Name: ______________________________ Period: __________

BC Calculus

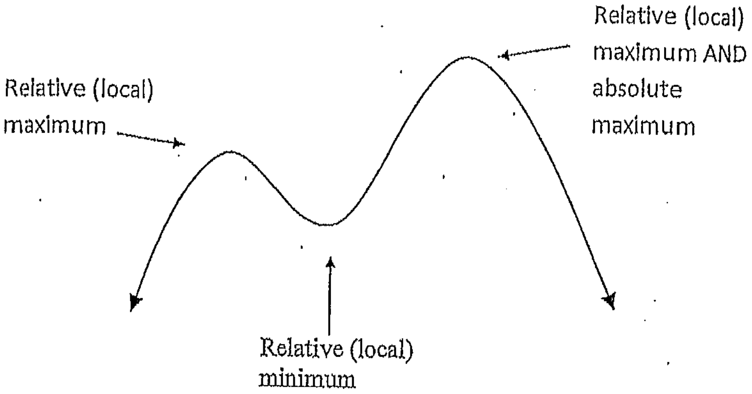
Unit 5

Applications of Differentiation (Part 2)

(Extreme Value Theorem, Mean Value Theorem, Curve Sketching, Derivative Graphs, & Optimization)

Calculus Ch. 5.1 Notes: Extrema and Critical Points

Extrema : maximums and minimums

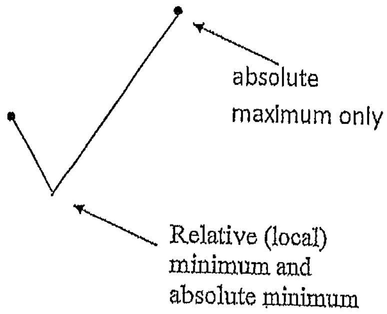


Relative (local) extrema: any "hills and valleys" of graph

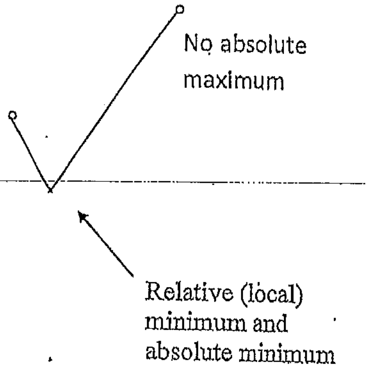
Absolute (global) extrema: highest or lowest points on the entire graph

*holes and ±∞ can not be considered as absolute extrema.

Closed interval



Open Interval



(EVT)

Extreme Value Theorem: If a function is continuous on a closed interval, then it has **both** an (absolute) minimum and an (absolute) maximum on that interval.

Fermat's Theorem: If a function is continuous on a closed interval, then the absolute extreme will **either** be at the **a)** critical numbers or **b)** at an endpoint.

Critical numbers (values) : x-values in the domain of a function where the derivative of a function is either 0 or undefined.

*Relative extrema ONLY occur at critical numbers, but not all critical numbers are where relative extrema occur.

*Maximum and minimum values refer to the **y-values** of the point.

②

Steps: * Confirm continuous function on closed interval

1. Find critical points
   a. Set $f'(x) = 0$
   b. Find where $f'(x)$ is undefined (Set denominator of $f'(x) = 0$)
2. Plug all critical points and endpoints into $f(x)$
3. Compare y-values to determine absolute maximum(s) and absolute minimum(s)

Find all critical numbers for each. What are the values of the absolute extrema?

Example 1: $f(x) = 3x^4 - 4x^3$ on $[0, 2]$

Example 2: $f(x) = (x-1)^{\frac{2}{3}}$ on $[-1, 0]$

Example 3: $f(x) = \frac{4}{3}x\sqrt{3-x}$ on $[0, 3]$

③

Chapter 5 Curve Sketching 5.1 EVT Classwork Problems

Finding Extrema on a Closed Interval In Exercises 17–36, find the absolute extrema of the function on the closed interval.

19. $g(x) = 2x^2 - 8x,\ [0, 6]$

21. $f(x) = x^3 - \frac{3}{2}x^2,\ [-1, 2]$

23. $y = 3x^{2/3} - 2x,\ [-1, 1]$

24. $g(x) = \sqrt[3]{x},\ [-8, 8]$

26. $f(x) = \frac{2x}{x^2 + 1},\ [-2, 2]$

28. $h(t) = \frac{t}{t + 3},\ [-1, 6]$

4

Now find the absolute maximum of V on the interval $\left[\frac{r_0}{2}, r_0\right]$.

$$V'(r) = \frac{4k\,r_0}{c}r^3 - \frac{5k}{c}r^4 = \frac{k}{c}r^3(4r_0 - 5r)$$

The only critical number in the interval $\left(\frac{r_0}{2}, r_0\right)$ is $r = \frac{4r_0}{5}$.

Evaluate V at the critical number and at the endpoints, $\frac{r_0}{2}$ and r_0.

r	$V(r) = k\left(\frac{r_0 - r}{c}\right)r^4$
$\frac{r_0}{2}$	$k\left(\frac{r_0 - \frac{r_0}{2}}{c}\right)\left(\frac{r_0}{2}\right)^4 = \frac{k\,r_0^5}{32c} \approx \frac{0.031\,k\,r_0^5}{c}$
$\frac{4r_0}{5}$	$k\left(\frac{r_0 - \frac{4r_0}{5}}{c}\right)\left(\frac{4r_0}{5}\right)^4 = \frac{k}{c}\cdot\frac{4^4\,r_0^5}{5^5} = \frac{256\,kr_0^5}{3125c} \approx \frac{0.082\,k\,r_0^5}{c}$
r_0	0

The largest of these three values is $\frac{256\,kr_0^5}{3125c}$. So, the maximum air flow occurs when the radius of the windpipe is $\frac{4r_0}{5}$, that is, when the windpipe contracts by 20%. ■

5.1 Assess Your Understanding

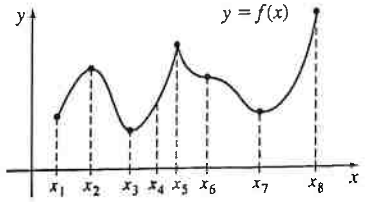
Concepts and Vocabulary

1. *True or False* Any function f that is defined on a closed interval $[a, b]$ has both an absolute maximum value and an absolute minimum value.
2. *Multiple Choice* A number c in the domain of a function f is called a(n)
[(**a**) extreme value (**b**) critical number (**c**) local number]
of f if either $f'(c) = 0$ or $f'(c)$ does not exist.
3. *True or False* At a critical number, there is a local extreme value.
4. *True or False* If a function f is continuous on a closed interval $[a, b]$, then its absolute maximum value is found at a critical number.
5. *True or False* The Extreme Value Theorem tells us where the absolute maximum and absolute minimum can be found.
6. *True or False* If f is differentiable on the interval $(0, 4)$ and $f'(2) = 0$, then f has a local maximum or a local minimum at 2.

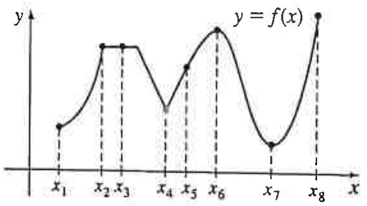
Skill Building

In Problems 7 and 8, use the graphs to determine whether the function f has an absolute extremum and/or a local extremum or neither at x_1, x_2, x_3, x_4, x_5, x_6, x_7, and x_8.

PAGE 309 **7.**



8.



In Problems 9–12, provide a graph of a continuous function f that has the following properties:

PAGE 309 **9.** domain $[0, 8]$, absolute maximum at 0, absolute minimum at 3, local minimum at 7

10. domain $[-5, 5]$, absolute maximum at 3, absolute minimum at -3

11. domain $[3, 10]$ and has no local extreme points

12. has no absolute extreme values, is differentiable at 4 and has a local minimum at 4, is not differentiable at 0 but has a local maximum at 0

In Problems 13–36, find the critical numbers, if any, of each function.

13. $f(x) = x^2 - 8x$

14. $f(x) = 1 - 6x + x^2$

15. $f(x) = x^3 - 3x^2$

16. $f(x) = x^3 - 6x$

17. $f(x) = x^4 - 2x^2 + 1$

18. $f(x) = 3x^4 - 4x^3$

19. $f(x) = x^{2/3}$

20. $f(x) = x^{1/3}$

21. $f(x) = 2\sqrt{x}$

22. $f(x) = 4 - \sqrt{x}$

23. $f(x) = x + \sin x$, $0 \le x \le \pi$

24. $f(x) = x - \cos x$, $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

5

25. $f(x)=x\sqrt{1-x^2}$

26. $f(x)=x^2\sqrt{2-x}$

27. $f(x)=\dfrac{x^2}{x-1}$

28. $f(x)=\dfrac{x}{x^2-1}$

29. $f(x)=(x+3)^2(x-1)^{2/3}$

30. $f(x)=(x-1)^2(x+1)^{1/3}$

31. $f(x)=\dfrac{(x-3)^{1/3}}{x-1}$

32. $f(x)=\dfrac{(x+3)^{2/3}}{x+1}$

33. $f(x)=\dfrac{\sqrt[3]{x^2-9}}{x}$

34. $f(x)=\dfrac{\sqrt[3]{4-x^2}}{x}$

35. $f(x)=\begin{cases} 3x & \text{if } 0\le x<1 \\ 4-x & \text{if } 1\le x\le 2 \end{cases}$

36. $f(x)=\begin{cases} x^2 & \text{if } 0\le x<1 \\ 1-x^2 & \text{if } 1\le x\le 2 \end{cases}$

In Problems 37–64, find the absolute maximum value and absolute minimum value of each function on the indicated interval. Notice that the functions in Problems 37–58 are the same as those in Problems 13–34 above.

37. $f(x)=x^2-8x$ on $[-1, 10]$

38. $f(x)=1-6x+x^2$ on $[0, 4]$

39. $f(x)=x^3-3x^2$ on $[1, 4]$

40. $f(x)=x^3-6x$ on $[-1, 1]$

41. $f(x)=x^4-2x^2+1$ on $[0, 2]$

42. $f(x)=3x^4-4x^3$ on $[-2, 0]$

43. $f(x)=x^{2/3}$ on $[-1, 1]$

44. $f(x)=x^{1/3}$ on $[-1, 1]$

45. $f(x)=2\sqrt{x}$ on $[1, 4]$

46. $f(x)=4-\sqrt{x}$ on $[0, 4]$

47. $f(x)=x+\sin x$ on $[0, \pi]$

48. $f(x)=x-\cos x$ on $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

49. $f(x)=x\sqrt{1-x^2}$ on $[-1, 1]$

50. $f(x)=x^2\sqrt{2-x}$ on $[0, 2]$

51. $f(x)=\dfrac{x^2}{x-1}$ on $\left[-1, \dfrac{1}{2}\right]$

52. $f(x)=\dfrac{x}{x^2-1}$ on $\left[-\dfrac{1}{2}, \dfrac{1}{2}\right]$

53. $f(x)=(x+3)^2(x-1)^{2/3}$ on $[-4, 5]$

54. $f(x)=(x-1)^2(x+1)^{1/3}$ on $[-2, 7]$

55. $f(x)=\dfrac{(x-3)^{1/3}}{x-1}$ on $[2, 11]$

56. $f(x)=\dfrac{(x+3)^{2/3}}{x+1}$ on $[-4, -2]$

57. $f(x)=\dfrac{\sqrt[3]{x^2-9}}{x}$ on $[3, 6]$

58. $f(x)=\dfrac{\sqrt[3]{4-x^2}}{x}$ on $[-4, -1]$

59. $f(x)=e^x-3x$ on $[0, 1]$

60. $f(x)=e^{\cos x}$ on $[-\pi, 2\pi]$

61. $f(x)=\begin{cases} 2x+1 & \text{if } 0\le x<1 \\ 3x & \text{if } 1\le x\le 3 \end{cases}$

62. $f(x)=\begin{cases} x+3 & \text{if } -1\le x\le 2 \\ 2x+1 & \text{if } 2<x\le 4 \end{cases}$

63. $f(x)=\begin{cases} x^2 & \text{if } -2\le x<1 \\ x^3 & \text{if } 1\le x\le 2 \end{cases}$

64. $f(x)=\begin{cases} x+2 & \text{if } -1\le x<0 \\ 2-x & \text{if } 0\le x\le 1 \end{cases}$

Applications and Extensions

In Problems 65–68, for each function f:

(a) Find the derivative f'.

(b) Use technology to find the critical numbers of f.

(c) Graph f and describe the behavior of f suggested by the graph at each critical number.

65. $f(x)=3x^4-2x^3-21x^2+36x$

66. $f(x)=x^2+2x-\dfrac{2}{x}$

67. $f(x)=\dfrac{(x^2-5x+2)\sqrt{x+5}}{\sqrt{x^2+2}}$

68. $f(x)=\dfrac{(x^2-9x+16)\sqrt{x+3}}{\sqrt{x^2-4x+6}}$

In Problems 69 and 70, for each function f:

(a) Find the derivative f'.

(b) Use technology to find the absolute maximum value and the absolute minimum value of f on the closed interval [0, 5].

(c) Graph f. Are the results from (b) supported by the graph?

69. $f(x)=x^4-12.4x^3+49.24x^2-68.64x$

70. $f(x)=e^{-x}\sin(2x)+e^{-x/2}\cos(2x)$

71. **Cost of Fuel** A truck has a top speed of 75 mi/h, and when traveling at the rate of x mi/h, it consumes fuel at the rate of $\dfrac{1}{200}\left(\dfrac{2500}{x}+x\right)$ gal/mi. If the price of fuel is \$3.60/ gal, the cost C (in dollars) of driving 200 mi is given by

$$C(x)=3.60\cdot\left(\frac{2500}{x}+x\right)$$

(a) What is the most economical speed for the truck to travel? Use the interval [10, 75].

(b) Graph the cost function C.

⑥

5.1 – Extreme Value Theorem (EVT) - AP Practice Problems (p. 319)

1. The critical numbers of $g(x) = \sin x + \cos x$ on the open interval $(0, 2\pi)$ are

(A) $\frac{\pi}{4}$ (B) $\frac{3\pi}{4}$ and $\frac{7\pi}{4}$

(C) $\frac{\pi}{4}$ and $\frac{5\pi}{4}$ (D) $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$ and $\frac{7\pi}{4}$

2. On the closed interval $[0, 2\pi]$, the absolute minimum of $f(x) = e^{\sin x}$ occurs at

(A) 0 (B) $\frac{\pi}{2}$ (C) $\frac{3\pi}{2}$ (D) 2π

3. The maximum value of $f(x) = 2x^3 - 15x^2 + 36x$ on the closed interval $[0, 4]$ is

(A) 28 (B) 30 (C) 32 (D) 48

4. On the closed interval $[0, 5]$, the function $f(x) = 3 - |x - 1|$ has:

(A) both an absolute maximum and an absolute minimum.
(B) an absolute maximum but no absolute minimum.
(C) no absolute maximum but an absolute minimum.
(D) an absolute maximum and two absolute minima.

5. The critical numbers of the function

$$f(x) = \begin{cases} x^2 + 1 & \text{if } -2 \le x \le 1 \\ 3x^2 - 4x + 3 & \text{if } 1 < x \le 3 \end{cases} \quad \text{are}$$

(A) 0 and 1 (B) 0 and $\frac{2}{3}$ (C) 0, $\frac{2}{3}$, and 1 (D) 0

6. $f'(x) = x \sin^2 x - \frac{1}{x}$ is the derivative of a function f. How many critical numbers does f have on the open interval $(0, 2\pi)$?

(A) 1 (B) 3 (C) 4 (D) 5

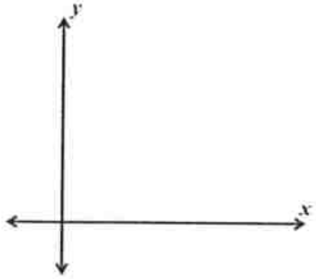
⑧

AP Calculus – 5.2 Notes – Mean Value Theorem (MVT) and Rolle's Theorem

We use the MVT to justify conclusions about a function over an interval.

Mean Value Theorem:

If a function f is continuous over the interval and differentiable over the interval then there exists a point within that open interval where the instantaneous rate of change equals the average rate of change over the interval.



1. Use the function $f(x) = -x^2 + 3x + 10$ to answer the following.
   a. On the interval $[2, 6]$, what is the average rate of change?

   b. On the interval $(2, 6)$, when does the instantaneous rate of change equal the average rate of change?

MVT vs IVT

Mean Value Theorem MVT	Intermediate Value Theorem IVT
• The derivative (instantaneous rate of change) must equal the average rate of change somewhere in the interval.	• On a given interval, you will have a y-value at each of the end points of the interval. Every y-value exists between these two y-values at least once in the interval.

2.

t minutes	0	5	15	20	30
$h(t)$ feet	0	40	70	65	80

A hot air balloon is launched into the air with a human pilot. The twice-differentiable function h models the balloon's height, measured in feet, at time t, measured in minutes. The table above gives values of the $h(t)$ of the balloon at selected times t.

a. For $5 \le t \le 15$, must there be a time t when the balloon is 50 feet in the air? Justify your answer.

b. For $20 \le t \le 30$, must there be a time t when the balloon's velocity is 1.5 feet per minute? Justify your answer.

Practice:

Skater Sully is riding a skateboard back and forth on a street that runs north/south. The twice-differentiable function S models Sully's position on the street, measured by how many meters north he is from his starting point, at time t, measured in seconds from the start of his ride. The table below gives values of $S(t)$ at selected times t.

t seconds	0	20	30	60
$S(t)$ meters	0	-5	7	40

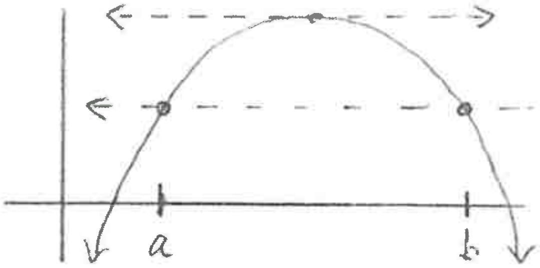
a. For $0 \le t \le 20$, must there be a time t when Sully is 2 meters south of his starting point? Justify your answer.

b. For $30 \le t \le 60$, must there be a time t when Sully's velocity is 1.1 meters per second? Justify your answer.

Rolle's Theorem: If a function, f(x) is continuous on [a, b], differentiable on (a, b), and f(a) = f(b), then there must be at least one point on the function where the slope of the tangent (derivative) is 0.

*In other words, if the endpoints have the same y-values, then we can guarantee a relative maximum or relative minimum somewhere between the endpoints

*Rolle's Theorem is just a specific case of the Mean Value Theorem



$$\frac{f(b)-f(a)}{b-a} = 0$$

Rolle's Theorem Steps:

1. Check Continuity (no breaks between endpoints)
2. Check Differentiability (smooth curve between endpoints)

Note, all polynomials are continuous and differentiable everywhere

3. Test endpoints. Does f(a) = f(b)? If not ,then Rolle's fails / does not apply
4. If yes, then set f'(x) = 0 and solve for x

Example 2: Determine if Rolle's theorem can be applied to $f(x) = x^2 - 3x + 2$ on the interval [1, 2]. If so, find the value of c such that f '(c) = 0.

Using the Mean Value Theorem, find where the instantaneous rate of change is equivalent to the average rate of change.

4. $y = x^2 - 5x + 2$ on $[-4, -2]$	5. $y = \sin 3x$ on $[0, \pi]$

6. $y = (-5x + 15)^{\frac{1}{2}}$ on $[1, 3]$

7. $y = e^x$ on $[0, \ln 2]$

Calculator active problem

8. A particle moves along the x-axis so that its position at any time $t \geq 0$ is given by $x(t) = t^3 - 3t^2 + t + 1$. For what values of t, $0 \leq t \leq 2$, is the particle's instantaneous velocity the same as its average velocity on the closed interval $[0, 2]$?

No calculator on this problem.

9. The table below gives selected values of a function f. The function is twice differentiable with $f''(x) > 0$.

x	$f(x)$
3	12.5
5	13.9
7	16.1

Which of the following could be the value of $f'(5)$?

(A) 0.5 (B) 0.7 (C) 0.9 (D) 1.1 (E) 1.3

11.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	3	8	2	4
2	6	3	1	2
3	5	-3	6	3
4	-2	6	3	5

The functions f and g are differentiable for all real numbers. The table above gives values of the functions and their first derivatives at selected values of x. The function h is given by $h(x) = f(g(x)) + 2$. Must there be a value c for $2 < c < 4$ such that $h'(c) = 1$.

⑫

Chapter 5 Curve Sketching 5.2 MVT and Rolle's Classwork Problems

If the Mean Value Theorem cannot be applied, explain why not.

39. $f(x) = x^3 + 2x, \quad [-1, 1]$

40. $f(x) = x^4 - 8x, \quad [0, 2]$

41. $f(x) = x^{2/3}, \quad [0, 1]$

42. $f(x) = \dfrac{x+1}{x}, \quad [-1, 2]$

35. Mean Value Theorem Consider the graph of the function $f(x) = -x^2 + 5$ (see figure on next page).

(a) Find the equation of the secant line joining the points $(-1, 4)$ and $(2, 1)$.

(b) Use the Mean Value Theorem to determine a point c in the interval $(-1, 2)$ such that the tangent line at c is parallel to the secant line.

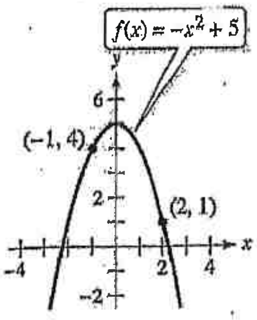


Figure for 35

(f) The graph of f is increasing when $f'(x) > 0$; that is, when the graph of f' is above the x-axis. This occurs on the intervals $(-2, 0)$ and $(0, 4)$. Since f is continuous for all real numbers, f is increasing on $[-2, 4]$.

(g) The graph of f is decreasing when $f'(x) < 0$; that is, when the graph of f' is below the x-axis. This occurs on the intervals $(-\infty, -2)$ and $(4, \infty)$. Since f is continuous for all x, f is decreasing on $(-\infty, -2]$ and $[4, \infty)$. ■

NOW WORK Problem 47 and AP® Practice Problems 2 and 6.

Application: Agricultural Economics

EXAMPLE 7 Determining Crop Yield*

A variation of the von Liebig model states that the yield $f(x)$ of a plant, measured in bushels, responds to the amount x of potassium in a fertilizer according to the following square root model:

$$f(x) = -0.057 - 0.417x + 0.852\sqrt{x}$$

For what amounts of potassium will the yield increase? For what amounts of potassium will the yield decrease?

Solution

The yield is increasing when $f'(x) > 0$.

$$f'(x) = -0.417 + \frac{0.426}{\sqrt{x}} = \frac{-0.417\sqrt{x} + 0.426}{\sqrt{x}}$$

Now $f'(x) > 0$ when

$$\begin{aligned} -0.417\sqrt{x} + 0.426 &> 0 \\ 0.417\sqrt{x} &< 0.426 \\ \sqrt{x} &< 1.022 \\ x &< 1.044 \end{aligned}$$

The crop yield is increasing when the amount of potassium in the fertilizer is less than 1.044 and is decreasing when the amount of potassium in the fertilizer is greater than 1.044. ■

5.2 Assess Your Understanding

Concepts and Vocabulary

1. *True or False* If a function f is defined and continuous on a closed interval $[a, b]$, differentiable on the open interval (a, b), and if $f(a) = f(b)$, then Rolle's Theorem guarantees that there is at least one number c in the interval (a, b) for which $f(c) = 0$.
2. In your own words, give a geometric interpretation of the Mean Value Theorem.
3. *True or False* If two functions f and g are differentiable on an open interval (a, b) and if $f'(x) = g'(x)$ for all numbers x in (a, b), then f and g differ by a constant on (a, b).
4. *True or False* When the derivative f' is positive on an open interval I, then f is positive on I.

Skill Building

In Problems 5–16, verify that each function satisfies the three conditions of Rolle's Theorem on the given interval. Then find all numbers c in (a, b) guaranteed by Rolle's Theorem.

5. $f(x) = x^2 - 3x$ on $[0, 3]$

6. $f(x) = x^2 + 2x$ on $[-2, 0]$

7. $g(x) = x^2 - 2x - 2$ on $[0, 2]$

8. $g(x) = x^2 + 1$ on $[-1, 1]$

PAGE 320 **9.** $f(x) = x^3 - x$ on $[-1, 0]$

10. $f(x) = x^3 - 4x$ on $[-2, 2]$

11. $f(t) = t^3 - t + 2$ on $[-1, 1]$

12. $f(t) = t^4 - 3$ on $[-2, 2]$

13. $s(t) = t^4 - 2t^2 + 1$ on $[-2, 2]$

14. $s(t) = t^4 + t^2$ on $[-2, 2]$

**Source:* Quirino Paris. (1992), The von Liebig Hypothesis, *American Journal of Agricultural Economics*, *74*(4), 1019–1028.

⑭

15. $f(x) = \sin(2x)$ on $[0, \pi]$

16. $f(x) = \sin x + \cos x$ on $[0, 2\pi]$

In Problems 17–20, state why Rolle's Theorem cannot be used with the function f on the given interval.

17. $f(x) = x^2 - 2x + 1$ on $[-2, 1]$ **18.** $f(x) = x^3 - 3x$ on $[2, 4]$

19. $f(x) = x^{1/3} - x$ on $[-1, 1]$ **20.** $f(x) = x^{2/5}$ on $[-1, 1]$

In Problems 21–30,

(a) Verify that each function satisfies the conditions of the Mean Value Theorem on the indicated interval.

(b) Find the number(s) c guaranteed by the Mean Value Theorem.

(c) Interpret the number(s) c geometrically.

21. $f(x) = x^2 + 1$ on $[0, 2]$

22. $f(x) = x + 2 + \dfrac{3}{x - 1}$ on $[2, 7]$

23. $f(x) = \ln \sqrt{x}$ on $[1, e]$ **24.** $f(x) = xe^x$ on $[0, 1]$

PAGE 322 **25.** $f(x) = x^3 - 5x^2 + 4x - 2$ on $[1, 3]$

26. $f(x) = x^3 - 7x^2 + 5x$ on $[-2, 2]$

27. $f(x) = \dfrac{x+1}{x}$ on $[1, 3]$ **28.** $f(x) = \dfrac{x^2}{x+1}$ on $[0, 1]$

29. $f(x) = \sqrt[3]{x^2}$ on $[1, 8]$ **30.** $f(x) = \sqrt{x - 2}$ on $[2, 4]$

In Problems 31–46, determine where each function is increasing and where each is decreasing.

PAGE 325 **31.** $f(x) = x^3 + 6x^2 + 12x + 1$ **32.** $f(x) = -x^3 + 3x^2 + 4$

33. $f(x) = x^3 - 3x + 1$ **34.** $f(x) = x^3 - 6x^2 - 3$

35. $f(x) = x^4 - 4x^2 + 1$ **36.** $f(x) = x^4 + 4x^3 - 2$

PAGE 326 **37.** $f(x) = x^{2/3}(x^2 - 4)$ **38.** $f(x) = x^{1/3}(x^2 - 7)$

39. $f(x) = |x^3 + 3|$ **40.** $f(x) = |x^2 - 4|$

41. $f(x) = 3 \sin x$ on $[0, 2\pi]$

42. $f(x) = \cos(2x)$ on $[0, 2\pi]$

43. $f(x) = xe^x$ **44.** $g(x) = x + e^x$

45. $f(x) = e^x \sin x, 0 \le x \le 2\pi$

46. $f(x) = e^x \cos x, 0 \le x \le 2\pi$

In Problems 47–50, the graph of the derivative function f' of a function f that is continuous for all real numbers is given.

(a) What is the domain of the derivative function f'?

(b) List the critical numbers of f.

(c) At what numbers x, if any, does the graph of f have a horizontal tangent line?

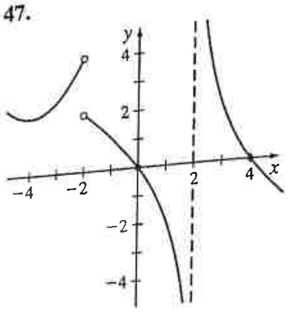
(d) At what numbers x, if any, does the graph of f have a vertical tangent line?

(e) At what numbers x, if any, does the graph of f have a corner?

(f) On what intervals is the graph of f increasing?

(g) On what intervals is the graph of f decreasing?

PAGE 327 **47.**



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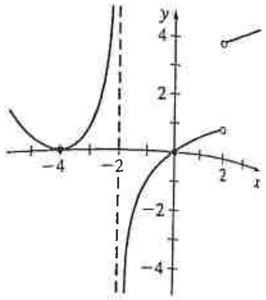

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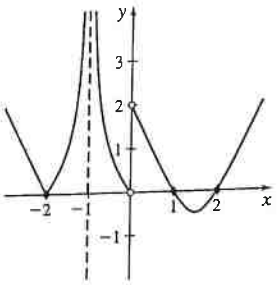

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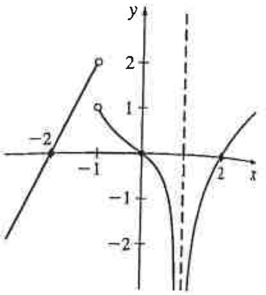


Applications and Extensions

51. Show that the function $f(x) = 2x^3 - 6x^2 + 6x - 5$ is increasing for all x.

52. Show that the function $f(x) = x^3 - 3x^2 + 3x$ is increasing for all x.

53. Show that the function $f(x) = \dfrac{x}{x+1}$ is increasing on any interval not containing $x = -1$.

54. Show that the function $f(x) = \dfrac{x+1}{x}$ is decreasing on any interval not containing $x = 0$.

55. Mean Value Theorem Draw the graph of a function f that is continuous on $[a, b]$ but is not differentiable on (a, b), and for which the conclusion of the Mean Value Theorem does not hold.

56. Mean Value Theorem Draw the graph of a function f that is differentiable on (a, b) but is not continuous on $[a, b]$, and for which the conclusion of the Mean Value Theorem does not hold.

PAGE 323 **57. Rectilinear Motion** An automobile travels 20 mi down a straight road at an average velocity of 40 mi/h. Show that the automobile must have a velocity of exactly 40 mi/h at some time during the trip. (Assume that the position function is differentiable.)

58. Rectilinear Motion Suppose a car is traveling on a highway. At 4:00 p.m., the car's speedometer reads 40 mi/h. At 4:12 p.m., it reads 60 mi/h. Show that at some time between 4:00 p.m. and 4:12 p.m., the acceleration was exactly 100 mi/p.m.

59. Rectilinear Motion Two stock cars start a race at the same time and finish in a tie. If $f_1(t)$ is the position of one car at time t and $f_2(t)$ is the position of the second car at time t, show that at some time during the race they have the same velocity. *Hint:* Set $f(t) = f_2(t) - f_1(t)$.

60. Rectilinear Motion Suppose $s = f(t)$ is the position of an object from the origin at time t. If the object is at a specific location at $t = a$, and returns to that location at $t = b$, then $f(a) = f(b)$. Show that there is at least one time $t = c$, $a < c < b$ for which $f'(c) = 0$. That is, show that there is a time c when the velocity of the object is 0.

61. **Loaded Beam** The vertical deflection d (in feet), of a particular 5-foot-long loaded beam can be approximated by

$$d = d(x) = -\frac{1}{192}x^4 + \frac{25}{384}x^3 - \frac{25}{128}x^2$$

where x (in feet) is the distance from one end of the beam.

(a) Verify that the function $d = d(x)$ satisfies the conditions of Rolle's Theorem on the interval $[0, 5]$.

(b) What does the result in (a) say about the ends of the beam?

(c) Find all numbers c in $(0, 5)$ that satisfy the conclusion of Rolle's Theorem. Then find the deflection d at each number c.

(d) Graph the function d on the interval $[0, 5]$.

62. For the function $f(x) = x^4 - 2x^3 - 4x^2 + 7x + 3$:

(a) Find the critical numbers of f rounded to three decimal places.

(b) Find the intervals where f is increasing and decreasing.

63. **Rolle's Theorem** Use Rolle's Theorem with the function $f(x) = (x - 1)\sin x$ on $[0, 1]$ to show that the equation $\tan x + x = 1$ has a solution in the interval $(0, 1)$.

64. **Rolle's Theorem** Use Rolle's Theorem to show that the function $f(x) = x^3 - 2$ has exactly one real zero.

65. **Rolle's Theorem** Use Rolle's Theorem to show that the function $f(x) = (x - 8)^3$ has exactly one real zero.

66. **Rolle's Theorem** Without finding the derivative, show that if $f(x) = (x^2 - 4x + 3)(x^2 + x + 1)$, then $f'(x) = 0$ for at least one number between 1 and 3. Check by finding the derivative and using the Intermediate Value Theorem.

67. **Rolle's Theorem** Consider $f(x) = |x|$ on the interval $[-1, 1]$. Here $f(1) = f(-1) = 1$ but there is no c in the interval $(-1, 1)$ at which $f'(c) = 0$. Explain why this does not contradict Rolle's Theorem.

68. **Mean Value Theorem** Consider $f(x) = x^{2/3}$ on the interval $[-1, 1]$. Verify that there is no c in $(-1, 1)$ for which

$$f'(c) = \frac{f(1) - f(-1)}{1 - (-1)}$$

Explain why this does not contradict the Mean Value Theorem.

69. **Mean Value Theorem** The Mean Value Theorem guarantees that there is a real number N in the interval $(0, 1)$ for which $f'(N) = f(1) - f(0)$ if f is continuous on the interval $[0, 1]$ and differentiable on the interval $(0, 1)$. Find N if $f(x) = \sin^{-1} x$.

70. **Mean Value Theorem** Show that when the Mean Value Theorem is applied to the function $f(x) = Ax^2 + Bx + C$ in the interval $[a, b]$, the number c referred to in the theorem is the midpoint of the interval.

71. (a) Apply the Increasing/Decreasing Function Test to the function $f(x) = \sqrt{x}$. What do you conclude?

(b) Is f increasing on the interval $[0, \infty)$? Explain.

72. Explain why the function $f(x) = ax^4 + bx^3 + cx^2 + dx + e$ must have a zero between 0 and 1 if

$$\frac{a}{5} + \frac{b}{4} + \frac{c}{3} + \frac{d}{2} + e = 0$$

73. **Put It Together** If $f'(x)$ and $g'(x)$ exist and $f'(x) > g'(x)$ for all real x, then which of the following statements must be true about the graph of $y = f(x)$ and the graph of $y = g(x)$?

(a) They intersect exactly once.

(b) They intersect no more than once.

(c) They do not intersect.

(d) They could intersect more than once.

(e) They have a common tangent at each point of intersection.

74. Prove that there is no k for which the function

$$f(x) = x^3 - 3x + k$$

has two distinct zeros in the interval $[0, 1]$.

75. Show that $e^x > x^2$ for all $x > 0$.
Hint: Show that $f(x) = e^x - x^2$ is an increasing function for $x > 0$.

76. Show that $e^x > 1 + x$ for all $x > 0$.

77. Show that $0 < \ln x < x$ for $x > 1$.

78. Show that $\tan\theta \geq \theta$ for all θ in the open interval $\left(0, \frac{\pi}{2}\right)$.

79. Establish the identity $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$ by showing that the derivative of $y = \sin^{-1} x + \cos^{-1} x$ is 0. Then use the fact that when $x = 0$, then $y = \frac{\pi}{2}$.

80. Establish the identity $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$ by showing that the derivative of $y = \tan^{-1} x + \cot^{-1} x$ is 0. Then use the fact that when $x = 1$, then $y = \frac{\pi}{2}$.

81. Let f be a function that is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). If $f(x) = 0$ for three different numbers x in (a, b), show that there must be at least two numbers in (a, b) at which $f'(x) = 0$.

82. **Proof of the Increasing/Decreasing Function Test** Let f be a function that is differentiable on an open interval (a, b). Show that if $f'(x) < 0$ for all numbers in (a, b), then f is a decreasing function on (a, b). (See the Corollary on p. 324.)

83. Suppose that the domain of f is an open interval (a, b) and $f'(x) > 0$ for all x in the interval. Show that f cannot have an extreme value on (a, b).

Challenge Problems

84. Use Rolle's Theorem to show that between any two real zeros of a polynomial function f, there is a real zero of its derivative function f'.

85. Find where the general cubic $f(x) = ax^3 + bx^2 + cx + d$ is increasing and where it is decreasing by considering cases depending on the value of $b^2 - 3ac$.
Hint: $f'(x)$ is a quadratic function; examine its discriminant.

86. Explain why the function $f(x) = x^n + ax + b$, where n is a positive even integer, has at most two distinct real zeros.

87. Explain why the function $f(x) = x^n + ax + b$, where n is a positive odd integer, has at most three distinct real zeros.

88. Explain why the function $f(x) = x^n + ax^2 + b$, where n is a positive odd integer, has at most three distinct real zeros.

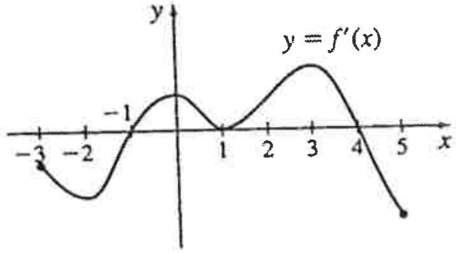
89. Explain why the function $f(x) = x^n + ax^2 + b$, where n is a positive even integer, has at most four distinct real zeros.

5.2 - Mean Value Theorem (MVT) & Rolle's - AP Practice Problems (p.330)

1. A function f is continuous on the closed interval $[-2, 5]$, differentiable on the open interval $(-2, 5)$, and $f(-2) = f(5) = 3$. Which of the following statements must be true?

(A) There is a number c in the interval $(-2, 5)$ for which $f'(c) = 0$.
(B) $f'(x) > 0$ for all numbers in the interval $(-2, 5)$.
(C) $f'(x) = 3$ for all numbers in the interval $(-2, 5)$.
(D) None of the above

2. The graph of f' for the interval $[-3, 5]$ is shown below. On what interval(s) is(are) f decreasing?



(A) $[-3, -2]$, $[0, 1]$ and $[3, 5]$ (B) $[-3, -1]$ and $[4, 5]$

(C) $[-3, -2]$ and $[3, 5]$ (D) $[-3, -2]$ and $[4, 5]$

3. For the function $f(x) = \sqrt{x}$, find the value(s) of c that satisfy the conclusion of the Mean Value Theorem on the interval $[0, 4]$.

(A) 0 (B) $\frac{1}{2}$ (C) 1 (D) 2

4. On what interval(s) is the function $f(x) = e^{3x^3-x}$ increasing?

(A) $(-\infty, \infty)$ (B) $[0, \infty)$

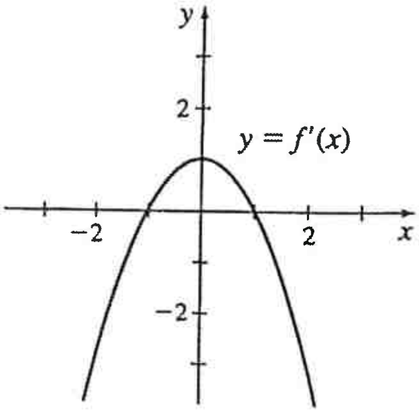
(C) $\left(-\infty, -\frac{1}{3}\right]$ and $\left[\frac{1}{3}, \infty\right)$ (D) $\left[-\frac{1}{3}, \infty\right)$

5. For which values of x is the function $f(x) = x^4 - 4x^3 + 4x^2 + 1$ decreasing?

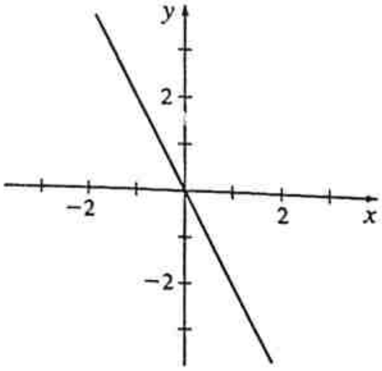
(A) $x < 0$ or $1 < x < 2$ (B) $x < 0$ or $x > 2$

(C) $0 < x < 2$ (D) $x > 2$ only

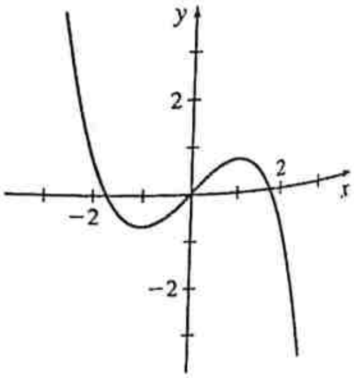
6. The graph of the derivative of f is shown. Which of the following can be the graph of f?



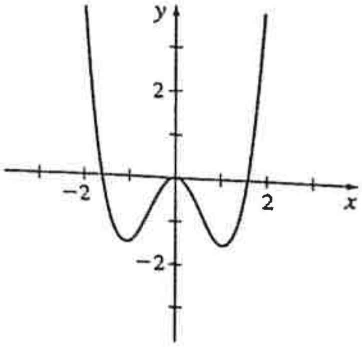
(A)



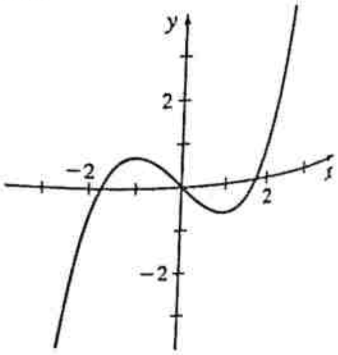
(B)



(C)



(D)



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7. Suppose f and g are differentiable functions for which
   - $f(x) > 0$ for all real numbers
   - $g(0) = 4$

   If $h(x) = f(x)g(x)$ and $h'(x) = f'(x)g(x)$ for all real numbers, then $g(x)$ equals

   (A) $f(x)$ (B) $g'(x)$ (C) 0 (D) 4

8. The domain of the function f is the set of all real numbers.

   If $f'(x) = \dfrac{|x^2 - 9|}{x - 3}$, then f is increasing on the interval

   (A) $(-\infty, \infty)$. (B) $[-3, 3]$.

   (C) $[3, 9]$. (D) $[3, \infty)$.

9. An object moves along the x-axis so that its distance from the origin at any time $t \geq 0$ is given by $x(t) = t^3 + \frac{3}{2}t^2 - 18t + 4$. At what times t is the object at rest?

(A) 2 and -3 (B) 2 only (C) 3 only (D) 0 and 2

10. Which of the following statements is true for the function

$$f(x) = \frac{\ln x}{x} \quad x > 0$$

(A) f is increasing on the interval $(0, \infty)$.
(B) f is increasing on the interval $[e, \infty)$.
(C) f is decreasing on the interval $[1, e]$.
(D) f is decreasing on the interval $[e, \infty)$.

⑳

Calculus Ch. 5.3a **Notes: First Derivative Test**

What does the derivative represent? ______________________________

When the function is **increasing**, what is common about the derivatives at those points? ______________

When the function is **decreasing**, what is common about the derivatives at those points? ______________

When $f'(x) > 0$, ______________________

When $f'(x) < 0$, ______________________

When $f'(x) = 0$, ______________________

First Derivative Test Steps (Finds inc/dec and relative max/min)

1. Find f '(x), set equal to zero
   a. Find critical points from BOTH numerator and denominator
   b. Remember, critical points also exist where function is not differentiable (sharp point)
2. Put all critical points on sign line
3. Test intervals
   a. Plug values into f '(x) to determine slope
      i. Positive (+) means increasing slope
      ii. Negative(–) means decreasing slope
4. Write Because Statements
   a. f(x) increasing in interval (a,b) b/c $f'(x) > 0$
   b. f(x) decreasing in interval (a,b) b/c $f'(x) < 0$
   c. Relative max at (**a, f(a)**) b/c f '(x) changes from + to –
   d. Relative min at (**a, f(a)**) b/c f '(x) changes from – to +

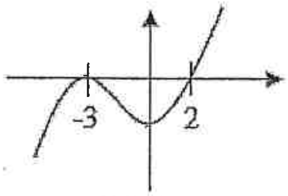
Example 1: Determine the intervals at which the function $f(x) = \frac{1}{3}x^3 + \frac{5}{2}x^2 - 6x - 3$ is increasing and decreasing.. Locate the relative extrema.

First Derivative Test Steps (Finds inc/dec and relative max/min).

1. Find f '(x), set equal to zero
   - a. Find critical points from BOTH numerator and denominator
   - b. Remember, critical points also exist where function is not differentiable (sharp point)
2. Put all critical points on sign line
3. Test intervals
   - a. Plug values into f '(x) to determine slope
     - i. Positive (+) means increasing slope
     - ii. Negative(–) means decreasing slope
4. Write Because Statements
   - a. f(x) increasing in interval (a,b) b/c f '(x) > 0
   - b. f(x) decreasing in interval (a,b) b/c f '(x) < 0
   - c. Relative max at (a, f(a)) b/c f '(x) changes from + to –
   - d. Relative min at (a, f(a)) b/c f '(x) changes from – to +

Example 2: Determine the intervals at which the function $f(x)=\dfrac{5x+2}{x-3}$ is increasing and decreasing. Locate the relative extrema.

Example 3: Make a first derivative sign line for the following graph of f '(x):



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C

Calculus Ch. 3.3b Notes: "Concavity Test" and the 2^{nd} Derivative Test

Are both of these functions increasing? ________ What do we know about their derivatives? ________

1) If f "(x) is > 0, then f '(x) is increasing and f(x) is concave up.
2) If f "(x) is < 0, then f '(x) is decreasing and f(x) is concave down.
3) A Point of Inflection (POI) occurs whenever f "(x) changes sign. (f(x) changes concavity)

"Concavity Test" Steps (Finding interval Concave Up/Down and POI)

1. Find $f''(x)$, set equal to zero
   a. Find critical points from BOTH numerator and denominator
2. Put all critical points on sign line
3. Test intervals
   a. Plug values into $f''(x)$ to determine concavity
      i. Positive (+) means concave up
      ii. Negative(–) means concave down
4. Write Because Statements
   a. Concave up in interval (a,b) b/c $f''(x) > 0$
   b. Concave down in interval (a,b) b/c $f''(x) < 0$
5. Point of Inflection at (a, f(a)) b/c $f''(x)$ changes signs

*Note: POI may exist on graph where $f''(x)$ does not exist (sharp point). POI exists as long as graph is continuous and $f''(x)$ changes concavity (change of signs)

Example 1: Find the points of inflection if $f(x) = -2x^5 + \frac{5}{3}x^3$

The 2nd Derivative Test

The 2nd derivative test is a test for relative extrema (max/min) and NOT for Point of Inflection

*The 2nd derivative test achieves the same as the 1st derivative test.

1) If you plug a critical number from $f'(x)$ into $f''(x)$ and if $f''(x) > 0$, then that is the x-value of the relative **minimum**

2) If you plug a critical number from $f'(x)$ into $f''(x)$ and if $f''(x) < 0$, then that is the x-value of the relative **maximum**

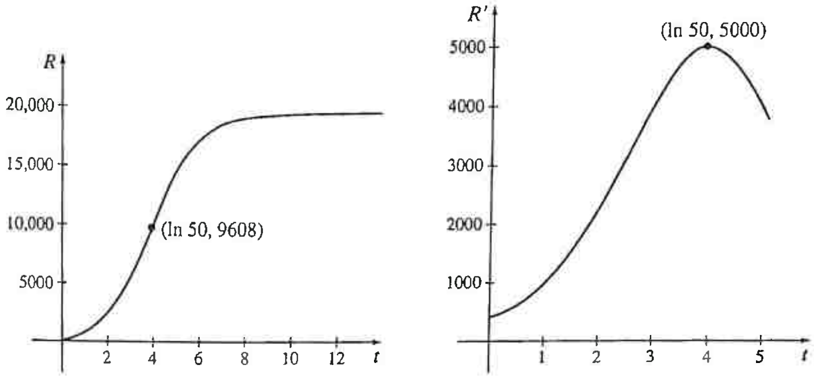
3) If you plug a critical number from $f'(x)$ into $f''(x)$ and if $f''(x) = 0$, then the test is inconclusive. We need the first derivative test to determine if critical number is a relative extrema.

2nd Derivative Test Steps (Test for Relative Extrema, **NOT** Point of Inflection)

1. Find $f'(x)$, set equal to zero
   a. Find critical points. Set numerator and denominator of $f'(x) = 0$. (These are candidates for relative max/min)
2. Find $f''(x)$
3. Plug the critical points (from step #1) into $f''(x)$.
   a. If result is positive value, then $f''(x) > 0$, concave up, and therefore relative minimum exists at x-value
   b. If result is negative value, then $f''(x) < 0$, concave down, and therefore relative maximum exists at x-value
   c. f result is zero, then since $f''(x) = 0$, then this test is inconclusive. We cannot determine whether relative extrema exists. (Use First Derivative Test)

Example 2: Find the relative extrema of $f(x) = x^3 - 4x^2 - 3x$

See Figure 42 for the graphs of R and R'.



(a) $R(t) = \frac{20{,}000}{1 + 50e^{-t}} - \frac{20{,}000}{51}$

(b) $R'(t) = \frac{1{,}000{,}000\,e^{-t}}{(1 + 50e^{-t})^2}$

Figure 42

NOW WORK Problem 93.

5.3 Assess Your Understanding

Concepts and Vocabulary

1. *True or False* If a function f is continuous on the interval $[a, b]$, differentiable on the interval (a, b), and changes from an increasing function to a decreasing function at the point $(c, f(c))$, then $(c, f(c))$ is an inflection point of f.
2. *True or False* Suppose c is a critical number of f and (a, b) is an open interval containing c. If $f'(x)$ is positive on both sides of c, then $f(c)$ is a local maximum value.
3. *Multiple Choice* Suppose a function f is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b). If the graph of f lies above each of its tangent lines on the interval (a, b), then on (a, b) f is

   [(a) concave up (b) concave down (c) neither].
4. *Multiple Choice* If the acceleration of an object in rectilinear motion is negative, then the velocity of the object is

   [(a) increasing (b) decreasing (c) neither].
5. *Multiple Choice* Suppose f is a function that is differentiable on an open interval containing c and the concavity of f changes at the point $(c, f(c))$. Then the point $(c, f(c))$ on the graph of f is a(n)

   [(a) inflection point (b) critical point (c) both (d) neither].
6. *Multiple Choice* Suppose a function f is continuous on a closed interval $[a, b]$ and both f' and f'' exist on the open interval (a, b). If $f''(x) > 0$ on the interval (a, b), then on (a, b) f is

   [(a) increasing (b) decreasing (c) concave up (d) concave down].
7. *True or False* Suppose f is a function for which f' and f'' exist on an open interval (a, b) and suppose c, $a < c < b$, is a critical number of f. If $f''(c) = 0$, then the Second Derivative Test cannot be used to determine if there is a local extremum at c.
8. *True or False* Suppose a function f is differentiable on the open interval (a, b). If either $f''(c) = 0$ or f'' does not exist at the number c in (a, b), then $(c, f(c))$ is an inflection point of f.

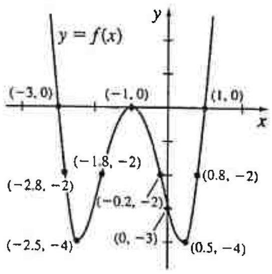
Skill Building

In Problems 9–12, the graph of a function f is given.

(a) Identify the points where each function has a local maximum value, a local minimum value, or an inflection point.

(b) Identify the intervals on which each function is increasing, decreasing, concave up, or concave down.

9.



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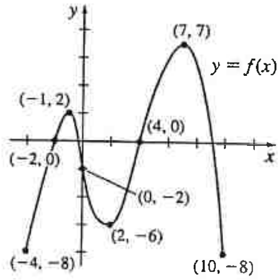

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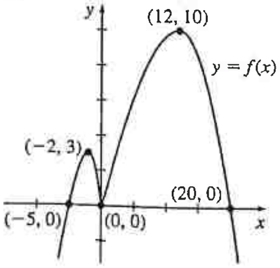

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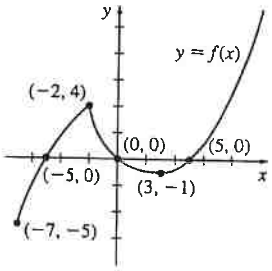


In Problems 13–26, for each function:

(a) Find the critical numbers.

(b) Use the First Derivative Test to find any local extrema.

PAGE 332 **13.** $f(x)=x^3-6x^2+2$

14. $f(x)=x^3+6x^2+12x+1$

15. $f(x)=3x^4-4x^3$

16. $h(x)=x^4+2x^3-3$

17. $f(x)=(5-2x)e^x$

18. $f(x)=(x-8)e^x$

19. $f(x)=x^{2/3}+x^{1/3}$

20. $f(x)=\frac{1}{2}x^{2/3}-x^{1/3}$

PAGE 333 **21.** $g(x)=x^{2/3}(x^2-4)$

22. $f(x)=x^{1/3}(x^2-9)$

23. $f(x)=\frac{\ln x}{x^3}$

24. $h(x)=\frac{\ln x}{\sqrt{x^3}}$

25. $f(\theta)=\sin\theta-2\cos\theta$

26. $f(x)=x+2\sin x$

In Problems 27–34, an object in rectilinear motion moves along a horizontal line with the positive direction to the right. The position s of the object from the origin at time $t\geq 0$ (in seconds) is given by the function $s=s(t)$.

(a) Determine the intervals during which the object moves to the right and the intervals during which it moves to the left.

(b) When does the object reverse direction?

(c) When is the velocity of the object increasing and when is it decreasing?

(d) Draw a figure to illustrate the motion of the object.

(e) Draw a figure to illustrate the velocity of the object.

27. $s=t^2-2t+3$

28. $s=2t^2+8t-7$

PAGE 335 **29.** $s=2t^3+6t^2-18t+1$

30. $s=3t^4-16t^3+24t^2$

31. $s=2t-\frac{6}{t},\quad t>0$

32. $s=3\sqrt{t}-\frac{1}{\sqrt{t}},\quad t>0$

33. $s=2\sin(3t),\quad 0\leq t\leq\frac{2\pi}{3}$

34. $s=3\cos(\pi t),\quad 0\leq t\leq 2$

In Problems 35–38, the function f is continuous for all real numbers and the graph of its derivative function f' is given.

(a) Determine the critical numbers of f.

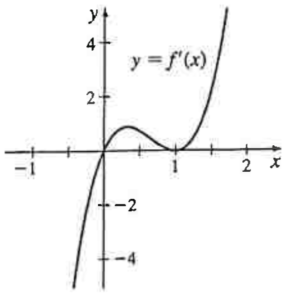
(b) Where is f increasing?

(c) Where is f decreasing?

(d) At what numbers x, if any, does f have a local minimum?

(e) At what numbers x, if any, does f have a local maximum?

PAGE 333 **35.**



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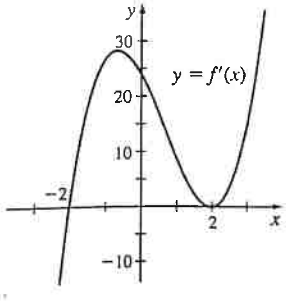

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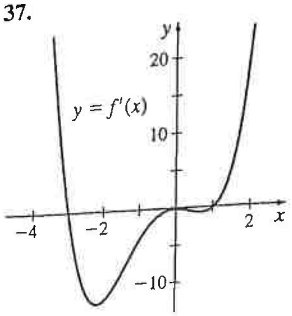

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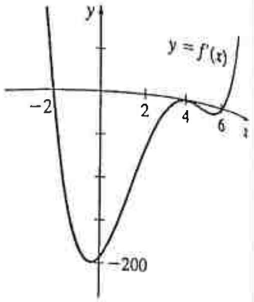


In Problems 39–62:

(a) Find the local extrema of f.

(b) Determine the intervals on which f is concave up and on which is concave down.

(c) Find any points of inflection.

39. $f(x)=2x^3-6x^2+6x-3$

40. $f(x)=2x^3+9x^2+12$

41. $f(x)=x^4-4x$

42. $f(x)=x^4+4x$

43. $f(x)=5x^4-x^5$

44. $f(x)=4x^6+6x^4$

45. $f(x)=3x^5-20x^3$

46. $f(x)=3x^5+5x^3$

47. $f(x)=x^2e^x$

48. $f(x)=x^3e^x$

PAGE 338 **49.** $f(x)=6x^{4/3}-3x^{1/3}$

50. $f(x)=x^{2/3}-x^{1/3}$

PAGE 333 **51.** $f(x)=x^{2/3}(x^2-8)$

52. $f(x)=x^{1/3}(x^2-2)$

53. $f(x)=x^2-\ln x$

54. $f(x)=\ln x-x$

55. $f(x)=\frac{x}{(1+x^2)^{5/2}}$

56. $f(x)=\frac{\sqrt{x}}{1+x}$

57. $f(x)=x^2\sqrt{1-x^2}$

58. $f(x)=x\sqrt{1-x}$

PAGE 341 **59.** $f(x)=x-2\sin x,\quad 0\leq x\leq 2\pi$

60. $f(x)=x+2\cos x,\quad 0\leq x\leq 2\pi$

61. $f(x)=2\cos^2 x-\sin^2 x,\quad 0<x<2\pi$

62. $f(x)=2\sin^2 x-\cos^2 x,\quad 0<x<2\pi$

In Problems 63–66, the function f is continuous for all real numbers and the graphs of f' and f'' are given.

(a) Determine the critical numbers of f.

(b) Where is f increasing?

(c) Where is f decreasing?

(d) At what numbers x, if any, does f have a local minimum?

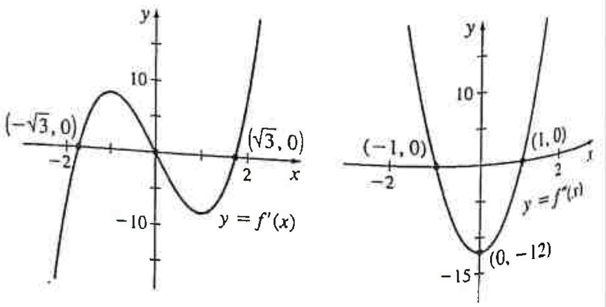
(e) At what numbers x, if any, does f have a local maximum?

(f) Where is f concave up?

(g) Where is f concave down?

(h) List any inflection points.

PAGE 339 **63.**



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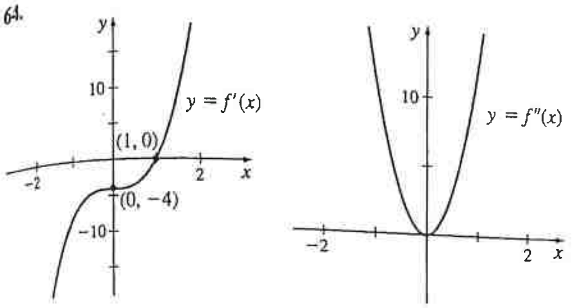

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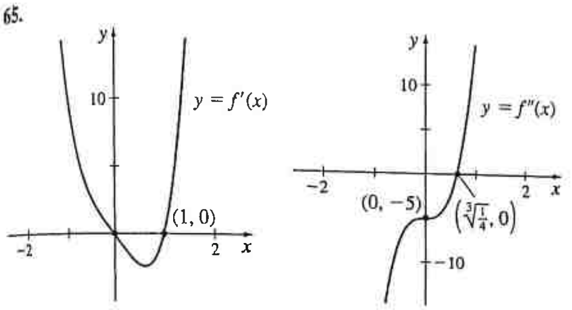

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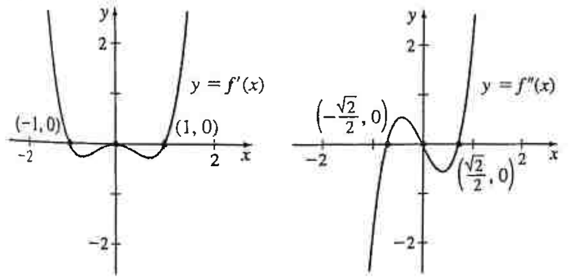


In Problems 67–74, find the local extrema of each function f by:

(a) Using the First Derivative Test.

(b) Using the Second Derivative Test.

(c) Discuss which of the two tests you found easier.

67. $f(x) = -2x^3 + 15x^2 - 36x + 7$

68. $f(x) = x^3 + 10x^2 + 25x - 25$

69. $f(x) = x^4 - 8x^2 - 5$

70. $f(x) = x^4 + 2x^2 + 2$

71. $f(x) = 3x^5 + 5x^4 + 1$

72. $f(x) = 60x^5 + 20x^3$

73. $f(x) = (x-3)^2 e^x$

74. $f(x) = (x+1)^2 e^{-x}$

Applications and Extensions

In Problems 75–86, sketch the graph of a continuous function f that has the given properties. Answers will vary.

75. f is concave up on $(-\infty, \infty)$, increasing on $(-\infty, 0)$, decreasing on $(0, \infty)$, and $f(0) = 1$.

76. f is concave up on $(-\infty, 0)$, concave down on $(0, \infty)$, decreasing on $(-\infty, 0)$, increasing on $(0, \infty)$, and $f(0) = 1$.

77. f is concave down on $(-\infty, 1)$, concave up on $(1, \infty)$, decreasing on $(-\infty, 0)$, increasing on $(0, \infty)$, $f(0) = 1$, and $f(1) = 2$.

78. f is concave down on $(-\infty, 0)$, concave up on $(0, \infty)$, increasing on $(-\infty, \infty)$, and $f(0) = 1$ and $f(1) = 2$.

79. $f'(x) > 0$ if $x < 0$; $f'(x) < 0$ if $x > 0$; $f''(x) > 0$ if $x < 0$; $f''(x) > 0$ if $x > 0$ and $f(0) = 1$.

80. $f'(x) > 0$ if $x < 0$; $f'(x) < 0$ if $x > 0$; $f''(x) > 0$ if $x < 0$; $f''(x) < 0$ if $x > 0$ and $f(0) = 1$.

81. $f''(0) = 0$; $f'(0) = 0$; $f''(x) > 0$ if $x < 0$; $f''(x) > 0$ if $x > 0$ and $f(0) = 1$.

82. $f''(0) = 0$; $f'(x) > 0$ if $x \neq 0$; $f''(x) < 0$ if $x < 0$; $f''(x) > 0$ if $x > 0$ and $f(0) = 1$.

83. $f'(0) = 0$; $f'(x) < 0$ if $x \neq 0$; $f''(x) > 0$ if $x < 0$; $f''(x) < 0$ if $x > 0$; $f(0) = 1$.

84. $f''(0) = 0$; $f'(0) = \frac{1}{2}$; $f''(x) > 0$ if $x < 0$; $f''(x) < 0$ if $x > 0$ and $f(0) = 1$.

85. $f'(0)$ does not exist; $f''(x) > 0$ if $x < 0$; $f''(x) > 0$ if $x > 0$ and $f(0) = 1$.

86. $f'(0)$ does not exist; $f''(x) < 0$ if $x < 0$; $f''(x) > 0$ if $x > 0$ and $f(0) = 1$.

CAS *In Problems 87–90, for each function:*

(a) Determine the intervals on which f is concave up and on which it is concave down.

(b) Find any points of inflection.

(c) Graph f and describe the behavior of f at each inflection point.

87. $f(x) = e^{-(x-2)^2}$

88. $f(x) = x^2\sqrt{5-x}$

89. $f(x) = \dfrac{2-x}{2x^2 - 2x + 1}$

90. $f(x) = \dfrac{3x}{x^2 + 3x + 5}$

91. **Inflection Point** For the function $f(x) = ax^3 + bx^2$, find a and b so that the point $(1, 6)$ is an inflection point of f.

92. **Inflection Point** For the cubic polynomial function $f(x) = ax^3 + bx^2 + cx + d$, find a, b, c, and d so that 0 is a critical number, $f(0) = 4$, and the point $(1, -2)$ is an inflection point of f.

PAGE 343 93. **Public Health** In a town of 50,000 people, the number of people at time t who have the flu is $N(t) = \dfrac{10{,}000}{1 + 9999e^{-t}}$, where t is measured in days. Note that the flu is spread by the one person who has it at $t = 0$.

(a) Find the rate of change of the number of infected people.

(b) When is N' increasing? When is it decreasing?

(c) When is the rate of change of the number of infected people a maximum?

(d) Find any inflection points of N.

(e) Interpret the result found in (d) in the context of the problem.

94. Business The profit P (in millions of dollars) generated by introducing a new technology is expected to follow the logistic function $P(t) = \frac{300}{1 + 50e^{-0.2t}}$, where t is the number of years after its release.

(a) When is annual profit increasing? When is it decreasing?
(b) Find the rate of change in profit.
(c) When is the rate of change in profit increasing? When is it decreasing?
(d) When is the rate of change in profit a maximum?
(e) Find any inflection points of P.
(f) Interpret the result found in (e) in the context of the problem.

95. Population Model The following data represent the population of the United States:

Year	Population	Year	Population
1900	76,212,168	1960	179,323,175
1910	92,228,486	1970	203,302,031
1920	106,021,537	1980	226,542,203
1930	123,202,624	1990	248,709,873
1940	132,164,569	2000	281,421,906
1950	151,325,798	2010	308,745,538

Source: U.S. Census Bureau.

An ecologist finds the data fit the logistic function

$$P(t) = \frac{762{,}176{,}717.8}{1 + 8.743e^{-0.0162t}}.$$

(a) Draw a scatterplot of the data using the years since 1900 as the independent variable and population as the dependent variable.
(b) Verify that P is the logistic function of best fit.
(c) Find the rate of change in population.
(d) When is P' increasing? When is it decreasing?
(e) When is the rate of change in population a maximum?
(f) Find any inflection points of P.
(g) Interpret the result found in (f) in the context of the problem.

96. Biology The amount of yeast biomass in a culture after t hours is given in the table below.

Time (in hours)	Yeast Biomass	Time (in hours)	Yeast Biomass	Time (in hours)	Yeast Biomass
0	9.6	7	257.3	13	629.4
1	18.3	8	350.7	14	640.8
2	29.0	9	441.0	15	651.1
3	47.2	10	513.3	16	655.9
4	71.1	11	559.7	17	659.6
5	119.1	12	594.8	18	661.8
6	174.6				

Source: Tor Carlson, *Uber Geschwindigkeit und Grosse der Hefevermehrung in Wurze, Biochemische Zeitschrift, Bd. 57*, 1913, pp. 313–334.

The logistic function $y = \frac{663.0}{1 + 71.6e^{-0.5470t}}$, where t is time, models the data.

(a) Draw a scatterplot of the data using time t as the independent variable.
(b) Verify that y is the logistic function of best fit.
(c) Find the rate of change in biomass.
(d) When is y' increasing? When is it decreasing?
(e) When is the rate of change in the biomass a maximum?
(f) Find any inflection points of y.
(g) Interpret the result found in (f) in the context of the problem.

97. U.S. Budget The United States budget documents the amount of money (revenue) the federal government takes in (through taxes, for example) and the amount (expenses) it pays out (for social programs, defense, etc.). When revenue exceeds expenses, we say there is a **budget surplus**, and when expenses exceed revenue, there is a **budget deficit**. The function

$$B = B(t) = -12.8t^3 + 163.4t^2 - 614.0t + 390.6$$

where $0 \le t \le 12$ approximates revenue minus expenses for the years 2000 to 2012, with $t = 0$ representing the year 2000 and B in billions of dollars.

(a) Find all the local extrema of B. (Round the answers to two decimal places.)
(b) Do the local extreme values represent a budget surplus or a budget deficit?
(c) Find the intervals on which B is concave up or concave down. Identify any inflection points of B.
(d) What does the concavity on either side of the inflection point(s) indicate about the rate of change of the budget? Is it increasing at an increasing rate? Increasing at a decreasing rate?
(e) Graph the function B. Given that the budget for 2015 was 3.9 trillion dollars, does B seem to be an accurate predictor for the budget for the years 2015 and beyond?

98. If $f(x) = ax^3 + bx^2 + cx + d$, $a \neq 0$, how does the quantity $b^2 - 3ac$ determine the number of potential local extrema?

99. If $f(x) = ax^3 + bx^2 + cx + d$, $a \neq 0$, find a, b, c, and d so that f has a local minimum at 0, a local maximum at 4, and the graph contains the points (0, 5) and (4, 33).

100. Find the local minimum of the function

$$f(x) = \frac{2}{x} + \frac{8}{1 - x}, \quad 0 < x < 1.$$

101. Find the local extrema and the inflection points of $y = \sqrt{3}\sin x + \cos x$, $0 \le x \le 2\pi$.

102. If $x > 0$ and $n > 1$, can the expression $x^n - n(x - 1) - 1$ ever be negative?

103. Why must the First Derivative Test be used to find the local extreme values of the function $f(x) = x^{2/3}$?

104. Put It Together Which of the following is true of the function

$$f(x) = x^2 + e^{-2x}$$

(a) f is increasing (b) f is decreasing (c) f is discontinuous at 0
(d) f is concave up (e) f is concave down

28

5.3 -First Derivative Test, Test for Concavity - AP Practice Problems (p. 347-348)

1. If $f''(x) = x(x+1)^2(x-2)$ is the second derivative of the function f, then list the x-coordinate(s) of the inflection point(s) of f.

   (A) $-1, 0$, and 2 (B) 0 only

   (C) 2 only (D) 0 and 2 only

2. If $f(x) = x + \sin x$, find the smallest positive number x at which the function changes concavity.

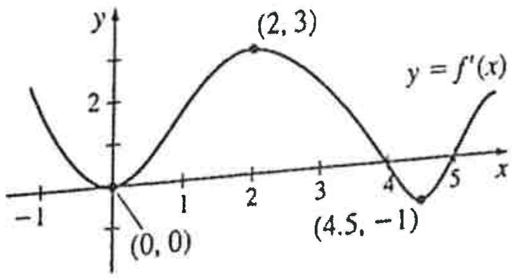
   (A) $\frac{\pi}{4}$ (B) $\frac{\pi}{2}$ (C) π (D) $\frac{3\pi}{2}$

3. The function $g(x) = x^5 + x^3 - 2x - 1$ has a local minimum at

   (A) $x = -\frac{\sqrt{10}}{5}$ (B) $x = \frac{\sqrt{10}}{5}$

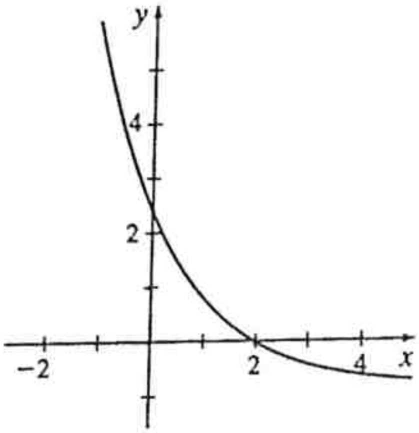
   (C) $x = \frac{2}{5}$ (D) $x = 0$

4. A function f is continuous for all real numbers. The graph of its derivative function f' is shown. The graph has horizontal tangent lines at (0, 0), (2, 3), and (4.5, −1). At which number(s) x does f have a local minimum?



(A) 0, 4, and 5
(B) 4 only
(C) 5 only
(D) 0, 2, and 4.5

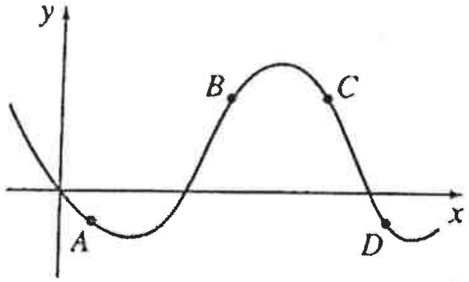
5. For the function f graphed below, both f' and f'' exist for all numbers x. Which of the following is true?



(A) $f(2) < f'(2) < f''(2)$
(B) $f'(2) < f(2) < f''(2)$
(C) $f''(2) < f'(2) < f(2)$
(D) $f''(2) < f(2) < f'(2)$

30

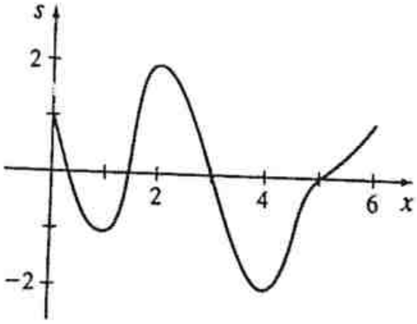
6. At which point on the graph of f do both f' and f'' have the same sign?



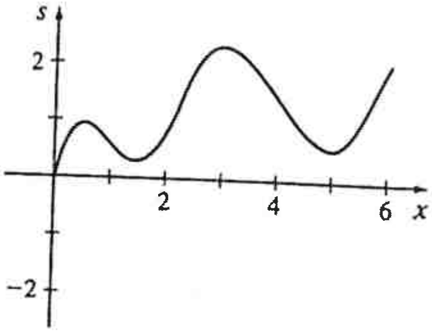
(A) A (B) B (C) C (D) D

7. An object in rectilinear motion has position function $s = s(t)$. At time $t = 0$, the object is at the origin. Several values of the velocity v of the object are listed in the table. Which of the following graphs could be the graph of the position of the object for $0 \le t \le 6$?

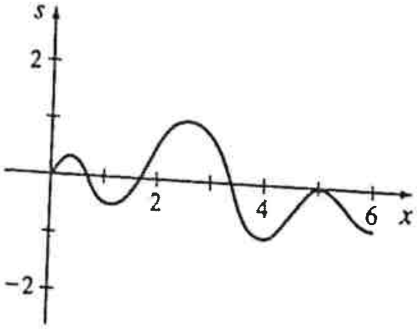
t	0	1	2	3	4	5	6
$v(t)$	1	−1	2	0	−2	0	1



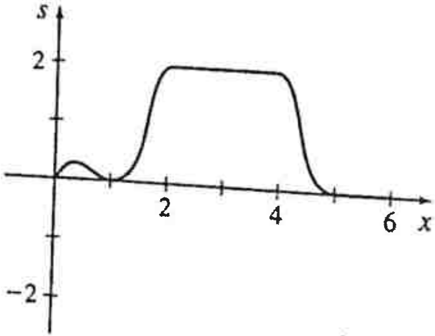
(A)



(B)



(C)



(D)

8. Find the inflection points, if any, of the function $f(x) = 3x^4 - 2x^2$.

(A) $(0, 0)$

(B) $\left(-\frac{1}{3}, 0\right)$ and $\left(\frac{1}{3}, 0\right)$

(C) $\left(-\frac{1}{3}, -\frac{5}{27}\right)$ and $\left(\frac{1}{3}, \frac{5}{27}\right)$

(D) $\left(-\frac{1}{3}, -\frac{5}{27}\right)$ and $\left(\frac{1}{3}, -\frac{5}{27}\right)$

9. For what interval(s) is the function

$$f(x) = \frac{x^4}{2} - 2x^3 - 9x^2 - 12x + 5$$

concave down?

(A) $(-1, 3)$

(B) $(-\infty, -1)$ or $(3, \infty)$

(C) $(-\infty, -3)$ or $(1, \infty)$

(D) Nowhere; the function is always concave up.

10. The derivative of a function f is

$$f'(x) = x^2(x - 1)(x + 2)(x + 3)$$

At what number(s) x does f have a relative minimum?

(A) -3 only

(B) 0 and 1

(C) -3 and -2

(D) -3 and 1

32

11. Which statement regarding the function

$$f(x) = (x-1)^{4/5} - 2$$

is not true?

(A) The point $(1, -2)$ is a local minimum of f.

(B) f is concave down on the intervals $(-\infty, 1)$ and $(1, \infty)$.

(C) The point $(1, -2)$ is an inflection point of f.

(D) f has a vertical tangent line at $(1, -2)$.

12. Find the minimum value, if any, of the function

$$f(x) = x \ln x.$$

(A) $-\frac{1}{e}$ (B) -1 (C) $\frac{1}{e}$ (D) f has no minimum value.

13. A polynomial function f has only three critical numbers -5, 0, and 2 and has three local extrema: a local minimum point at $(-5, -3)$, a local maximum point at $(0, -1)$, and a local minimum point at $(2, -6)$. How many zeros does f have?

(A) One (B) Two (C) Three (D) Four

14. Determine where the function $f(x) = \dfrac{2}{x+3}$ is concave up.

(A) $(-3, \infty)$ (B) $(2, \infty)$ (C) $(-2, \infty)$ (D) $(-\infty, -3)$

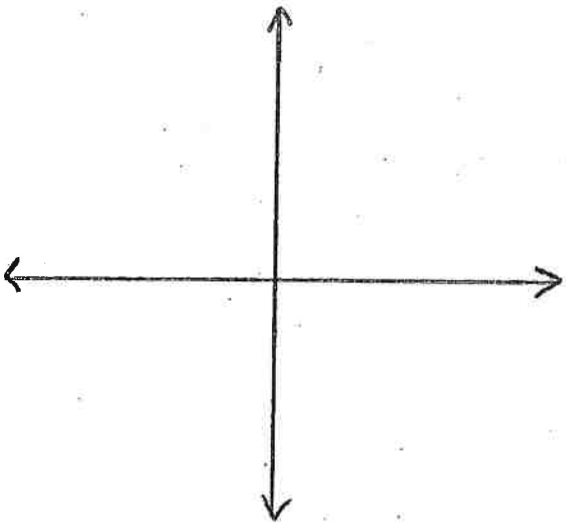
15. An object in rectilinear motion moves along the x-axis so that its position at time $t \geq 1$ (in seconds) is $x(t) = \dfrac{\ln t}{t}$. At what time t is the object farthest from the origin?

(A) $2e$ seconds (B) 1 second (C) 2 seconds (D) e seconds

(34)

5.4 Curve Sketching

1. Sketch the graph of the function and find the below information: $f(x) = -3x^5 + 5x^3$



x-ints: ______________

y-ints: ______________

V.A. ______________

H.A. ______________

Domain:

Interval Increasing

Interval Decreasing

Relative Maximum

Relative Minimum:

Points of Inflection:

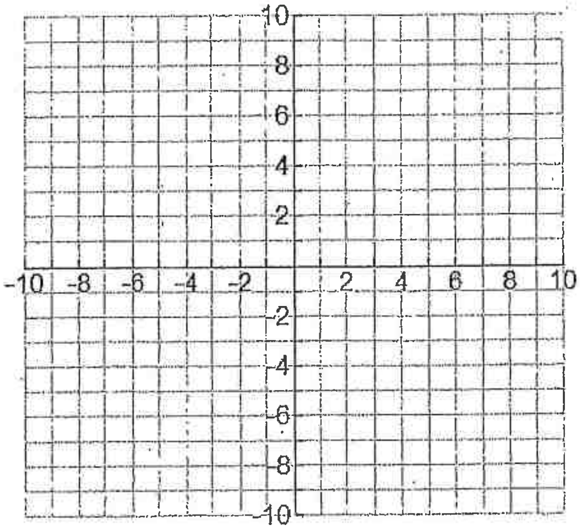
Interval Concave Up:

Interval Concave Down:

②

㉟

2. Sketch the graph of the function and find the below information: $f(x) = \frac{2x^2}{9-x^2}$



x-ints: ____________

y-ints: ____________

V.A. ____________

H.A. ____________

Domain:

Interval Increasing

Interval Decreasing

Relative Maximum

Relative Minimum:

Points of Inflection:

Interval Concave Up:

Interval Concave Down:

AP Calculus AB Chapter 3 Curve Sketching Sketching and interpreting Derivative Graphs

1. **Sketching 1st Derivative and 2nd Derivative Graphs (Given the f (x) graph)**
   1. Given the f(x) graph
   2. Make a sign line for f ‘(x) graph
      - a. Label Critical points (relative max, relative min, or where slope = 0) on sign line
      - b. Find intervals where graph is increasing (rising) and decreasing (falling)
      - c. Use + and ↗ arrow on the sign line to indicate increasing slope
      - d. Use – and ↘ arrow on the sign line to indicate decreasing slope
   3. Sketch f ‘ (x) graph
      - a. Plot critical points on the graph as x – intercepts (where slope = 0)
      - b. Sketch portions of graph above the x-axis (positive slope) or below x-axis (negative slope) using the information on your sign line.
   4. Make a sign line for f ’’(x) graph
      - a. Locate Points of Inflection on your f (x) graph.
        - i. This is where graph transitions from concave up to down or from concave down to up.
      - b. Label critical point on your sign line
        - i. Where graph resembles parabola opening up , use + and ∪ to indicate concave up
        - ii. Where graph resembles parabola opening up , use – and ∩ to indicate concave down
   5. Sketch f ’’(x) graph
      - a. Plot critical points on the graph as x – intercepts (POI and where f ’’ (x) = 0)
      - b. Sketch portions of graph above the x-axis (concave up) or below x-axis (concave down) using the information on your sign line.
2. **Sketching f(x) graph and 2nd Derivative Graph (Given the f ‘(x) graph)**
   1. Given the f ‘(x) graph
   2. Make a sign line for f ‘(x) graph
      - a. Label Critical points (x-intercepts) on sign line
      - b. Find intervals where graph is increasing(above x-axis) and decreasing(below x-axis)
      - c. Use + and ↗ arrow on the sign line to indicate increasing slope
      - d. Use – and ↘ arrow on the sign line to indicate decreasing slope
   3. Sketch f (x) graph
      - a. Follow the directional arrows on your sign line to draw the f(x) graph, along with the relative max (hills) and relative min (valleys) of your graph
   4. Make a sign line for f ’’(x) graph
      - a. Locate critical points (Points of Inflection) on your f ’(x) graph
        - i. Points of Inflections are the relative max (hills) and relative mins (valleys) of your f ’(x) graph
      - b. Label critical point on your sign line
        - i. Where f ’(x) graph is increasing(rising), use + and ∪ to indicate concave up
        - ii. Where f ’(x) graph is decreasing(falling) , use – and ∩ to indicate concave down
   5. Sketch f ’’(x) graph
      - a. Plot critical points on the graph as x – intercepts (POI and where f ’’ (x) = 0)
      - b. Sketch portions of graph above the x-axis (concave up) or below x-axis (concave down) using the information on your sign line.
3. **“Morgan’s Method”**

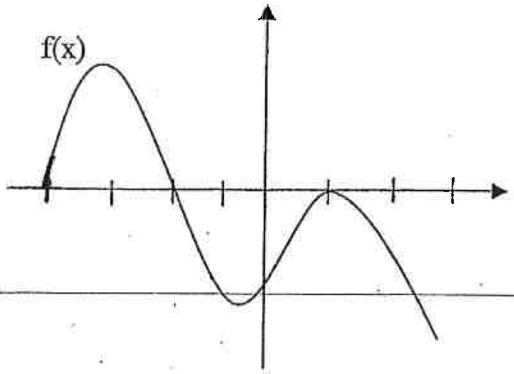
X – x-ints
M – max & mins
P – POI

f(x)	f ’(x)	f ’’(x)
X		
M	X	
P	M	X
	P	M
		P

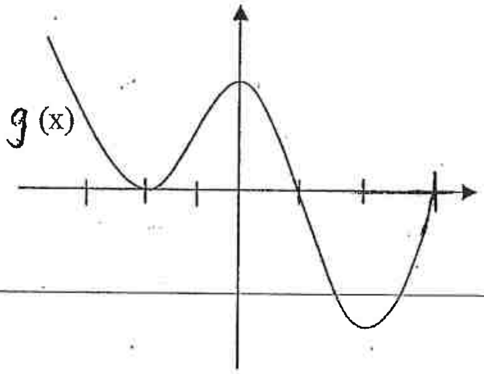
3.36 Interpreting Derivative Graphs

Make a sign line for slope and concavity for each of the following graphs

1)



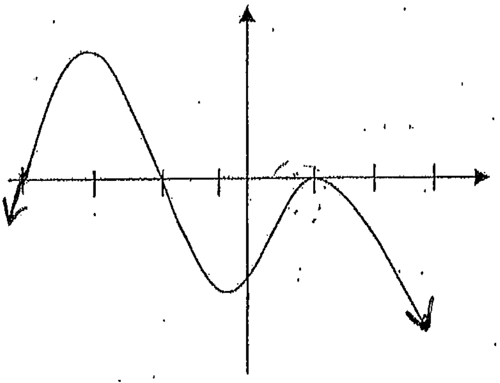
2)



(38)

3. f '(x) graph shown

Sketch f(x) graph :



Sketch f''(x) graph:

Characteristics of f(x)

increasing: ____________ decreasing______________

rel. max______________rel. min________________

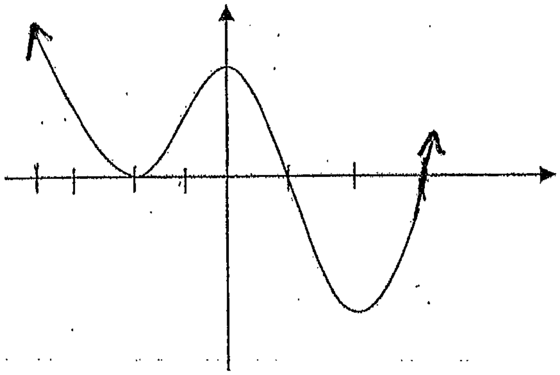
Concave up___________ Concave Down__________

POI____________

---

4. g'(x) graph shown:

Sketch f(x) graph :



Sketch f''(x) graph:

Characteristics of g(x)

increasing: ____________ decreasing______________

rel. max______________rel. min________________

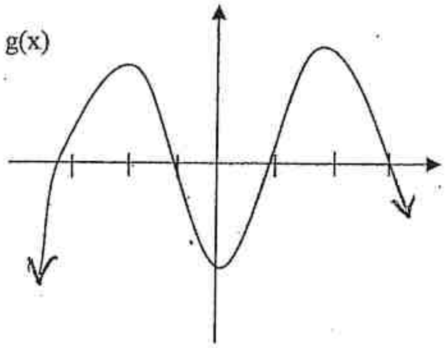
Concave up___________ Concave Down__________

POI____________

3.6b Interpreting Derivative Graphs – More Practice

Sketch the derivative graphs for the below f(x)

1)

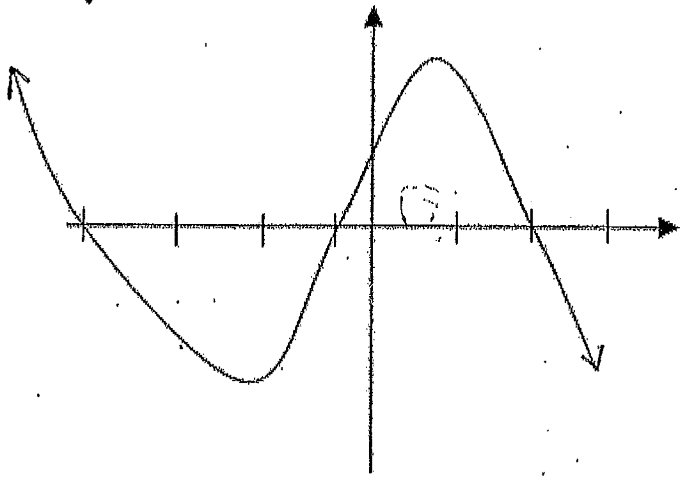


Sketch g '(x) graph:

Sketch g ''(x) graph: (POI at x = -1 and x = 1)

(40)

2) f'(x) graph show



Sketch the f (x) graph:

Sketch the f ''(x) graph:

Characteristics of f(x)

increasing:_____________ decreasing_______________

rel. max_______________ rel. min__________________

Concave up____________ Concave Down___________

POI_____________

Since $\lim_{x\to\infty}[e^x(x^2-3)]=\infty$, there is no horizontal asymptote as $x\to\infty$. Draw the asymptote on the graph.

Step 3 $f'(x)=\frac{d}{dx}[e^x(x^2-3)]=e^x(2x)+e^x(x^2-3)=e^x(x^2+2x-3)$

$$=e^x(x+3)(x-1)$$

$$f''(x)=\frac{d}{dx}[e^x(x^2+2x-3)]=e^x(2x+2)+e^x(x^2+2x-3)$$

$$=e^x(x^2+4x-1)$$

Solving $f'(x)=0$, we find that the critical numbers are -3 and 1.

Step 4 Use the critical numbers -3 and 1 to form three intervals on the x-axis: $(-\infty,-3)$, $(-3,1)$ and $(1,\infty)$. Then determine the sign of $f'(x)$ on each interval and whether f is increasing or decreasing on the interval.

Interval	Sign of f'	Conclusion
$(-\infty,-3)$	Positive	f is increasing on the interval $(-\infty,-3)$
$(-3,1)$	Negative	f is decreasing on the interval $(-3,1)$
$(1,\infty)$	Positive	f is increasing on the interval $(1,\infty)$

Step 5 Use the First Derivative Test to identify the local extrema. From the table in Step 4, there is a local maximum at -3 and a local minimum at 1. Then $f(-3)=6e^{-3}\approx 0.30$ is a local maximum value and $f(1)=-2e\approx -5.44$ is a local minimum value. Plot the local extrema.

Step 6 To determine the concavity of f, first we solve $f''(x)=0$. We find $x=-2\pm\sqrt{5}$. Now use the numbers $-2-\sqrt{5}\approx -4.24$ and $-2+\sqrt{5}\approx 0.24$ to form three intervals on the x-axis: $(-\infty,-2-\sqrt{5})$, $(-2-\sqrt{5},-2+\sqrt{5})$ and $(-2+\sqrt{5},\infty)$. Then determine the sign of $f''(x)$ on each interval and whether f is concave up or concave down on the interval.

Interval	Sign of f''	Conclusion
$(-\infty,-2-\sqrt{5})$	Positive	f is concave up on the interval $(-\infty,-2-\sqrt{5})$
$(-2-\sqrt{5},-2+\sqrt{5})$	Negative	f is concave down on the interval $(-2-\sqrt{5},-2+\sqrt{5})$
$(-2+\sqrt{5},\infty)$	Positive	f is concave up on the interval $(-2+\sqrt{5},\infty)$

The inflection points are $(-4.24, 0.22)$ and $(0.24, -3.73)$. Plot the inflection points.

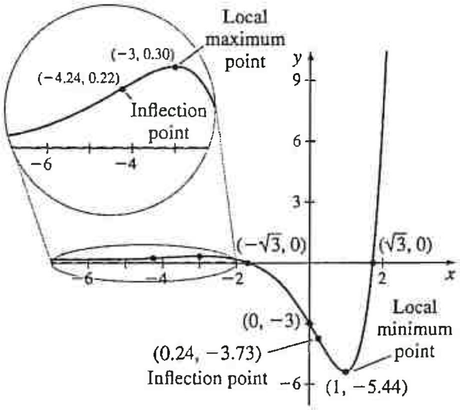


Figure 49 $f(x)=e^x(x^2-3)$

Step 7 The graph of f is given in Figure 49. ■

NOW WORK Problem 39.

5.4 Assess Your Understanding

Skill Building

In Problems 1–42, use calculus to graph each function. Follow the steps for graphing a function on page 349.

PAGE 350 **1.** $f(x)=x^4-6x^2+10$

2. $f(x)=x^4-4x$

3. $f(x)=x^5-10x^2$

4. $f(x)=x^5-3x^3+4$

5. $f(x)=3x^5+5x^4$

6. $f(x)=60x^5+20x^3$

7. $f(x)=\frac{2}{x^2-4}$

8. $f(x)=\frac{1}{x^2-1}$

9. $f(x)=\frac{2x-1}{x+1}$

10. $f(x)=\frac{x-2}{x}$

PAGE 352 **11.** $f(x)=\frac{x}{x^2+1}$

12. $f(x)=\frac{2x}{x^2-4}$

42

13. $f(x)=\dfrac{x^2+1}{2x}$ 14. $f(x)=\dfrac{x^2-1}{2x}$

15. $f(x)=\dfrac{x^2-1}{x^2+2x-3}$ 16. $f(x)=\dfrac{x^2-x-12}{x^2-9}$

17. $f(x)=\dfrac{x(x^3+1)}{(x^2-4)(x+1)}$ 18. $f(x)=\dfrac{x(x^3-1)}{(x+1)^2(x-1)}$

19. $f(x)=1+\dfrac{1}{x}+\dfrac{1}{x^2}$ 20. $f(x)=\dfrac{2}{x}+\dfrac{1}{x^2}$

21. $f(x)=\sqrt{3-x}$ 22. $f(x)=x\sqrt{x+2}$

23. $f(x)=x+\sqrt{x}$ 24. $f(x)=\sqrt{x}-\sqrt{x+1}$

25. $f(x)=\dfrac{x^2}{\sqrt{x+1}}$ 26. $f(x)=\dfrac{x}{\sqrt{x^2+2}}$

27. $f(x)=\dfrac{1}{(x+1)(x-2)}$ 28. $f(x)=\dfrac{1}{(x-1)(x+3)}$

29. $f(x)=x^{2/3}+3x^{1/3}+2$ 30. $f(x)=x^{5/3}-5x^{2/3}$

31. $f(x)=\sin x-\cos x$ 32. $f(x)=\sin x+\tan x$

33. $f(x)=\sin^2 x-\cos x$ 34. $f(x)=\cos^2 x+\sin x$

35. $f(x)=\ln x-x$ 36. $f(x)=x\ln x$

37. $f(x)=\ln(4-x^2)$ 38. $f(x)=\ln(x^2+2)$

39. $f(x)=3e^{3x}(5-x)$ 40. $f(x)=3e^{-3x}(x-4)$

41. $f(x)=e^{-x^2}$ 42. $f(x)=e^{1/x}$

Applications and Extensions

In Problems 43–52, for each function:

(a) Graph the function.

(b) Identify any asymptotes.

(c) Use the graph to identify intervals on which the function increases or decreases and the intervals where the function is concave up or down.

(d) Approximate the local extreme values using the graph.

(e) Compare the approximate local maxima and local minima to the exact local extrema found by using calculus.

(f) Approximate any inflection points using the graph.

43. $f(x)=\dfrac{x^{2/3}}{x-1}$ 44. $f(x)=\dfrac{5-x}{x^2+3x+4}$

45. $f(x)=x+\sin(2x)$ 46. $f(x)=x-\cos x$

47. $f(x)=\ln(x\sqrt{x}-1)$ 48. $f(x)=\ln(\tan^2 x)$

49. $f(x)=\sqrt[3]{\sin x}$ 50. $f(x)=e^{-x}\cos x$

51. $y^2=x^2(6-x),\ y\ge 0$ 52. $y^2=x^2(4-x^2),\ y\ge 0$

In Problems 53–56, graph a function f that is continuous on the interval $[1, 6]$ and satisfies the given conditions.

53. $f'(2)$ does not exist
$f'(3)=-1$
$f''(3)=0$
$f'(5)=0$
$f''(x)<0,\quad 2<x<3$
$f''(x)>0,\quad x>3$

54. $f'(2)=0$
$f''(2)=0$
$f'(3)$ does not exist
$f'(5)=0$
$f''(x)>0,\quad 2<x<3$
$f''(x)>0,\quad x>3$

55. $f'(2)=0$
$\lim_{x\to 3^-} f'(x)=\infty$
$\lim_{x\to 3^+} f'(x)=\infty$
$f'(5)=0$
$f''(x)>0,\quad x<3$
$f''(x)<0,\quad x>3$

56. $f'(2)=0$
$\lim_{x\to 3^-} f'(x)=-\infty$
$\lim_{x\to 3^+} f'(x)=-\infty$
$f'(5)=0$
$f''(x)<0,\quad x<3$
$f''(x)>0,\quad x>3$

57. Graph a function f defined and continuous for $-1\le x\le 2$ that satisfies the following conditions:

$$f(-1)=1 \quad f(1)=2 \quad f(2)=3 \quad f(0)=0 \quad f\left(\frac{1}{2}\right)=3$$

$$\lim_{x\to -1^+} f'(x)=-\infty \quad \lim_{x\to 1^-} f'(x)=-1 \quad \lim_{x\to 1^+} f'(x)=\infty$$

f has a local minimum at 0. f has a local maximum at $\frac{1}{2}$.

58. **Graph of a Function** Which of the following is true about the graph of $f(x)=\ln|x^2-1|$ in the interval $(-1, 1)$?

(a) The graph is increasing.

(b) The graph has a local minimum at $(0, 0)$.

(c) The graph has a range of all real numbers.

(d) The graph is concave down.

(e) The graph has an asymptote $x=0$.

59. **Properties of a Function** Suppose $f(x)=\dfrac{1}{x}+\ln x$ is defined only on the closed interval $\dfrac{1}{e}\le x\le e$.

(a) Determine the numbers x at which f has its absolute maximum and absolute minimum.

(b) For what numbers x is the graph concave up?

(c) Graph f.

60. **Probability** The function $f(x)=\dfrac{1}{\sqrt{2\pi}}e^{-x^2/2}$, encountered in probability theory, is called the **standard normal density function**. Determine where this function is increasing and decreasing, find all local maxima and local minima, find all inflection points, and determine the intervals where f is concave up and concave down. Then graph the function.

In Problems 61–64, graph each function. Use L'Hôpital's Rule to find any asymptotes.

61. $f(x)=\dfrac{\sin(3x)}{x\sqrt{4-x^2}}$ 62. $f(x)=x^{\sqrt{x}}$

63. $f(x)=x^{1/x}$ 64. $f(x)=\dfrac{1}{x}\tan x \quad -\dfrac{\pi}{2}<x<\dfrac{\pi}{2}$

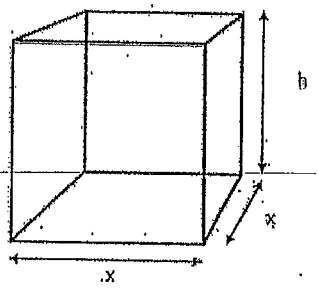
Calculus Optimization Notes

Optimization: Optimization is the process of finding the greatest (maximum optimal solution) or least value of a function (the minimum optimal solution) for some constraint, which must be true regardless of the solution. Optimization finds the most suitable value for a function within a given domain.

Optimization steps:

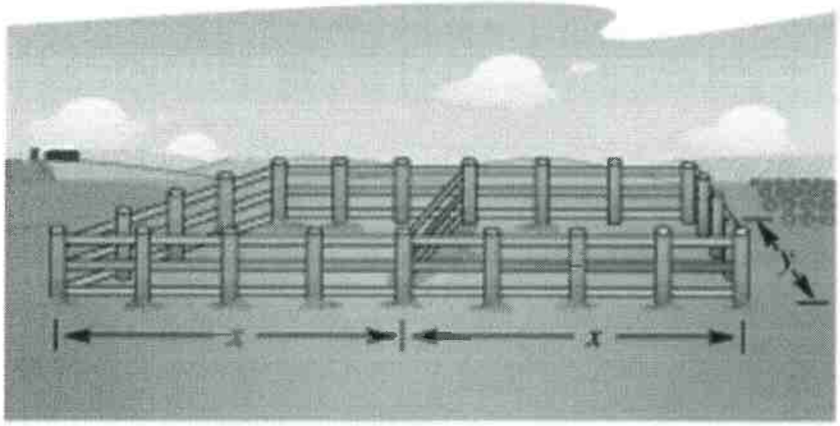
1. Write equation for variable you want to optimize
2. Substitute to get equation in terms of one variable on one side
3. Find derivative, set derivative = 0 and solve.

Example 1: A manufacturer wants to design an open box having a square base and a surface area of 108 in^2. What dimensions will produce a box with maximum volume?



2)

A rancher has 200 feet of fencing with which to enclose two adjacent rectangular corrals (see figure). What dimensions should be used so that the enclosed area will be a maximum?



Calculus AB Optimization Practice Problems

1. A rectangular field is to be fenced off along the bank of a river; no fence is required along the river. If the material for the fence costs $\$8$ per running foot for the two ends, and $\$12$ per foot for the side parallel to the river, find the dimensions of the field of largest possible area that can be enclosed with $\$3600$ worth of fence.

2. A rectangular storage container with an open top is to have a Volume of $10\ \text{m}^3$. The length of its base is twice its length. Material for the base costs $\$10/\ \text{m}^2$. Material for the sides cost $\$6/\text{m}^2$. Find the cost of material for the cheapest container. (Hint: Minimize surface area)

---

3. A piece of cardboard measures 10 by 15 in. For equal squares are removed from corners of all sides. Find the maximum volume.

4. 1988 multiple choice problem #45
The volume of a cylindrical tin can with a top and a bottom is to be 16π cubic inches. If a minimum amount of tin is to be used to construct the can, what must be the height, in inches, of the can? (A cylinder with radius r and height h has a volume of $V = \pi r^2 h$ and a surface area of $S = 2\pi r^2 + 2\pi r h$.)

46

5. Highway 400 averaged 54,000 cars per day for its first 5 years charging $0.50 per car. A scientific research study concludes that for every $0.05 increase in the toll, the number of cars will be reduced by 500. In order to maximize revenue, what toll should Highway 400 charge?

6. The manager of a department store wants to build a 600 square foot rectangular enclosure on the store's parking lot in order to display some equipment. Three sides of the enclosure will be built of redwood fencing, at a cost of $14 per running foot. The forth side will be built of cement blocks, at a cost of $28 per running foot. What dimensions will minimize the total cost of the building materials? What will this minimum cost be?

7. A 150-room resort hotel is filled at a room rate of $125 per day. For each $5 increase in the room rate, three fewer rooms are rented. What room rate will result in maximum daily revenue? How many rooms will be rented at that rate?

8. A farmer has 400 feet of fencing to make three rectangular pens. What dimensions x and y will maximize the total area?

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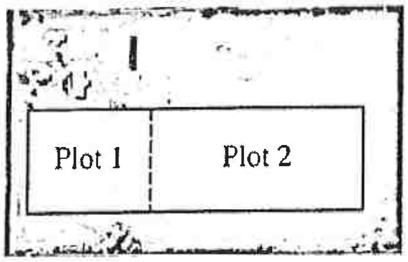
5.5 Optimization Homework

pg. 366-370 #5,6,7,9,12,14

5. **Maximizing Area**
A gardener with 200 m of available fencing wishes to enclose a rectangular field and then divide it into two plots with a fence parallel to one of the sides, as shown in the figure. What is the largest area that can be enclosed?



6. **Minimizing Fencing** A realtor wishes to enclose 600 m^2 of land in a rectangular plot and then divide it into two plots with a fence parallel to one of the sides. What are the dimensions of the rectangular plot that require the least amount of fencing?

7. **Maximizing the Volume of a Box** An open box with a square base is to be made from a square piece of cardboard that measures 12 cm on each side. A square will be cut out from each corner of the cardboard and the sides will be turned up to form the box. Find the dimensions that yield the maximum volume.

9. **Minimizing the Surface Area of a Box** An open box with a square base is to have a volume of 2000 cm^3. What should be the dimensions of the box if the amount of material used is to be a minimum?

12. **Minimizing the Cost of Fencing** A builder wishes to fence in 60,000 m^2 of land in a rectangular shape. For security reasons, the fence along the front part of the land will cost $20 per meter, while the fence for the other three sides will cost $10 per meter. How much of each type of fence should the builder buy to minimize the cost of the fence? What is the minimum cost?

14. **Maximizing Revenue** A charter flight club charges its members $200 per year. But for each new member in excess of 60, the charge for every member is reduced by $2. What number of members leads to a maximum revenue?

5.5 – Optimization - AP Practice Problems (p. 370)

1. Find the dimensions of the rectangle with the largest area that can be formed with its base on the x-axis and its upper vertices on the graph of the parabola $y = 9 - x^2$.

(A) base: $\sqrt{3}$, height: 6 (B) base: 3, height: 6

(C) base: 5, height: 5 (D) base: $2\sqrt{3}$, height: 6

2. A closed cylindrical aluminum can has a volume of 16π in^3. Find the dimensions of the can that minimizes the amount of aluminum used.

(A) height: 8 in; radius: 2 in (B) height: 4 in; radius: 2 in

(C) height: 2 in; radius: 4 in (D) height: $\frac{16}{9}$ in; radius: 3 in

3. What is the area of the largest rectangle with sides parallel to the coordinate axes that can be inscribed in the ellipse $4x^2 + y^2 = 16$?

(A) 4 (B) $2\sqrt{2}$ (C) $8\sqrt{2}$ (D) 16

4. If $y = 4x^2 - 3$, what is the minimum value of the product xy?

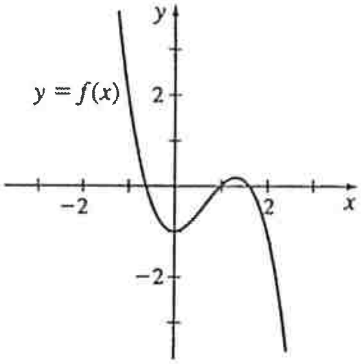
(A) -1 (B) 1 (C) -12 (D) -2

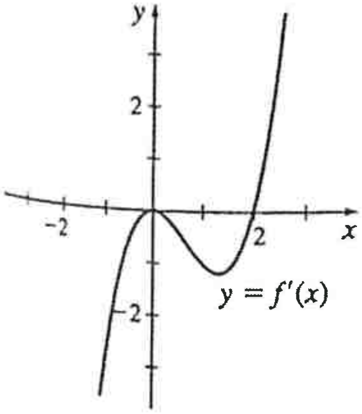
5. What point on the graph of $(x - 1)y = 4$, $x \geq 0$, is closest to the point $(1, 0)$?

(A) $(0, -4)$ (B) $\left(4, \frac{4}{3}\right)$ (C) $(3, 2)$ (D) $(5, 1)$

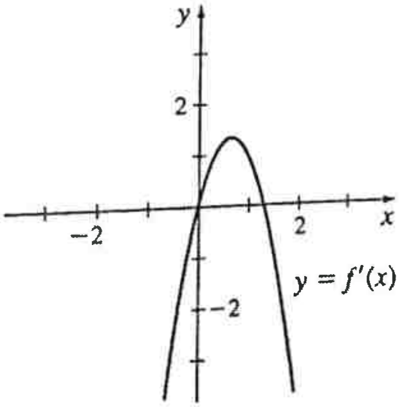
Ch. 5 Unit Review AP Practice Problems (p.385-386)

1. The graph of a function f is shown below. Which of the following could be the graph of the derivative function f'?

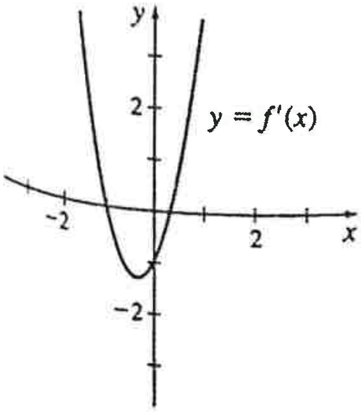




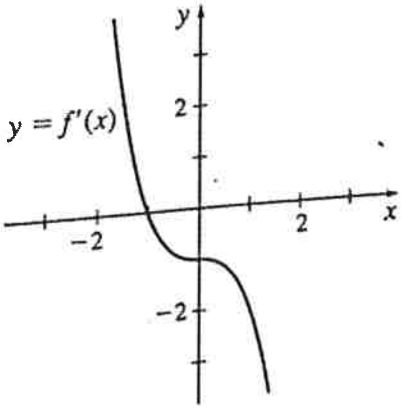
(A)



(B)



(C)



(D)

2. On which of the following intervals is the function $f(x) = x^3 - 2x^2 + x$ increasing?

(A) $(-\infty, \infty)$

(B) $(0, \infty)$

(C) $\left(-\infty, \frac{1}{3}\right)$ and $(1, \infty)$

(D) $\left(\frac{1}{3}, 1\right)$

3. An object moves along the y-axis, so that at any time $t \geq 0$, its position is given by $y(t) = te^{-t^2}$. At what time(s) t is the object at rest?

(A) $\frac{\sqrt{2}}{2}$ only (B) 0 only

(C) 0 and $\frac{\sqrt{2}}{2}$ (D) $\frac{\sqrt{2}}{2}$ and $-\frac{\sqrt{2}}{2}$

4. The polynomial function f has a second derivative function f''. Several values of f'' are given in the table. Which statement must be true?

x	0	1	2	3	4
$f''(x)$	1	-1	2	0	-2

(A) f changes concavity in the interval $(2, 4)$.
(B) f has a point of inflection at $x = 3$.
(C) f has a local maximum at $x = 3$.
(D) f is not increasing on the interval $(0, 2)$.

5. The function f is continuous for all real numbers and has a relative maximum at $(-4, 3)$ and a relative minimum at $(1, -8)$. Which of the following statements must be true?

(A) f has a point of inflection between $(-4, 3)$ and $(1, -8)$.
(B) f has a horizontal asymptote at either $(-4, 3)$ or $(1, -8)$.
(C) f has at least one zero.
(D) f is decreasing on the open interval $(-4, 1)$.

6. The function $f(x) = 3x^{1/3} - 4x + 1$ has a local minimum at $x =$

(A) 1 (B) $\frac{1}{8}$ (C) $-\frac{1}{8}$ (D) 0

7. Suppose f is a polynomial function of degree greater than 2 and $f(a) = f(b)$ where $a < b$. Then which of the following statements must be true for at least one number c in the interval (a, b)?

I. $f(c) = 0$

II. $f'(c) = 0$

III. $f''(c) = 0$

(A) I only (B) II only

(C) II and III only (D) I, II, and III

8. Find the absolute maximum value of the function $f(\theta) = \cos\theta - \cos^2\theta$ on the closed interval $[0, \pi]$.

(A) 1 (B) $\frac{1}{2}$ (C) $\frac{1}{4}$ (D) 0

(54)

) **9.** A water storage tank in the shape of a right circular cylinder is constructed so that the sum of the height and the circumference of the tank is 300 m. What are the radius r and the height h of the tank with maximum volume?

(A) radius: 50 m; height 200 m

(B) radius: $\frac{100}{\pi}$ m; height 100 m

(C) radius: 100π m; height 100 m

(D) radius: $\frac{50}{\pi}$ m; height 200 m

10. Find all the antiderivatives of the function $f(x) = 3x^2 + \sec^2 x - 4e^x$.

(A) $x^3 + \tan x - 4e^x + C$

(B) $x^3 + \tan x - 2e^{2x} + C$

(C) $x^3 + \frac{\sec^3 x}{3} - 2e^{x^2} + C$

(D) $x^3 + 2\sec^2 x \tan x - 4e^x + C$

11. For the function $f(x) = x^3 + 3x^2 + 2$

(a) Find the critical numbers of f.

(b) Find the intervals where the function is increasing and the intervals where it is decreasing.

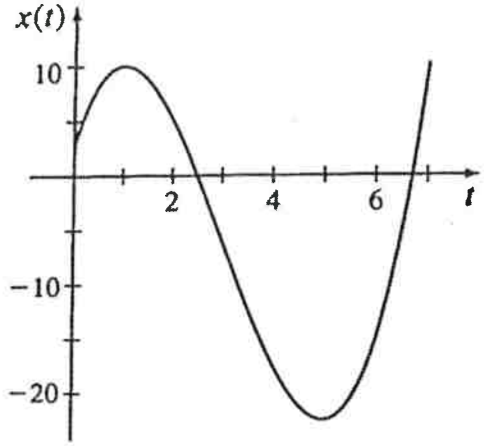
(c) Identify the local extreme points.

(d) Find the intervals where the function is concave up and concave down.

(e) Identify any inflection points.

(f) Find an equation of the tangent line to the graph of f at its inflection point(s).

12. The position $x(t)$ (in feet) of an object in rectilinear motion for time t, $0 \leq t \leq 7$ (in seconds), is shown in the graph. The graph has horizontal tangents at time $t = 1$ and $t = 5$ and has an inflection point at $(3, -6)$.



(a) At what time(s) t is the object at rest?

(b) On what interval(s) is the velocity of the object increasing?

13. (a) Find the general solution to the differential equation

$$\frac{d^2y}{dx^2} = 3x^2 - 6x$$

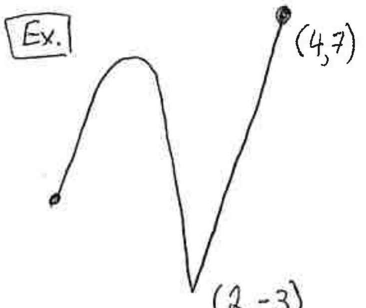
(b) Verify the solution found in (a).

(c) Find the particular solution to the differential equation $\frac{d^2y}{dx^2} = 3x^2 - 6x$ with the boundary conditions when $x = 0$, then $y = 2$ and when $x = 1$, then $y = 3$.

Ch. 3.1

Extreme Value Theorem (EVT) Ch. 3.1

Ex.

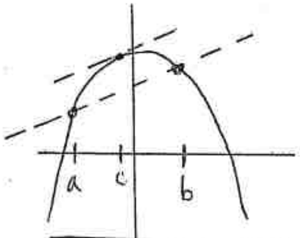


Purpose: Find Abs max/min on closed interval

* ✱ $f(x)$ continuous $[a,b]$
* ✱ find critical points
  - a) set $f'(x)=0$
  - b) set denominator of $f'(x)=0$
* ✱ test critical points and endpoints into $f(x)$ to find absolute max/min
* ✱ Abs max is 7 at $x=4$

  Abs min is -3 at $x=2$

3.2a Mean Value Theorem (MVT)



purpose: find the location on the curve where the guaranteed slope occurs.

$$\boxed{\text{MVT: } f'(c)=\frac{f(b)-f(a)}{b-a}}$$

Conditions:

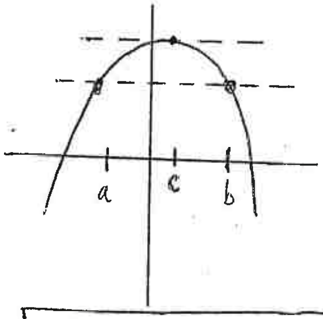
* ✱ $f(x)$ continuous on $[a,b]$ (no VA, no holes on interval)
* ✱ $f(x)$ differentiable on (a,b) (no sharp turns, no slope undefined on (a,b))

Steps:

1) find slope between endpoints $\left[\frac{f(b)-f(a)}{b-a}\right]$
2) find $f'(x)$
3) set $f'(x)=$ slope value, solve for x (c-value)
4) Keep the c-values in interval (a,b)

(17)

3.2b Rolle's Theorem



Purpose: Find the location on the curve where the guaranteed slope of 0 occurs.

Conditions:

- * f(x) continuous [a,b] (no breaks, no vertical asymptote, no holes)
- * f(x) differentiable (a,b) (no sharp turns, no location with undefined slope)
- * f(a) = f(b) (endpoints with same y-value)

$\boxed{\text{Rolle's Theorem: } f'(c)=0}$

Steps:

1) confirm endpts have same y-values
2) find f'(x)
3) set numerator of f'(x) = 0, solve for x (c-value)
4) Keep the c-values in interval (a,b)

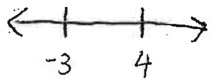
(18)

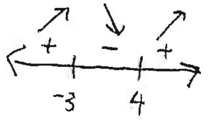
Ch. 3.3 1st Derivative Test

Purpose: Use f'(x) to determine slope behavior of graph and find relative max, relative mins of graph

1) Find critical points
   a) find f'(x)
   b) set numerator of f'(x) = 0
   c) set denominator of f'(x) = 0
2) Place critical values on f'(x) sign line
3) Test intervals, plug in x-values into f'(x)
   a) Rel. max at (-3,_) b/c f'(x) changes from + to −
   b) Rel. min at (4,_) b/c f'(x) changes from − to +
   c) f(x) increasing (-∞,3),(4,∞) b/c f'(x) > 0
   d) f(x) decreasing (-3,4) b/c f'(x) < 0

Ex:



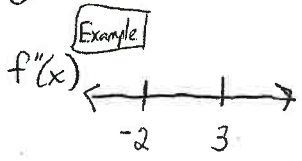


Ch. 3.4 Concavity Test:

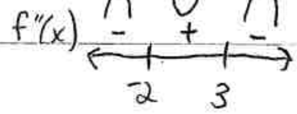
Purpose: Use $f''(x)$ to determine concavity behavior of graph and find Points of Inflection (POI)

1) Find critical points
   a) find $f''(x)$
   b) set numerator, denominator of $f''(x) = 0$

2) Place critical points on $f''(x)$ sign line.



3) Test intervals, plug x-values into $f''(x)$



a) POI at (-2,_) and (3,_) b/c $f''(x)$ change signs.

b) $f(x)$ concave up (-2,3) b/c $f''(x) > 0$

c) $f(x)$ concave down $(-\infty,-2)$, $(3,\infty)$ b/c $f''(x) < 0$

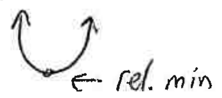
Ch. 3.4 2nd derivative test

Purpose: Use $f''(x)$ to determine relative max/mins of graph

Steps:

example

1) find $f'(x)$ and critical points (set $f'(x)=0$) $x = a, b$
2) find $f''(x)$
3) plug in critical points from $f'(x)$ into $f''(x)$
4) If $f''(a) > 0$, concave up, Rel. min at $x=a$



5) If $f''(b) < 0$, concave down, Rel. max at $x=b$