$c^2 = (6+h)^2 + \left(\sqrt{36-h^2}\right)^2$

$c^2 = 36 + 12h + h^2 + 36 - h^2 = 12h + 72$

$c = \sqrt{12(h+6)}$

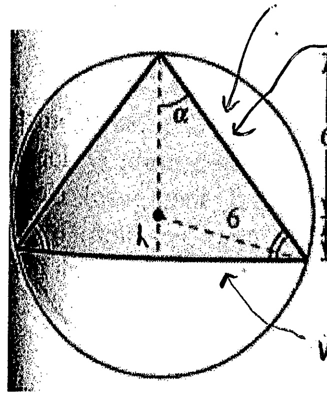
$= 2\sqrt{3} \cdot \sqrt{6+h}$

24. Maximum Area Find the area of the largest isosceles triangle that can be inscribed in a circle of radius 6 (see figure).

(a) Solve by writing the area as a function of h.

(b) Solve by writing the area as a function of α.

(c) Identify the type of triangle of maximum area.



a) $\text{Area} = \frac{1}{2} b \cdot h = \frac{1}{2}\left[2\sqrt{36-h^2}\right]\left[6+h\right] = \sqrt{36-h^2}\,(6+h) \rightarrow A = (36-h^2)^{1/2} \cdot (6+h)$

$\frac{dA}{dh} = \frac{1}{2}(36-h^2)^{-1/2}(-2h)\cdot(6+h) + (36-h^2)^{1/2}(1) = \frac{-h(6+h)}{\sqrt{36-h^2}} + \frac{\sqrt{36-h^2}}{1}$

$0 = \frac{-h(6+h) + 36 - h^2}{\sqrt{36-h^2}} \rightarrow -h^2 - 6h + 36 - h^2 = -2h^2 - 6h + 36 = -2(h^2 + 3h - 18)$

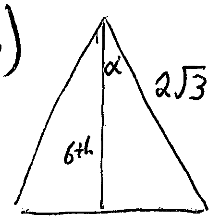
$= -2(h+6)(h-3)$

$\underline{h = 3}$

$\frac{dA}{dh} = 0$ when $h = 3$, sides are $6\sqrt{3}$

Max Area $= 27\sqrt{3}$ units^2

b)



$\cos\alpha = \frac{6+h}{2\sqrt{3}\cdot\sqrt{6+h}} = \frac{\sqrt{6+h}}{2\sqrt{3}}$ $\quad \tan\alpha = \frac{\sqrt{36-h^2}}{6+h}$

$\rightarrow 2\sqrt{3}\cdot\cos\alpha = \sqrt{6+h}$

$\text{Area} = \sqrt{36-h^2}\,(6+h) = (6+h)^2 \tan\alpha \leftarrow (2\sqrt{3}\cos\alpha)^4 = (6+h)^2$

$\text{Area} = 144\cos^4\alpha \tan\alpha$

$A'(\alpha) = 144 \cdot 4\cos^3\alpha \cdot -\sin\alpha \tan\alpha + 144\cos^4\alpha \cdot \sec^2\alpha$

$0 = 144\left[\cos^3(-\sin\alpha)\tan\alpha + \cos^4\alpha \sec^2\alpha\right) \rightarrow \cos^4\alpha \sec^2\alpha = 4\cos^3\alpha \sin\alpha \tan\alpha$

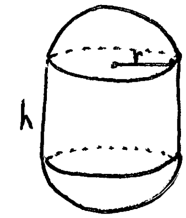
$1 = 4\cos\alpha \sin\alpha \tan\alpha$

$\frac{1}{4} = \sin^2\alpha, \quad \sin\alpha = \frac{1}{2}$

$\alpha = 30^\circ \quad A = 27\sqrt{3}$

c) Equilateral triangle

34. Minimum Cost An industrial tank of the shape described in Exercise 33 must have a volume of 4000 cubic feet. The hemispherical ends cost twice as much per square foot of surface area as the sides. Find the dimensions that will minimize cost.



$$V = \frac{4}{3}\pi r^3 + \pi r^2 h = 4000$$

$$\pi r^2 h = 4000 - \frac{4}{3}\pi r^3 \qquad h = \frac{4000}{\pi r^2} - \frac{4\pi r^3}{3\pi r^2}$$

$$h = \frac{4000}{\pi r^2} - \frac{4}{3}r$$

K = cost per square foot $\quad 2K$ = cost per square foot

$$C = 2K(4\pi r^2) + K(2\pi r h) = K\left[8\pi r^2 + 2\pi r\left(\frac{4000}{\pi r^2} - \frac{4}{3}r\right)\right] = K\left[\frac{16}{3}\pi r^2 + \frac{8000}{r}\right]$$

$$\frac{dC}{dr} = K\left[\frac{32}{3}\pi r - \frac{8000}{r^2}\right] = 0 \qquad r = \sqrt[3]{\frac{750}{\pi}} \approx 6.204\text{ ft.} \quad h \approx 24.814\text{ ft.}$$

Cost is minimum when $r = \sqrt[3]{\frac{750}{\pi}}$ and $h \approx 24.814$ ft.

36. Maximum Area Twenty feet of wire is to be used to form two figures. In each of the following cases, how much wire should be used for each figure so that the total enclosed area is maximum?

(a) Equilateral triangle and square

(b) Square and regular pentagon

(c) Regular pentagon and regular hexagon

(d) Regular hexagon and circle

What can you conclude from this pattern? {*Hint:* The area of a regular polygon with n sides of length x is $A = (n/4)[\cot(\pi/n)]x^2$.}

$\cot(\pi/3) = \frac{1}{\tan \pi/3} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

a) let x = side of triangle, y = side of square

$A = \frac{3}{4}\left[\cot\left(\frac{\pi}{3}\right)x^2\right] + \frac{4}{4}\left(\cot \frac{\pi}{4}\right)y^2$ and $3x + 4y = 20$

$A = \frac{\sqrt{3}}{4}x^2 + y^2$

$4y = 20 - 3x$

$y = 5 - \frac{3}{4}x$

$A = \frac{\sqrt{3}}{4}x^2 + \left[5 - \frac{3}{4}x\right]^2$

$A'(x) = \frac{\sqrt{3}}{4} \cdot 2x + 2\left[5 - \frac{3}{4}x\right]\left(-\frac{3}{4}\right) = \frac{\sqrt{3}}{2}x - \frac{15}{2} + \frac{9}{8}x = \frac{4\sqrt{3}x - 60 + 9x}{8} = 0$

$4\sqrt{3}x + 9x - 60 = 0$

$x(4\sqrt{3} + 9) = 60$

$x = \frac{60}{4\sqrt{3} + 9}$

Test $x = 0, 20/3, \frac{60}{4\sqrt{3}+9}$

$\boxed{A(0) = 25 \ast}$

$A\left(\frac{20}{3}\right) = 19.245$

$A\left(\frac{60}{4\sqrt{3}+9}\right) = 10.847$

b) $A = \frac{4}{4}\left(\cot \pi/4\right)x^2 + \frac{5}{4}\left(\cot\left(\frac{\pi}{5}\right)\right)y^2$ where $4x + 5y = 20$

$= x^2 + 1.72\left(4 - \frac{4}{5}x\right)^2$

$A'(x) = 2x - 2.752\left(4 - \frac{4}{5}x\right) = 0$ $x \approx 2.62$

$A(0) = 27.528$

$A(2.62) = 13.102$

$\ast$ $A(5) = 25$

36) continued...

c). let x = side of pentagon, y = side of hexagon:

$$A = \frac{5}{4}\left(\cot\frac{\pi}{5}\right)x^2 + \frac{6}{4}\left(\cot \pi/6\right)y^2 \quad \text{and} \quad 5x+6y=20$$

$$= \frac{5}{4}\cot \pi/5\, x^2 + \frac{3}{2}\sqrt{3}\left[\frac{20-5x}{6}\right]^2$$

$$A'(x) = \frac{5}{2}\cot(\pi/5)x + 3\sqrt{3}\left(\frac{-5}{6}\right)\left(\frac{20-5x}{6}\right) = 0 \qquad x \approx 2.0475$$

$A(0) = 28.868$ * | All 20 ft used on hexagon.

$A(2.0475) = 14.091$

$A(4) = 27.528$

d) x = side of hexagon r = radius of circle

$$A = \frac{6}{4}\cot \pi/6\, x^2 + \pi r^2 \rightarrow 6x + 2\pi r = 20$$

$$= \frac{3\sqrt{3}}{2}x^2 + \pi\left(\frac{10}{\pi} - \frac{3x}{\pi}\right)^2$$

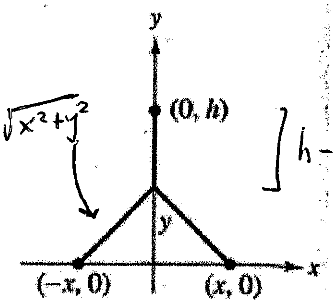
$$A'(x) = 3\sqrt{3} - 6\left(\frac{10}{\pi} - \frac{3x}{\pi}\right) = 0 \qquad x \approx 1.748$$

* $A(0) = 31.831$ | All used on circle

$A(1.748) = 15.138$

$A\left(\frac{10}{3}\right) = 28.868$

38. Minimum Length Two factories are located at the coordinates $(-x, 0)$ and $(x, 0)$, and their power supply is at $(0, h)$ (see figure). Find y such that the total length of power line from the power supply to the factories is a minimum.



$\sqrt{x^2+y^2}$ $\quad h-y$

$$A = h - y + 2\sqrt{x^2+y^2} = h - y + 2(x^2+y^2)^{1/2}$$

$$A'(y) = 0 - 1 + 2\cdot\tfrac{1}{2}(x^2+y^2)^{-1/2}(2y) = -1 + \frac{2y}{\sqrt{x^2+y^2}} = 0$$

$$\frac{2y}{\sqrt{x^2+y^2}} = 1 \rightarrow 2y = \sqrt{x^2+y^2} \rightarrow 4y^2 = x^2 + y^2 \quad 3y^2 = x^2 \quad y = \pm\sqrt{\frac{x^2}{3}} = \frac{x}{\sqrt{3}}$$

$\boxed{\text{The amt of power line is minimum when } y = \frac{x}{\sqrt{3}}}$

$$\frac{d^2A}{dy^2} = \frac{2x^2}{(x^2+y^2)^{3/2}} > 0$$

40. Illumination A light source is located over the center of a circular table of diameter 4 feet (see figure). Find the height h of the light source such that the illumination I at the perimeter of the table is maximum when

$$I = \frac{k \sin \alpha}{s^2}$$

where s is the slant height, α is the angle at which the light strikes the table, and k is a constant.

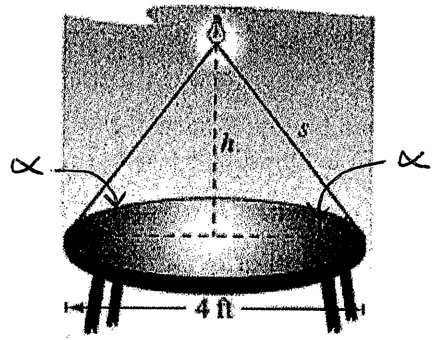


Figure for 40

$\sin \alpha = \frac{h}{s}$ $\tan \alpha = \frac{h}{2}$, $h = 2\tan\alpha$ $\rightarrow s = \frac{2\tan\alpha}{\sin\alpha} = 2\sec\alpha$

$s = \frac{h}{\sin\alpha}$

$$I = \frac{K\sin\alpha}{s^2} = \frac{K\sin\alpha}{(2\sec\alpha)^2} = \frac{K}{4}\sin\alpha\cos^2\alpha$$

$$\frac{dI}{d\alpha} = \frac{K}{4}\left[\sin\alpha(-2\sin\alpha\cos\alpha) + \cos^2\alpha(\cos\alpha)\right]$$

$$= \frac{K}{4}\left[\sin\alpha(-2\sin\alpha\cos\alpha + \cos^2\alpha\cos\alpha\right] = \frac{K}{4}\cos\alpha\left[\cos^2\alpha - 2\sin^2\alpha\right]$$

$$0 = \frac{K}{4}\cos\alpha\left[1 - 3\sin^2\alpha\right]$$

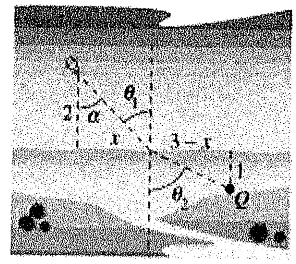
$\alpha = \pi/2, 3\pi/2$ or $\sin\alpha = \frac{\pm 1}{\sqrt{3}}$

$\sin\alpha = \frac{1}{\sqrt{3}}$, $\tan\alpha = \frac{1}{\sqrt{2}}$

$h = 2\tan\alpha$

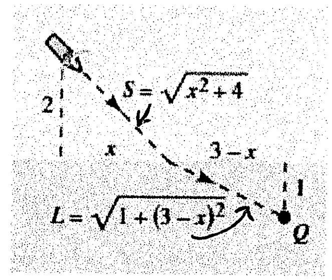
$$\boxed{h = 2\left(\frac{1}{\sqrt{2}}\right) = \sqrt{2} \text{ ft.}}$$

41. Minimum Time A man is in a boat 2 miles from the nearest point on the coast. He is to go to a point Q, located 3 miles down the coast and 1 mile inland (see figure). He can row at 2 miles per hour and walk at 4 miles per hour. Toward what point on the coast should he row in order to reach point Q in the least time? optimize Time:



$d = r \cdot t$

$t = \frac{d}{r}$



$$S = \sqrt{x^2+4} \qquad L = \sqrt{1+(3-x)^2}$$

$$\text{Time } T = \frac{S}{2} + \frac{L}{4}$$

$$T = \frac{\sqrt{x^2+4}}{2} + \frac{\sqrt{1+(3-x)^2}}{4}$$

$$\frac{dT}{dx} = \frac{1}{2}\cdot\frac{1}{2}(x^2+4)^{-1/2}(2x) + \frac{1}{4}\cdot\frac{1}{2}\left(1+(3-x)^2\right)^{1/2}\cdot 2(3-x)'(-1)$$

$$0 = \frac{x}{2\sqrt{x^2+4}} + \frac{x-3}{4\sqrt{x^2-6x+10}}$$

$$x^4 - 6x^3 + 9x^2 + 8x - 12 = 0 \qquad \boxed{x = 1}$$

The man should row to a point 1 mile from the nearest point on the coast.

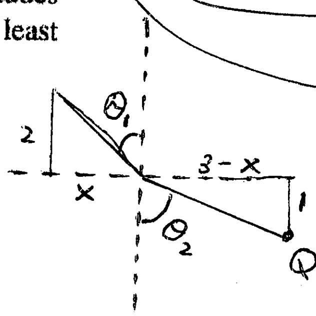
42)

[redacted] **Minimum Time** The conditions are the same as in Exercise 41 except that the man can row at v_1 miles per hour and walk at v_2 miles per hour. If θ_1 and θ_2 are the magnitudes of the angles, show that the man will reach point Q in the least time when

$$\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}.$$

$$\sqrt{1+(3-x)^2}$$

$$\downarrow$$



$$T = \frac{\sqrt{x^2+4}}{v_1} + \frac{\sqrt{x^2-6x+10}}{v_2}$$

$$\frac{dT}{dx} = \frac{x}{v_1\sqrt{x^2+4}} + \frac{x-3}{v_2\sqrt{x^2-6x+10}} = 0$$

$$\frac{x}{\sqrt{x^2+4}} = \sin\theta_1 \qquad \frac{x-3}{\sqrt{x^2-6x+10}} = -\sin\theta_2$$

therefore $\frac{\sin\theta_1}{v_1} - \frac{\sin\theta_2}{v_2} = 0$

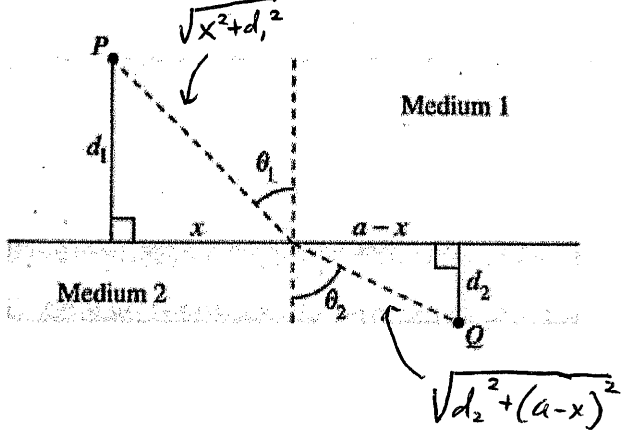
$$\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}$$

44)

44. Minimum Time When light waves traveling in a transparent medium strike the surface of a second transparent medium, they change direction. This change of direction is called *refraction* and is defined by **Snell's Law of Refraction,**

$$\frac{\sin \theta_1}{v_1} = \frac{\sin \theta_2}{v_2}$$

where θ_1 and θ_2 are the magnitudes of the angles shown in the figure and v_1 and v_2 are the velocities of light in the two media. Show that this problem is equivalent to that in Exercise 42, and that light waves traveling from P to Q follow the path of minimum time.



$$T = \frac{\sqrt{x^2+d_1^2}}{V_1} + \frac{\sqrt{d_2^2+(a-x)^2}}{V_2}$$

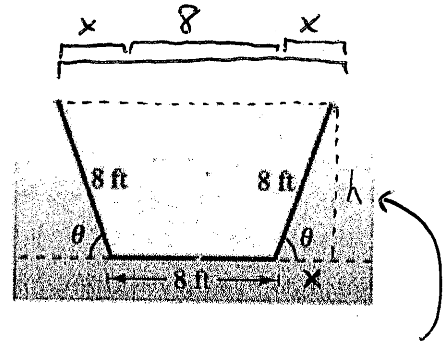
$$\frac{dT}{dx} = \frac{x}{V_1\sqrt{x^2+d_1^2}} + \frac{x-a}{V_2\sqrt{d_2^2+(a-x)^2}} = 0$$

$$\frac{x}{\sqrt{x^2+d_1^2}} = \sin\theta_1$$

$$\frac{x-a}{\sqrt{d_2^2+(a-x)^2}} = -\sin\theta_2$$

$$\frac{\sin\theta_1}{V_1} - \frac{\sin\theta_2}{V_2} = 0, \quad \frac{\sin\theta_1}{V_1} = \frac{\sin\theta_2}{V_2}$$

46. Numerical, Graphical, and Analytic Analysis The cross sections of an irrigation canal are isosceles trapezoids of which three sides are 8 feet long (see figure). Determine the angle of elevation θ of the sides such that the area of the cross sections is a maximum by completing the following.



$\sin\theta = \frac{h}{8} \rightarrow h = 8\sin\theta$

$\cos\theta = \frac{x}{8} \rightarrow x = 8\cos\theta$

(a) Analytically complete six rows of a table such as the one below. (The first two rows are shown.)

Base 1	Base 2	Altitude	Area
8	$8 + 16\cos 10°$	$8\sin 10°$	≈ 22.1
8	$8 + 16\cos 20°$	$8\sin 20°$	≈ 42.5

(b) Use a graphing utility to generate additional rows of the table and estimate the maximum cross-sectional area. (*Hint:* Use the *table* feature of the graphing utility.)

(c) Write the cross-sectional area A as a function of θ.

(d) Use calculus to find the critical number of the function in part (c) and find the angle that will yield the maximum cross-sectional area.

(e) Use a graphing utility to graph the function in part (c) and verify the maximum cross-sectional area.

c) $\text{Area(trapezoid)} = \frac{h}{2}[b_1 + b_2] = \frac{8\sin\theta}{2}[8 + 8 + 2(8\cos\theta)]$

$A = 4\sin\theta(16 + 16\cos\theta)$

$A = 64\sin\theta(1 + \cos\theta)$

d) $\frac{dA}{d\theta} = 64\cos\theta \cdot (1 + \cos\theta) + 64\sin\theta \cdot (-\sin\theta)$

$= 64\cos\theta + 64\cos^2\theta - 64\sin^2\theta = 64(\cos\theta + \overbrace{\cos^2\theta - \sin^2\theta}^{2\cos^2\theta - 1})$

$= 64(2\cos^2\theta + \cos\theta - 1)$

$= 64(2\cos\theta - 1)(\cos\theta + 1)$

$\cos\theta = 1/2$ | $\cos\theta = -1$

$\theta = \frac{\pi}{3}, \frac{5\pi}{3}$ | $\theta = \pi$

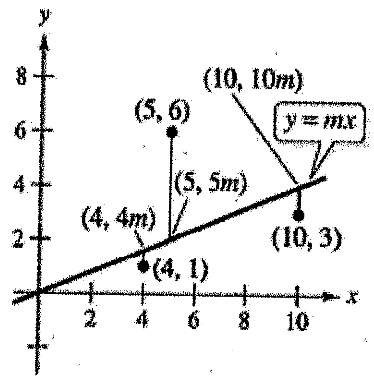
max occurs when $\theta = \pi/3$ or $60°$

Minimum Distance **In Exercises 49–51, consider a fuel distribution center located at the origin of the rectangular coordinate system (units in miles; see figures). The center supplies three factories with coordinates (4, 1), (5, 6), and (10, 3). A trunk line will run from the distribution center along the line $y = mx$, and feeder lines will run to the three factories. The objective is to find m such that the lengths of the feeder lines are minimized.**

50. Minimize the sum of the absolute values of the lengths of the vertical feeder lines (see figure) given by

$$S_2 = |4m - 1| + |5m - 6| + |10m - 3|.$$

Find the equation of the trunk line by this method and then determine the sum of the lengths of the feeder lines. (*Hint:* Use a graphing utility to graph the function S_2 and approximate the required critical number.)

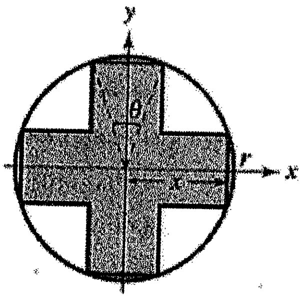


minimum occurs when $m = 0.3$

Line $y = 0.3x$

$$\boxed{S_2(0.3) = 4.7 \text{ mi}}$$

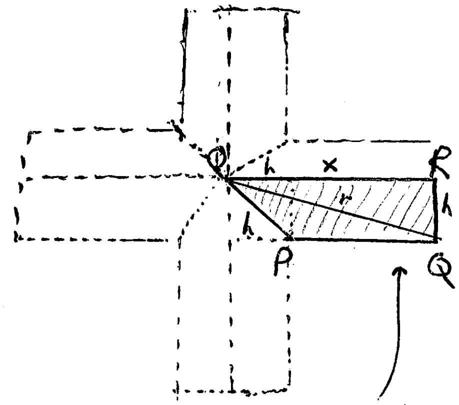
52. Maximum Area Consider a symmetric cross inscribed in a circle of radius r (see figure).



(a) Write the area A of the cross as a function of x and find the value of x that maximizes the area.

(b) Write the area A of the cross as a function of θ and find the value of θ that maximizes the area.

(c) Show that the critical numbers of parts (a) and (b) yield the same maximum area. What is that area?



triangle + rectangle

$\frac{1}{2}h^2+(x-h)(h)$

a) $r^2=x^2+h^2 \quad h=\sqrt{r^2-x^2}$

$$A=8\left[\tfrac{1}{2}h^2+(x-h)h\right]=8\left[\tfrac{1}{2}(r^2-x^2)+\left(x-\sqrt{r^2-x^2}\right)\sqrt{r^2-x^2}\right]=8x\sqrt{r^2-x^2}+4x^2-4r^2$$

$$A'(x)=8\sqrt{r^2-x^2}-\frac{8x^2}{\sqrt{r^2-x^2}}+8x=0$$

$$\frac{8x^2}{\sqrt{r^2-x^2}}=8x+8\sqrt{r^2-x^2} \qquad x^2=x\sqrt{r^2-x^2}+(r^2-x^2)$$

$$2x^2-r^2=x\sqrt{r^2-x^2}$$

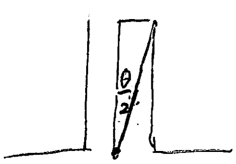
$$4x^4-4x^2r^2+r^4=x^2(r^2-x^2)$$

$$5x^4-5x^2r^2+r^4=0$$

$$x^2=\frac{5r^2\pm\sqrt{25r^4-20r^4}}{10}=\frac{r^2}{10}\left[5\pm\sqrt{5}\right]$$

$$x=r\sqrt{\frac{5+\sqrt{5}}{10}}\approx 0.8506r$$

b) $\sin\left(\frac{\theta}{2}\right)=\frac{h}{r} \qquad \cos\left(\frac{\theta}{2}\right)=\frac{x}{r}$



2 rectangles − common square $\qquad h=r\sin\left(\frac{\theta}{2}\right) \qquad x=r\cos\left(\frac{\theta}{2}\right)$

$$A=2(2x)(2h)-4h^2$$

$$A=8xh-4h^2$$

$$A=8r^2\sin\frac{\theta}{2}\cos\frac{\theta}{2}-4r^2\sin^2\frac{\theta}{2}=4r^2\left(\sin\theta-\sin^2\frac{\theta}{2}\right)$$

$$A'(\theta)=4r^2\left(\cos\theta-\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)=0$$

$$\cos\theta=\sin\frac{\theta}{2}\cos\frac{\theta}{2}=\frac{1}{2}\sin\theta$$

$$\tan\theta=2$$

$$\theta=\arctan(2)\approx 1.10715 \text{ or } \boxed{63.4^\circ}$$

52) continued...

c) $x^2 = \frac{r^2}{10}(5+\sqrt{5})$ and $r^2 - x^2 = \frac{r^2}{10}(5-\sqrt{5})$

$$A(x) = 8x\sqrt{r^2-x^2} + 4x^2 - 4r^2$$

$$= 8\left[\frac{r^2}{10}(5+\sqrt{5})\frac{r^2}{10}(5-\sqrt{5})\right]^{1/2} + 4\frac{r^2}{10}(5+\sqrt{5}) - 4r^2$$

$$= 8\left[\frac{r^4}{10}(20)\right]^{1/2} + 2r^2 + \frac{2}{5}\sqrt{5}r^2 - 4r^2$$

$$= \frac{8}{5}r^2\sqrt{5} - 2r^2 + \frac{2\sqrt{5}}{5}r^2 = 2r^2\left[\frac{4}{5}\sqrt{5} - 1 + \frac{\sqrt{5}}{5}\right] = 2r^2(\sqrt{5}-1)$$

$\tan\theta = 2 \quad \sin\theta = \frac{2}{\sqrt{5}}$, $\sin^2\left(\frac{\theta}{2}\right) = \frac{1}{2}(1-\cos\theta) = \frac{1}{2}\left(1-\frac{1}{\sqrt{5}}\right)$

$$A(\theta) = 4r^2\left(\sin\theta - \sin^2\frac{\theta}{2}\right) = 4r^2\left(\frac{2}{\sqrt{5}} - \frac{1}{2}\left(1-\frac{1}{\sqrt{5}}\right)\right) = \frac{4r^2(\sqrt{5}-1)}{2} = 2r^2(\sqrt{5}-$$