_{n=1}^{\infty}\frac{(-1)^{n+1}(-2)^n}{n2^n}$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cdot(-1)^n\cdot(2)^n}{n\cdot 2^n} = \frac{(-1)^{2n+1}}{n}$$

$$\sum_{n=1}^{\infty}\frac{-1}{n}$$ diverges

Interval of Convergence: $(3, 7]$

$f(x)=$ (b) $\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n = \sum_{n=0}^{\infty}\frac{x^n}{3^n}$ $\quad$ center $=0$ $\quad |x-0|<r$

$$\lim_{n\to\infty}\left|\frac{x^{n+1}}{3^{n+1}}\cdot\frac{3^n}{x^n}\right| = \lim_{n\to\infty}\left|\frac{x}{3}\right| = \left|\frac{x}{3}\right| < 1$$

$\frac{|x|}{3}<1 \rightarrow |x|<3$ $\quad |x-0|<3$ $\quad$ Radius $=3$

Possible I.O.C. $[(-3, 3)]$ $\quad$ center: $c=0$

Test $x=-3$

$$\sum_{n=0}^{\infty}\frac{(-3)^n}{3^n} = \sum_{n=0}^{\infty}\left(\frac{-3}{3}\right)^n = (-1)^n = 1-1+1-1+1-1$$

series diverges by n^{th} term test (oscillation)

Test $x=3$

$$\sum_{n=0}^{\infty}\frac{3^n}{3^n} = \sum_{n=0}^{\infty}(1)^n$$ diverges by n^{th} term test.

$\boxed{\text{Interval of convergence: } (-3, 3)}$

Root Test:

$$\lim_{n\to\infty}\sqrt[n]{\left(\frac{x}{3}\right)^n} = \left|\frac{x}{3}\right| < 1$$

Geometric (GST)

$$|r| = \left|\frac{x}{3}\right| < 1$$

$-3 < x < 3$

We will now look at a special family of power series for which you're almost already acquainted: **Taylor and Maclaurin Series.**

Taylor Series centered at $x=c$:

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdots = \sum_{n=0}^{\infty}\frac{f^{(n)}(c)}{n!}(x-c)^n$$

Once again, if $c=0$, the series is called a **Maclaurin series.**

Notice we now use an equal sign instead of an approximation sign. Do you know why???

(c) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

Ratio Test:

$$\lim_{n\to\infty} \left| \frac{x^{2(n+1)+1}}{(2(n+1)+1)!} \cdot \frac{(2n+1)!}{x^{2n+1}} \right|$$

$$\lim_{n\to\infty} \left| \frac{x^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{x^{2n+1}} \right| = \lim_{n\to\infty} \left| \frac{x^2}{(2n+3)(2n+2)} \right| = 0 < 1$$

(limit is always less than 1 regardless of what x is)

Radius: ∞

I.O.C.: $(-\infty, \infty)$ center: $c = 0$

(d) $\sum_{n=0}^{\infty} n!(x-3)^n$

* If rule of sequence does not go to zero, then the series has no chance of converging.

$$\lim_{n\to\infty} \left| \frac{(n+1)!(x-3)^{n+1}}{n!(x-3)^n} \right|$$

$$= \lim_{n\to\infty} \left| (n+1)(x-3) \right| = \infty > 1$$

Radius: 0

I.O.C.: none

* series only converges at center $x = 3$.

We will now look at a special family of power series for which you're almost already acquainted: **Taylor and Maclaurin Series.**

Taylor Series centered at $x = c$:

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \overbrace{\frac{f^{(n)}(c)}{n!}(x-c)^n}^{n^{th}\text{ term}} + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

Once again, if $c = 0$, the series is called a **Maclaurin series.**

Notice we now use an equal sign instead of an approximation sign. Do you know why??? Infinitely many terms

$\frac{f^{(n)}(c)}{n!}(x-c)^n$ (nth term)

Example 3:
Find a Taylor series for $f(x)=e^{5x}$ centered at $c=2$. Give the first four nonzero terms and the general term. *we can modify existing Maclaurin Series only when center = 0.

Use Taylor Rule to generate derivatives.

$f(x)=e^{5x}=e^{5x}\quad f(2)=e^{10}$

$f'(x)=5e^{5x}=5e^{5x}\quad f'(2)=5e^{10}$

$f''(x)=25e^{5x}=5^2e^{5x}\quad f''(2)=5^2e^{10}$

$f'''(x)=125e^{5x}=5^3e^{5x}\quad f'''(2)=5^3e^{10}$

$f(x)=e^{5x}=e^{10}+5e^{10}(x-2)+\frac{5^2e^{10}}{2!}(x-2)^2+\frac{5^3e^{10}}{3!}(x-2)^3+\ldots\boxed{+\frac{5^ne^{10}}{n!}(x-2)^n+\ldots}$

*we have to recognize the pattern to write the nth term

There are three special Maclaurin series you must know. These are the series for e^x, $\sin x$, and $\cos x$. They converge for all x.

Example 4:

If $f(x)=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots+\frac{x^n}{n!}+\cdots$, Find $f'(x)$ and $f'(0)$. Do you recognize this familiar function?

$f'(x)=0+1+\frac{2x}{2!}+\frac{3x^2}{3!}+\frac{n\cdot x^{n-1}}{n!}\rightarrow\frac{x^{n-1}}{(n-1)!}\quad=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots+\frac{x^{n-1}}{(n-1)!}$

$f'(x)=e^x$

$f'(0)=e^0=1\qquad f'(0)=1+0+0+0+\ldots=\boxed{1}$

Example 5:

If $f(x)=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots+\frac{x^n}{n!}+\cdots$, and $F(x)=\int f(x)dx$ and $\underbrace{F(0)=1}_{\text{Initial condition.}}$, find $F(x)$. Do you recognize this familiar function?

*Take antiderivative of this series

$\int f(x)dx=x+\frac{x^2}{2}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots\frac{x^{n+1}}{(n+1)!}+\ldots+C$

$\int f(x)dx=C+x+\frac{x^2}{2}+\ldots\frac{x^n}{n!}$

$F(x)=C+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots$

$1=C+0+0+\ldots$

$\boxed{C=1}$

$F(x)=1+x+\frac{x^2}{2}+\frac{x^3}{3!}+\ldots+\frac{x^n}{n!}+\ldots$

Example 6:
Derive a Maclaurin series for $\sin x$, then take its derivative to derive a Maclaurin series for $\cos x$. Be sure to include the general term. You may have to adjust our index for $\cos x$ to make it look "pretty." simpler

$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots\frac{(-1)^n x^{2n+1}}{(2n+1)!}$ or $\frac{x^{2n-1}}{(2n-1)!}$

$\frac{d}{dx}\sin x=\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}+\ldots+\frac{(2n-1)x^{2n-2}}{(2n-1)!}\rightarrow\frac{x^{2n-2}}{(2n-2)!}$ or $\boxed{\frac{x^{2n}}{(2n)!}}\rightarrow\frac{(-1)^n x^{2n}}{(2n)!}$

Once we have these series memorized these series (and perhaps those for $\frac{1}{1-x}$ centered at $c=0$ and $\ln x$ centered at $c=1$), we can conveniently manipulate them to suit other similar transcendental functions..

You can manipulate these three special series (or any series we are given) to find other series by using the following techniques. Note: the radius of convergence may change, though)

1) Substitute into a series for x
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4) Differentiate or integrate a series (may change the interval, but not the radius of convergence)
5) Recognize the series as the sum of a geometric power series

*Important! We cannot change the center on an existing Maclaurin Series! (stays $c=0$)

Example 7: centered at $x=0$

Find a Maclaurin series for $f(x)=\sin\left(x^2\right)$. Find the first four nonzero terms and the general term.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \dots \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \dots + \frac{(x^2)^{2n+1}}{(2n+1)!} \to \frac{(-1)^n x^{4n+2}}{(2n+1)!}$$

Example 8:

Find a Maclaurin series for $g(x)=x\cos x$. Find the first four nonzero terms and the general term.

$$f(x) = \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \frac{(-1)^n x^{2n}}{(2n)!}$$

$$g(x) = x\cos x = x - \frac{x^3}{2!} + \frac{x^5}{4!} - \frac{x^7}{6!} + \dots \frac{(-1)^n x^{2n+1}}{(2n)!}$$

Example 9:

Find a Maclaurin series for $h(x)=\frac{e^x+e^{-x}}{2}$. Find the first four nonzero terms and the general term.

*construct this series one portion at a time.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots + \frac{(-1)^n x^n}{n!} + \dots$$

$$e^x + e^{-x} = 2 + \frac{2x^2}{2!} + \frac{2x^4}{4!} + \frac{2x^6}{6!} + \dots + \frac{2x^{2n}}{(2n)!} + \dots$$

$$\frac{1}{2}\left(e^x + e^{-x}\right) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots + \frac{x^{2n}}{(2n)!} + \dots$$