AP Calculus BC   9.7 Notes   Taylor Polynomials and Approximations

Polynomial functions can be used to approximate other elementary functions such as $\sin x$, e^x, and $\ln x$.

Example 1:
Find the equation of the tangent line for $f(x) = \sin x$ at $x = 0$, then use it to approximate $\sin(0.2)$. Is this an over or an under approximation of $\sin(0.2)$?

The equation of the tangent line used in Ex. 1 is called a **first-degree Taylor polynomial**. Taylor polynomials of higher degree can be used to obtain increasingly better approximations of non-polynomial functions within a certain **radius** from a **center of approximation** $x = c$.

Example 2:
On your calculator graph $y1 = \sin x$. Use the following window: $X[-9,9]$, $Y[-4,4]$. Now in $y2 =$, graph successively, adding an extra term each time, the following: $y2 : x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$

What do you notice? What is $y1(0)$? $y2(0)$? What is $y1(0.2)$? $y2(0.2)$?

<u>Definition of an *n*th-degree Taylor polynomial:</u>

If f has n derivatives at $x = c$, then the polynomial

$$P_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

is called the <u>*n*th-degree Taylor polynomial for f centered at c</u>, named after Brook Taylor, an English mathematician.

Note 1: A first-degree Taylor polynomial is a tangent line to f at c.

Note 2: $\frac{f^{(n)}(c)}{n!}$ is the coefficient of the $(x-c)^n$ term

If $c = 0$, then $P_n(x) = f(0) + f'(0)(x) + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n$ is called the <u>*n*th-degree Maclaurin polynomial for f</u>, named after Scottish mathematician, Colin Maclaurin.

Note:

For Taylor & Mac Polynomials, you MUST use a "squiggle." For example, $f(x) \approx P_n(x) = \cdots$

Example 3:

Find the Maclaurin polynomial of degree $n = 5$ for $f(x) = \sin x$. Then use $P_5(x)$ to approximate the value of $\sin(0.1)$ using correct notation. Find the error for your approximation and determine an interval in which $\sin(0.1)$ could actually live. Finally, compare your approximation to the actual value of $\sin(0.1)$. Is it in your interval?

Example 4:
Find the Taylor polynomial of degree $n = 6$ for $f(x) = \ln x$ at $c = 1$. Then use $P_6(x)$ to approximate the value of $\ln(1.1)$

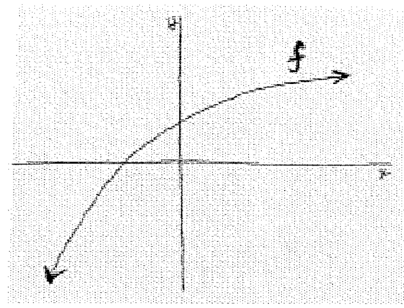
Example 5:
Suppose that g is a function which has continuous derivatives, and that $g(2) = 3$, $g'(2) = -4$, $g''(2) = 7$, $g'''(2) = -5$. Write the Taylor polynomial of degree 3 for g centered at 2.

Example 6:

Use a third-degree Taylor approximation of e^x for x near 0 to find $\lim_{x \to 0} \frac{e^x - 1}{2x}$, then compare it to the actual limit at zero..

Example 7:

Given that $P_2(x) = a + bx + cx^2$ is the second-degree Taylor polynomial for *f* about $x = 0$, what can you say about the signs of a, b, and c if *f* has the graph pictured at the right? Justify your answer.



Sometimes we can create Maclaurin Polynomials to approximate functions without having to derive them using Taylor's Theorem, but rather by modifying existing polynomials.

List the following Maclaurin Polynomials centered at c = 0:

a) $\sin x \approx$

b) $\cos x \approx$

c) $e^x \approx$

Example 8:

List the first four non-zero terms of the Maclaurin Polynomials for $f(x) = \sin x$, $f(x) = \cos x$, and $f(x) = e^x$, the find the following Maclaurin Polynomials.

(a) $g(x) = \sin(2x)$ (b) $g(x) = x\cos(x)$ (c) $g(x) = 4e^{x^2}$

AP Calculus BC 9.7 Notes Taylor Polynomials and Approximations

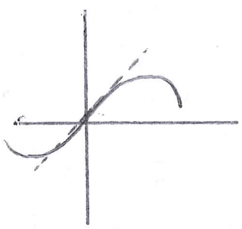
Key

(transcendental functions)

Polynomial functions can be used to approximate other elementary functions such as $\sin x$, e^x, and $\ln x$.

Example 1:
Find the equation of the tangent line for $f(x) = \sin x$ at $x = 0$, then use it to approximate $\sin(0.2)$. Is this an over or an under approximation of $\sin(0.2)$?

*Think linear approximation: $y - y_1 = f'(c)(x - x_1)$



$f'(x) = \cos x$

$f'(0) = \cos(0) = 1$

$f'(0) = 1$

$f(0) = 0$

$y - 0 = 1(x - 0)$

$y = x$

$y(0.2) = 0.2$ and $\sin(0.2) = 0.198$

The tangent line will produce an over-approximation of $\sin(0.2)$.

The equation of the tangent line used in Ex. 1 is called a **first-degree Taylor polynomial.** Taylor polynomials of higher degree can be used to obtain increasingly better approximations of non-polynomial functions within a certain **radius** from a **center of approximation** $x = c$.

Example 2:
On your calculator graph $y1 = \sin x$. Use the following window: $X[-9,9]$, $Y[-4,4]$. Now in $y2 =$, graph successively, adding an extra term each time, the following: $\boxed{y2 : x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}}$ ← Let's see if this polynomial is a better model, create a better copy of the sin x graph.

What do you notice? What is $y1(0)$? $y2(0)$? What is $y1(0.2)$? $y2(0.2)$?

$y_1 = \sin x$ $\qquad y_2 = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$

$y_1(0) = 0$ $\qquad y_2(0) = 0$

$y_1(0.2) = 0.1986$ $\qquad y_2(0.2) = 0.198669$

$y_1(\pi/2) = 1$ $\qquad y_2(\pi/2) = 0.9998$

*The more terms we add on the further we can get the polynomial to wrap itself around the curve over a larger radius (larger interval)

Taylor polynomial is a polynomial that will approximate these other function's values in a region that is nearby the "center"

*Tangent line is basically a first degree taylor polynomial

Definition of an *n*th-degree Taylor polynomial: center of approximation

n^{th} derivative evaluated at center "c" over n! times $(x-c)^n$

If f has n derivatives at $x=c$, then the polynomial

$$\underbrace{P_n(x)}_{n^{th}\text{ degree Taylor polynomial}} = \underbrace{f(c)+f'(c)(x-c)}_{\text{Tangent line}}+\frac{f''(c)}{2!}(x-c)^2+\cdots+\underbrace{\frac{f^{(n)}(c)}{n!}}_{\text{coefficients of polynomial}}(x-c)^n$$

is called the <u>*n*th-degree Taylor polynomial for f centered at c</u>, named after Brook Taylor, an English mathematician.

Note 1: A first-degree Taylor polynomial is a tangent line to f at c.

Note 2: $\frac{f^{(n)}(c)}{n!}$ is the coefficient of the $(x-c)^n$ term

If $c=0$, then $P_n(x)=f(0)+f'(0)(x)+\frac{f''(0)}{2!}x^2+\cdots+\frac{f^{(n)}(0)}{n!}x^n$ is called the <u>*n*th-degree Maclaurin polynomial for f</u>, named after Scottish mathematician, Colin Maclaurin.

Note:

For Taylor & Mac Polynomials, you MUST use a "squiggle." For example, $f(x)\approx P_n(x)=\cdots$

Example 3: center $c=0$ $\qquad y=y_0+m(x-x_0)$

Find the <u>Maclaurin polynomial</u> of degree $n=5$ for $f(x)=\sin x$. Then use $P_5(x)$ to approximate the value of $\sin(0.1)$ using correct notation. Find the error for your approximation and determine an interval in which $\sin(0.1)$ could actually live. Finally, compare your approximation to the actual value of $\sin(0.1)$. Is it in your interval?

$f(x)=\sin x$, $f(0)=0$
$f'(x)=\cos x$, $f'(0)=1$
$f''(x)=-\sin x$, $f''(0)=0$
$f'''(x)=-\cos x$, $f'''(0)=-1$
$f^4(x)=\sin x$, $f^4(0)=0$
$f^5(x)=\cos x$, $f^5(0)=1$

these values are the numerators of the coefficients of our terms

$$P_5(x)=0+1(x-0)+\frac{0}{2!}(x-0)^2+\frac{-1}{3!}(x-0)^3+\frac{0}{4!}(x-0)^4+\frac{1}{5!}(x-0)^5$$

$$P_5(x)=x-\frac{x^3}{3!}+\frac{x^5}{5!}$$

*We can always use Taylor's Rule to generate a polynomial of n^{th} degree. However, if we can recognize the pattern for a specific function, we can use the pattern to generate subsequent terms in the polynomial.

Example 4:

Find the Taylor polynomial of degree $n=6$ for $f(x)=\ln x$ at $c=1$. Then use $P_6(x)$ to approximate the value of $\ln(1.1)$

* Taylor Polynomial: $\dfrac{f^{(n)}(c)}{n!}(x-c)^n$

$f(x)=\ln x,\ f(1)=0$

$f'(x)=\frac{1}{x},\ f'(1)=1$

$f''(x)=\frac{-1}{x^2},\ f''(1)=-1$

$f'''(x)=\frac{2}{x^3},\ f'''(1)=2$

$f^4(x)=\frac{-6}{x^4},\ f^4(1)=-6$

$f^5(x)=24/x^5,\ f^{(5)}(1)=24$

$f^{(6)}(x)=\frac{-120}{x^6},\ f^{(6)}(1)=-120$

$$P_6(x)=0+1(x-1)-\frac{1}{2!}(x-1)^2+\frac{2}{3!}(x-1)^3-\frac{6}{4!}(x-1)^4+\frac{24}{5!}(x-1)^5-\frac{120}{6!}(x-1)^6$$

$$P_6(x)=(x-1)-\frac{1}{2}(x-1)^2+\frac{1}{3}(x-1)^3-\frac{1}{4}(x-1)^4+\frac{1}{5}(x-1)^5-\frac{1}{6}(x-1)^6$$

$$\ln(1.1)\approx P_6(1.1)=(0.1)-\frac{1}{2}(0.1)^2+\frac{1}{3}(0.1)^3-\frac{1}{4}(0.1)^4+\frac{1}{5}(0.1)^5-\frac{1}{6}(0.1)^6=0.09531$$

error bound $\rightarrow$ $R_6(1.1)\le\left|\dfrac{(0.1)^7}{7}\right|$

Example 5:

Suppose that g is a function which has continuous derivatives, and that $g(2)=3$, $g'(2)=-4$, $g''(2)=7$, $g'''(2)=-5$. Write the Taylor polynomial of degree 3 for g centered at 2.

$c=2$

* $\dfrac{f^n(c)}{n!}(x-c)^n$

$$\boxed{T_3(x)=3-4(x-2)+\frac{7}{2!}(x-2)^2-\frac{5}{3!}(x-2)^3}$$

Example 6:

Use a third-degree Taylor approximation of e^x for x near 0 to find $\lim_{x\to 0}\dfrac{e^x-1}{2x}$, then compare it to the actual limit at zero..

* nth term Taylor polynomial $=\dfrac{f^n(c)}{n!}(x-c)^n$ ← nth derivative

$f(x)=e^x,\ f(0)=1$

$f'(x)=e^x,\ f'(0)=1$

$f''(x)=e^x,\ f''(0)=1$

$f'''(x)=e^x,\ f'''(0)=1$

$$e^x\approx T_3(x)=1+1(x-0)+\frac{1}{2!}(x-0)^2+\frac{1}{3!}(x-0)^3$$

$$=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}$$ ← cubic Taylor polynomial

$$\lim_{x\to 0}\frac{e^x-1}{2x}=\lim_{x\to 0}\frac{1+x+\frac{x^2}{2!}+\frac{x^3}{3!}-1}{2x}$$

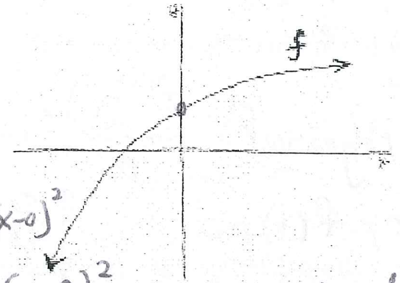
$$\lim_{x\to 0}\frac{x+\frac{x^2}{2!}+\frac{x^3}{3!}}{2x}=\lim_{x\to 0}\frac{1+\frac{x}{2!}+\frac{x^3}{3!}}{2}=\boxed{\frac{1}{2}}$$

$$\lim_{x\to 0}\frac{e^x-1}{2x}\xrightarrow{L'H}\lim_{x\to 0}\frac{e^x}{2}=\boxed{\frac{1}{2}}$$

centered at zero
$c=0$

Example 7:

Given that $P_2(x) = a + bx + cx^2$ is the second-degree Taylor polynomial for f about $x = 0$, what can you say about the signs of a, b, and c if f has the graph pictured at the right? Justify your answer.



* center at $x=0$ because $P_2(x) = a + b(x-0) + c(x-0)^2$

If centered at $x=3$, $P_2(x) = a + b(x-3) + c(x-3)^2$

function value at $x=0$ — slope of function at $x=0$

$$a = \frac{f^{(0)}(0)}{0!} = \frac{f(0)}{1} = f(0) > 0 \qquad b = \frac{f'(0)}{1!} = f'(0) > 0$$

coefficient of linear term

$$c = \frac{f''(0)}{2!} = \frac{1}{2}f''(0) < 0 \text{ (concave down)}$$

coefficient of squared term

Sometimes we can create Maclaurin Polynomials to approximate functions without having to derive them using Taylor's Theorem, but rather by modifying existing polynomials.

List the following Maclaurin Polynomials centered at c = 0:

a) $\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}$ (sinx is an odd function)

b) $\cos x \approx \frac{d}{dx}(\sin x) = 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \frac{7x^6}{7!} + \frac{9x^8}{9!}$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!}$$ (cosx is an even function)

c) $e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}$

Example 8:

List the first four non-zero terms of the Maclaurin Polynomials for $f(x) = \sin x$, $f(x) = \cos x$, and $f(x) = e^x$, the find the following Maclaurin Polynomials. *Modify existing taylor polynomials

(a) $g(x) = \sin(2x)$

$$\sin(2x) \approx (2x) - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \frac{(2x)^7}{7!}$$

$$g(x) \approx 2x - \frac{8}{3!}x^3 + \frac{32}{5!}x^5 - \frac{128}{7!}x^7$$

(b) $g(x) = x\cos(x)$

$$g(x) \approx x\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}\right)$$

$$g(x) \approx x - \frac{x^3}{2!} + \frac{x^5}{4!} - \frac{x^7}{6!}$$

(c) $g(x) = 4e^{x^2}$

$$e^{x^2} \approx 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{6!} + \frac{x^8}{4!}$$

$$4e^{x^2} \approx 4\left(1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{6!} + \frac{x^8}{4!}\right)$$

$$4e^{x^2} \approx 4 + 4x^2 + 2x^4 + \frac{4x^6}{3!} + \frac{4x^8}{4!}$$