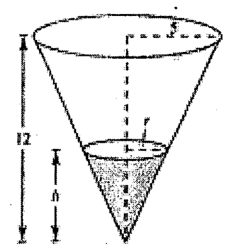
18. Depth A conical tank (with vertex down) is 10 feet across the top and 12 feet deep. Water is flowing into the tank at a rate of 10 cubic feet per minute. Find the rate of change of the depth of the water when the water is 8 feet deep.

$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \frac{25}{144} h^3 = \frac{25\pi}{3(144)} h^3$ $\left(\text{By similar triangles, } \frac{r}{5} = \frac{h}{12} \Rightarrow r = \frac{5}{12}h.\right)$

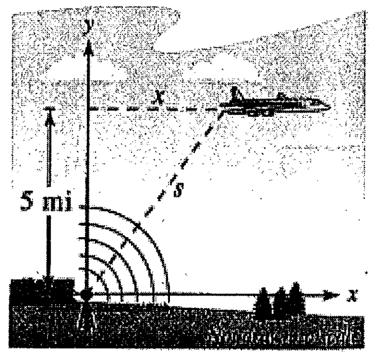


$\frac{dV}{dt} = 10$

$\frac{dV}{dt} = \frac{25\pi}{144} h^2 \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \left(\frac{144}{25\pi h^2}\right)\frac{dV}{dt}$

When $h = 8$, $\frac{dh}{dt} = \frac{144}{25\pi(64)}(10) = \frac{9}{10\pi}$ ft/min.

26. Air Traffic Control An airplane is flying at an altitude of 5 miles and passes directly over a radar antenna (see figure). When the plane is 10 miles away ($s = 10$), the radar detects that the distance s is changing at a rate of 240 miles per hour. What is the speed of the plane?



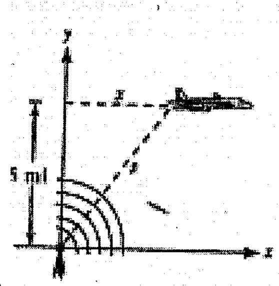
$x^2 + y^2 = s^2$

$2x\frac{dx}{dt} + 0 = 2s\frac{ds}{dt}$ $\left(\text{because } \frac{dy}{dt} = 0\right)$

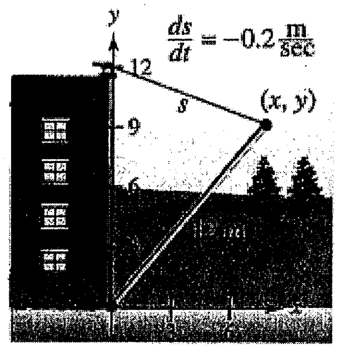
$\frac{dx}{dt} = \frac{s}{x}\frac{ds}{dt}$

When $s = 10$, $x = \sqrt{100 - 25} = \sqrt{75} = 5\sqrt{3}$,

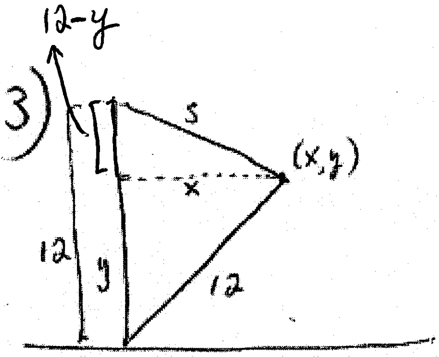
$\frac{dx}{dt} = \frac{10}{5\sqrt{3}}(240) = \frac{480}{\sqrt{3}} = 160\sqrt{3} \approx 277.13$ mi/h.



23. Construction A winch at the top of a 12-meter building pulls a pipe of the same length to a vertical position, as shown in the figure. The winch pulls in rope at a rate of -0.2 meter per second. Find the rate of vertical change and the rate of horizontal change at the end of the pipe when $y = 6$.



23)



$\frac{ds}{dt} = -0.2\ m/s$ Find $\frac{dx}{dt}, \frac{dy}{dt}$

$y = 6,\ x = 6\sqrt{3},\ s = 12$

$x^2 + (12-y)^2 = s^2$, and $\longrightarrow$ $x^2 + y^2 = 12^2$

$2x\left(\frac{dx}{dt}\right) + 2(12-y)(-1)\boxed{\left(\frac{dy}{dt}\right)} = 2s\left(\frac{ds}{dt}\right)$

$2x\left(\frac{dx}{dt}\right) + 2y\left(\frac{dy}{dt}\right) = 0$

$\cancel{2}x\left(\frac{dx}{dt}\right) + \cancel{2}(y-12)\boxed{\left[\frac{-x}{y}\left(\frac{dx}{dt}\right)\right]} = \cancel{2}s\left(\frac{ds}{dt}\right)$

$\frac{dy}{dt} = \boxed{\frac{-x}{y}\left(\frac{dx}{dt}\right)}$

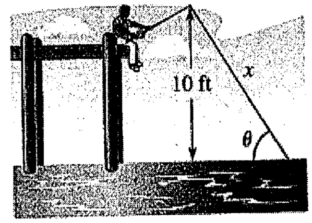
$\cancel{x\left(\frac{dx}{dt}\right)} - \cancel{x\left(\frac{dx}{dt}\right)} + \frac{12x}{y}\left(\frac{dx}{dt}\right) = s\left(\frac{ds}{dt}\right)$

$\frac{dx}{dt} = \frac{sy}{12x}\left(\frac{ds}{dt}\right)$

$\frac{dx}{dt} = \frac{(12)(6)}{12(6\sqrt{3})}(-0.2) = \frac{-1}{5\sqrt{3}} = \boxed{\frac{-\sqrt{3}}{15}\ m/sec}$ (horizontal)

$\frac{dy}{dt} = \frac{-6\sqrt{3}}{6}\left(\frac{-\sqrt{3}}{15}\right) = \frac{1}{5}\ m/sec$ (vertical)

39. Angle of Elevation A fish is reeled in at a rate of 1 foot per second from a point 10 feet above the water (see figure). At what rate is the angle θ between the line and the water changing when there is a total of 25 feet of line from the end of the rod to the water?



$$\sin\theta = \frac{10}{x}$$

$$\frac{dx}{dt} = (-1) \text{ ft/sec}$$

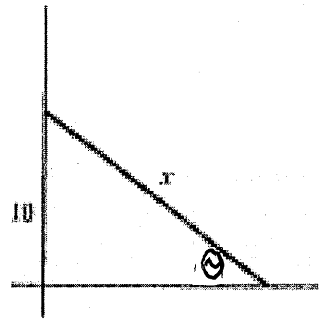
$$\cos\theta\left(\frac{d\theta}{dt}\right) = \frac{-10}{x^2}\cdot\frac{dx}{dt}$$

$$\frac{d\theta}{dt} = \frac{-10}{x^2}\frac{dx}{dt}(\sec\theta)$$

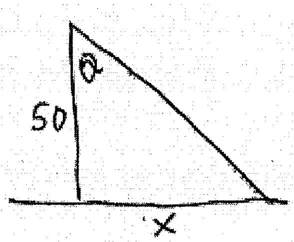
$$= \frac{-10}{25^2}(-1)\frac{25}{\sqrt{25^2-10^2}}$$

$$= \frac{10}{25}\frac{1}{5\sqrt{21}} = \frac{2}{25\sqrt{21}}$$

$$= \frac{2\sqrt{21}}{525} \approx 0.017 \text{ rad/sec}$$



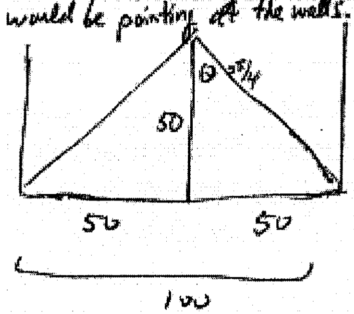
44. Security Camera A security camera is centered 50 feet above a 100-foot hallway (see figure). It is easiest to design the camera with a constant angular rate of rotation, but this results in recording the images of the surveillance area at a variable rate. So, it is desirable to design a system with a variable rate of rotation and a constant rate of movement of the scanning beam along the hallway. Find a model for the variable rate of rotation when $|dx/dt| = 2$ feet per second.



$$\tan\theta = \frac{x}{50} \qquad \frac{dx}{dt} = 2 \text{ ft/s}$$

$$\sec^2\theta\left(\frac{d\theta}{dt}\right) = \frac{1}{50}\left(\frac{dx}{dt}\right)$$

*If the angle extends beyond $\theta = \pi/4$ or $\theta = -\pi/4$, the security camera would be pointing at the walls.



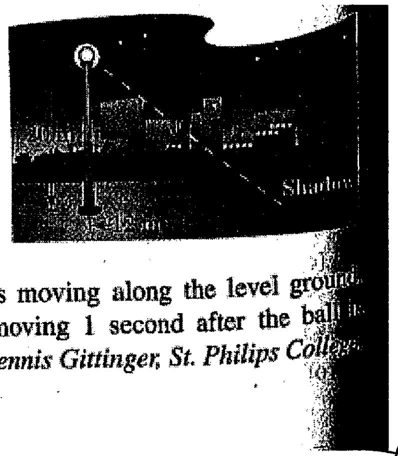
$$\frac{d\theta}{dt} = \cos^2\theta \cdot \frac{1}{50}\cdot\frac{dx}{dt}$$

$$\frac{d\theta}{dt} = \cos^2\theta \cdot \frac{1}{50}\cdot(2)$$

$$\boxed{\frac{d\theta}{dt} = \frac{1}{25}\cos^2\theta \quad \text{when } -\pi/4 \le \theta \le \pi/4}$$

49. Moving Shadow

A ball is dropped from a height of 20 meters, 12 meters away from the top of a 20-meter lamppost (see figure). The ball's shadow, caused by the light at the top of the lamppost, is moving along the level ground. How fast is the shadow moving 1 second after the ball is released? *(Submitted by Dennis Gittinger, St. Philips College, San Antonio, TX)*



Vertical motion equation:

$y(t) = \frac{1}{2}at^2 + v_0 t + y_0$

$y = -16t^2 + V_0 t + y_0$

OR $a = -9.8 m/s^2$

$y = -4.9t^2 + V_0 t + y_0$

$a = -32 ft/s^2$

V_0 = initial velocity

y_0 = initial height

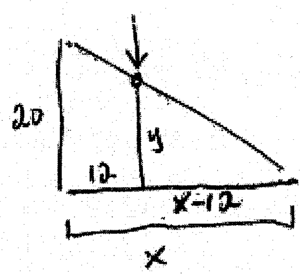
49) Moving Shadow

$y(t) = -4.9t^2 + 20 \leftarrow V_0 = 0, \; y_0 = 20$

$\frac{dy}{dt} = -9.8t$

$y(1) = -4.9 + 20 = 15.1$

$y'(1) = -9.8$



$\frac{y}{20} = \frac{x-12}{x}$

$\frac{15.1}{20} = \frac{x-12}{x}$

$15.1x = 20x - 240$

$-4.9x = -240$

$x = 48.979$

$\frac{y}{20} = \frac{x-12}{x}$

$yx = 20(x-12)$

$xy = 20x - 240$

$\frac{dx}{dt}y + x\left(\frac{dy}{dt}\right) = 20\left(\frac{dx}{dt}\right) - 0$

$\frac{dx}{dt}y - 20\left(\frac{dx}{dt}\right) = -x\left(\frac{dy}{dt}\right)$

$\frac{dx}{dt}(y - 20) = -x\left(\frac{dy}{dt}\right)$

$\frac{dx}{dt} = \frac{-x\left(\frac{dy}{dt}\right)}{y-20} = \frac{-48.979(-9.8)}{15.1-20} = \boxed{-97.958 \, m/sec}$