Key

Ch. 4 Review AP Practice Problems (p. 304)

1. If $y = \tan(3x + 2y)$, find the rate of change of y with respect to x at the origin.

$\boxed{\text{(A)}}$ -3 (B) 1 (C) 3 (D) 5

$y = \tan(3x+2y)$

$\frac{dy}{dx} = \sec^2(3x+2y)\cdot\left[3+2\left(\frac{dy}{dx}\right)\right]$

$\left.\frac{dy}{dx}\right|_{(0,0)} = \sec^2(0+0)\left[3+2\left(\frac{dy}{dx}\right)\right]$

$\frac{dy}{dx} = (1)\left(3+2\left(\frac{dy}{dx}\right)\right)$

$1\left(\frac{dy}{dx}\right) = 3+2\left(\frac{dy}{dx}\right)$

$-3 = \frac{dy}{dx}$

$\boxed{\frac{dy}{dx} = -3}$

2. The linear approximation for $f(x) = xe^x$ near $x = 2$ is

(A) $L(x) = 3e^2(x-2)$ (B) $L(x) = xe^x + xe^x(x-2)$

(C) $L(x) = 2e^2 + 3e^2(x-2)$ (D) $L(x) = 2e^2 + 2e^2(x-2)$

$f(2) = 2e^2$

$f'(x) = \underbrace{1}_{f'}\cdot\underbrace{e^x}_{g} + \underbrace{x}_{f}\cdot\underbrace{e^x(1)}_{g'}$

$f'(2) = e^2 + 2e^2$

$f'(2) = 3e^2$

point: $(2, 2e^2)$

slope: $m = 3e^2$

$y - y_1 = m(x - x_1)$

$y - 2e^2 = 3e^2(x-2)$

$\boxed{y = 2e^2 + 3e^2(x-2)}$

3. $\lim_{x\to\infty} \frac{x}{\ln x} =$

(A) 0 (B) 1 (C) e $\boxed{\text{(D)}}$ ∞

* comparative growth rate $L < R < P < E$

$\lim_{x\to\infty} \frac{\text{polynomial}}{\text{logarthm}} \to \boxed{+\infty}$

$\lim_{x\to\infty} \frac{x}{\ln x}$

4. The radius of a circle is increasing at a constant positive rate with respect to time. What is the radius of the circle when the rate of change of the area with respect to time is equal to twice the rate of change of the circumference with respect to time?

(A) 2 (B) $\frac{1}{2}$ (C) 1 (D) 4

$$\frac{dA}{dt} = 2\left(\frac{dC}{dt}\right)$$

$$A = \pi r^2 \qquad C = 2\pi r$$

$$\frac{dA}{dt} = 2\pi r\left(\frac{dr}{dt}\right) \qquad \frac{dC}{dt} = 2\pi\left(\frac{dr}{dt}\right)$$

$$\frac{dA}{dt} = 2\pi r\left(\frac{dr}{dt}\right)$$

$$2\left(\cancel{\frac{dC}{dt}}\right) = r\cdot\left(\cancel{\frac{dC}{dt}}\right)$$

$$\boxed{2 = r}$$

5. $\lim\limits_{x\to 0} \dfrac{xe^x}{\sin x} =$

(A) 0 (B) 1 (C) e (D) ∞

$$\lim_{x\to 0}\frac{xe^x}{\sin x} \to \frac{0}{0} \to \lim_{x\to 0}\frac{\overset{f'}{1}\cdot\overset{g}{e^x} + \overset{f}{x}\cdot\overset{g'}{e^x}}{\cos x} \to \frac{1e^0 + 0e^0}{\cos 0} \to \frac{1}{1} \to \boxed{1}$$

6. Use the linear approximation to $f(x) = \tan x$ at $x = 0$ to approximate $f(0.2)$.

(A) -0.2 (B) 0 (C) 0.2 (D) 0.8

$f(0) = \tan(0) = 0$

$f'(x) = \sec^2 x$

$f'(0) = \sec^2 0 \to 1$

point: $(0,0)$

slope: $m = 1$

$y - y_1 = m(x - x_1)$

$y - 0 = 1(x - 0)$

$y = 1x$

$y(0.2) = 1(0.2) = \boxed{0.2}$

Free Response Questions

7. Water is being pumped into a cylindrical tank that measures 20 m in height and 4 m in radius at a constant rate of 5 m^3/h. Ten meters from the bottom of the tank there is a hole in the tank. Water leaks from that hole at a rate of 1 m^3/h.

   (a) Find the rate at which the water is rising in the tank until it reaches the leak.

   (b) Find the rate at which the water is rising in the tank after it passes the leak.

   (c) What is the total time it will take for the tank to begin to overflow?

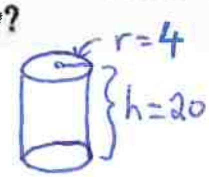
a) $\frac{dh}{dt} = \frac{5}{16\pi}$ m/h

b) $\frac{dh}{dt} = \frac{1}{4\pi}$ m/h

c) 226.195 h.

$V = \pi r^2 h$

*radius is constant



$\frac{dV}{dt} = 5\ m^3/hr$

$\frac{dV}{dt}(\text{hole}) = -1\ m^3/hr$

$V = \pi r^2 h$

$V = \pi (4)^2 h$

$V = 16\pi h$

$\frac{dV}{dt} = 16\pi \left(\frac{dh}{dt}\right)$

a) $5 = 16\pi \left(\frac{dh}{dt}\right)$

$\frac{5}{16\pi} = \frac{dh}{dt}$

$\boxed{\frac{dh}{dt} = \frac{5}{16\pi}\ m/h}$

b) $\frac{dV}{dt} = 5 - 1 = 4\ m^3/hr$

$\frac{dV}{dt} = 16\pi \left(\frac{dh}{dt}\right)$

$4 = 16\pi \left(\frac{dh}{dt}\right)$

$\frac{4}{16\pi} = \frac{dh}{dt}$

$\boxed{\frac{dh}{dt} = \frac{1}{4\pi}\ m/h}$

c) $\frac{1}{4\pi} \cdot t_1 = 10m$

$t_1 = 10 \cdot 4\pi = 40\pi$

$t_1 = 125.664$ hr.

(time taken to fill up first 10 meters of tank)

$\frac{5}{16\pi} \cdot t_2 = 10m$

$t_2 = 10 \cdot \frac{16\pi}{5} = 100.5312$ hrs.

(time taken to fill up upper half of tank)

Total time: $125.664 + 100.5312 = \boxed{226.195\text{ hrs.}}$

8. **The position function of an object moving along the x-axis is $s(t) = t\sin t + 3$, where s in meters and $t \geq 0$ in minutes.**

(a) Find the initial position of the object. Find its position at $t = \frac{2\pi}{3}$ min.

(b) Find and interpret $s'\left(\frac{2\pi}{3}\right)$ in the context of this problem.

(c) Find the average velocity of the object from $t = 0$ to $t = \frac{2\pi}{3}$.

a) $4.814m \left(\frac{\pi\sqrt{3}+9}{3}\right)$

b) $\frac{3\sqrt{3}-2\pi}{6}$

c) $\frac{\sqrt{3}}{2}$

a) $s(t) = t\sin t + 3$

$s(0) = 0\sin 0 + 3 = \boxed{3}$

$s\left(\frac{2\pi}{3}\right) = \frac{2\pi}{3}\sin\left(\frac{2\pi}{3}\right) + 3 = \frac{2\pi}{3}\left(\frac{\sqrt{3}}{2}\right) + 3 \rightarrow \boxed{\frac{\pi\sqrt{3}+9}{3} \approx 4.814m}$

b) $s'(t) = \underbrace{1}_{f'} \cdot \underbrace{\sin(t)}_{g} + \underbrace{t}_{f} \cdot \underbrace{\cos(t)}_{g'} + 0$

$s'\left(\frac{2\pi}{3}\right) = \sin\left(\frac{2\pi}{3}\right) + \frac{2\pi}{3}\cos\left(\frac{2\pi}{3}\right) \rightarrow \frac{\sqrt{3}}{2} + \frac{2\pi}{3} \cdot \frac{-1}{2} \rightarrow \frac{\sqrt{3}}{2} - \frac{\pi}{3}$

$s'\left(\frac{2\pi}{3}\right) = \frac{3\sqrt{3}-2\pi}{6} \approx -0.181$

The object is moving in the negative direction at 0.181 m/min at $t = \frac{2\pi}{3}$

c) $\text{Avg. velocity} = \frac{s\left(\frac{2\pi}{3}\right) - s(0)}{\frac{2\pi}{3} - 0}$

*Avg. velocity is $\frac{\text{change in position}}{\text{change in time}}$

$= \frac{\frac{\pi\sqrt{3}}{3} + 3 - 3}{\frac{2\pi}{3} - 0} \rightarrow \frac{\frac{\pi\sqrt{3}}{3}}{\frac{2\pi}{3}} \rightarrow \frac{\pi\sqrt{3}}{3} \cdot \frac{3}{2\pi} \rightarrow \boxed{\frac{\sqrt{3}}{2} \approx 0.866 \text{ m/min}}$