Name: ______________________________ Period: __________

BC Calculus

Unit 4

Applications of Differentiation (Part 1)

(Particle Motion, Related Rates, Linear Approximation, L'Hopital's Rule)

AP Calculus – 4.1 Notes Interpreting Derivatives & Particle Motion (Position-Velocity-Acceleration)

Recall:

Slope between two points: ― or ――――

Units for the Derivative:

The derivative of $f(x)$ is ――――

If $f'(x) > 0$, then $f(x)$ is

If $f'(x) < 0$, then $f(x)$ is

1. Mr. Sullivan wants Mr. Brust to finish creating his packets in Algebra 2. The number of packets Mr. Brust has completed is modeled by $p(w)$, where w is measured in weeks.
   a. Interpret $p(10) = 1$ in the context of the problem.

   b. Interpret $p'(39) = 4$ in the context of the problem.

2. The rate at which Mr. Kelly is buying baseball cards per year is modeled by $b(t)$, where t is measured in years.
   a. Interpret $b(3) = 150$ in the context of the problem.

   b. Interpret $b'(4) = 10$ in the context of the problem.

Practice problems:

For each problem, a differentiable function is given along with a definition of the variables. Interpret the values in the context of the problem.

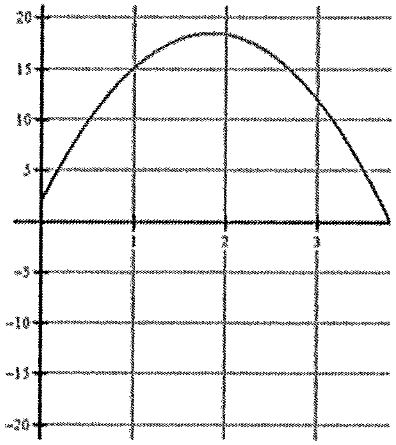
1. The percentage grade a student receives on a test, is modeled by $G(t)$ where t is the number of hours spent studying for the test. Interpret $G'(1) = 3$.	2. Mr. Bean rides his motor scooter to work some days. His distance from home can be modeled by $d(t)$ meters where t is measured in minutes. Interpret $d'(15) = 650$.

2

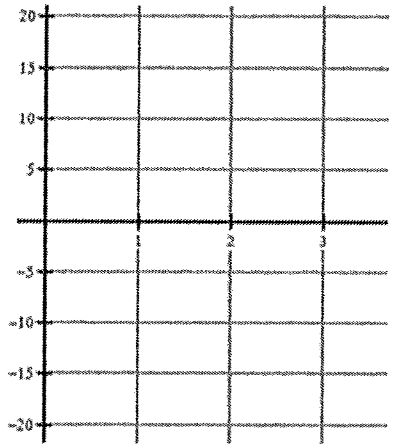
Particle Motion – Position – Velocity – Acceleration (PVA)

Mr. Brust is playing catch with his best friend, himself. He throws a tennis ball straight up into the air. The height of the ball is modeled by $s(t) = -4.9t^2 + 18t + 2$ where t is time in seconds and s is the height of the ball from the ground in meters.

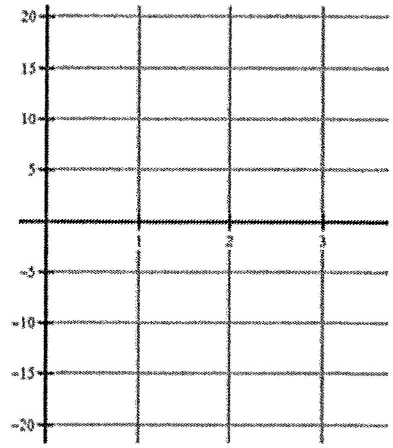
POSITION



VELOCITY



ACCELERATION



Position function: $s(t)$

Velocity function: $v(t) =$

Acceleration function: $a(t) =$

Velocity = Rate of Change of Position

$v(t) < 0$ means the particle is ______________________________

$v(t) > 0$ means the particle is ______________________________

$v(t) = 0$ means the particle is ____________

Average velocity =

Speed =

Speeding Up or Slowing Down?

If velocity and acceleration have the **same** sign, the particle is ____________________

If velocity and acceleration have **different** signs, the particle is __________________

t	-5	1	2	4
$v(t)$	3	-2	1	-1
$a(t)$	-4	7	0.1	-1
Conclusion				

Displacement:

Particle Motion from an equation.

The position (x-coordinate) of a particle moving on the x-axis is modeled by the function

$$x(t) = t^3 - 4t^2 + 3 \text{ for } t \geq 0,$$

Where x is measured in cm and t is measured in minutes.

1. Find the displacement of the particle during the first 2 minutes.

2. Find the average velocity of the particle during the first 2 minutes.

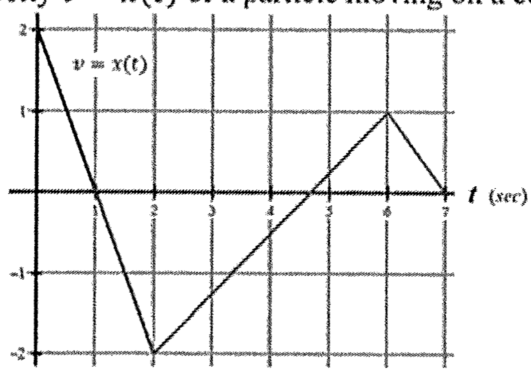
3. Find the velocity of the particle when $t = 4$.

4. Find the acceleration of the particle when $t = 4$.

5. Is the particle speeding up or slowing down at $t = 4$? Justify.

Particle Motion from a graph

The figure shows the velocity $v = x(t)$ of a particle moving on a coordinate line.



6. When is the particle moving right? Justify.

7. When is the particle moving left? Justify.

8. When is the particle's acceleration Positive? Negative? Zero?

9. When does the particle have the greatest speed?

10. When is the particle speeding up? Justify.

11. When is the particle slowing down? Justify.

(d) The acceleration $a = a(t)$ of an object in rectilinear motion is the rate of change of velocity with respect to time. That is,

$$a = a(t) = v'(t) = \frac{d}{dt}(3t^2 - 16t + 16) = 6t - 16 \text{ km/h}^2$$

At $t = 3$, $a(3) = 6 \cdot 3 - 16 = 2 \text{ km/h}^2$. ■

NOW WORK Problem 9 and AP® Practice Problem 3.

4.1 Assess Your Understanding

Concepts and Vocabulary

1. *True or False* The velocity of an object in rectilinear motion equals the change in its position.
2. *Multiple Choice* The acceleration of an object in rectilinear motion equals
   - **(a)** the change in the speed over an interval of time.
   - **(b)** the change in the velocity over an interval of time.
   - **(c)** the rate of change of the position of the object with respect to time.
   - **(d)** the rate of change of the velocity with respect to time.
3. *True or False* Both $f'(x)$ and $f''(x)$ are rates of change.
4. A(n) ____ measures change over an interval; a(n) ____ measures change at an instant.

Skill Building

In Problems 5–8, for each function,

(a) Find the derivative of f.
(b) Find the slope of the tangent line to the graph of f at the point $(c, f(c))$.
(c) Find an equation of the tangent line to the graph of f at the point $(c, f(c))$.

5. $f(x) = x^3 - 4x^2 - 2$ at $c = 3$
6. $f(x) = 4 - 3x^2$ at $c = -1$
7. (PAGE 268) $f(x) = \dfrac{e^x}{x^2 + 2}$ at $c = 0$
8. $f(x) = e^{2x}(2 - x^2)$ at $c = 0$

In Problems 9–12, the position function of an object in rectilinear motion is given. For each function, find the velocity and acceleration of the object at any time t.

9. (PAGE 270) $s(t) = -16t^2 + 120t + 43$, where s is in feet and t is in seconds
10. $s(t) = 3 - 12t + t^3$, where s is in meters and t is in seconds
11. $s(t) = t\cos^2 t$, where s is in kilometers and t is in hours
12. $s(t) = \dfrac{e^t}{t + 1}$, $t \geq 0$, where s is in inches and t is in minutes

Applications and Extensions

13. (PAGE 268) **Population Growth** A colony of fruit flies is growing in a controlled environment. The population is modeled by the function $P(t) = 30e^{0.2t}$, where t is measured in days.
    - **(a)** Find the initial number of fruit flies in the colony.
    - **(b)** How many fruit flies are in the colony after 5 days?
    - **(c)** Find the function that models the rate of change in population of the fruit flies at time t.
    - **(d)** Find and interpret the rate of change in the fruit fly population after 10 days.
14. **Market Penetration** The percentage A of the market penetrated by Blu-Ray players is modeled by the function $A(t) = 100 - 90e^{-0.12t}$, where t expresses the time in years since 2006 when Blu-Ray players were first introduced.
    - **(a)** Find $A'(t)$ and interpret its meaning.
    - **(b)** Evaluate $A'(5)$ and interpret its meaning.
    - **(c)** Evaluate $A'(15)$ and interpret its meaning.
15. **Units** The surface area S (in cm^2) of a right rectangular cylinder is increasing according to $S = S(t) = 4t - 1$, where t is the time in seconds. At what rate is the surface area changing with respect to time? What are the units of $S'(t)$?
16. **Units** The circumference C (in mi) of an oil spill is expanding according to $C = C(t) = 3t$, where t is measured in days. At what rate is the circumference changing with respect to time? What are the units of $C'(t)$?
17. **Units** The weekly revenue R (in dollars) is given by $R(x) = x^2 + 100x$, where x is the number of items sold. What is the rate of change of R with respect to x? What are the units of R'? Find and interpret $R'(75)$.
18. **Units** The daily cost C (in dollars) of producing x items is modeled by the function $C(x) = 0.1x^2 + 120x + 800$. Find the marginal cost, $C'(x)$, which is the cost of producing one additional item. What are the units of C'? Find and interpret $C'(100)$.

AP® Practice Problems

Preparing for the AP® Exam

1. (PAGE 268) An equation of the tangent line to the graph of $f(x) = 2x^3 - 3x + 6$ at the point $(0, f(0))$ is
   (A) $y = -3x - 6$ (B) $y = 3x + 6$
   (C) $y = -3x - 3$ (D) $y = -3x + 6$
2. (PAGE 268) If $y = f(x) = x^2 + \ln x$, what is the rate of change of f at 2?
   (A) $\frac{9}{2}$ (B) $\frac{9}{4}$ (C) $\frac{7}{2}$ (D) $\frac{7}{4}$
3. (PAGE 270) The position function s of an object in rectilinear motion is $s(t) = \dfrac{t^3}{6} - \dfrac{2t^2}{3} + 4t - 1$. At $t = 3$, the object is
   (A) accelerating. (B) decelerating.
   (C) neither accelerating nor decelerating. (D) stopped.

5

4.1 AP Practice Problems (p.270)

1. An equation of the tangent line to the graph of $f(x) = 2x^3 - 3x + 6$ at the point $(0, f(0))$ is

(A) $y = -3x - 6$ (B) $y = 3x + 6$

(C) $y = -3x - 3$ (D) $y = -3x + 6$

2. If $y = f(x) = x^2 + \ln x$, what is the rate of change of f at 2?

(A) $\frac{9}{2}$ (B) $\frac{9}{4}$ (C) $\frac{7}{2}$ (D) $\frac{7}{4}$

3. The position function s of an object in rectilinear motion is $s(t) = \frac{t^3}{6} - \frac{2t^2}{3} + 4t - 1$. At $t = 3$, the object is

(A) accelerating. (B) decelerating.

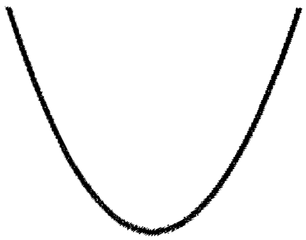
(C) neither accelerating nor decelerating. (D) stopped.

6

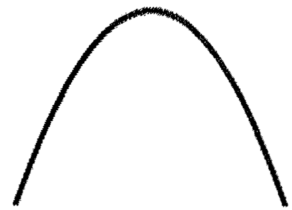
AP Calculus – 4.2 Notes Linear Approximation and rates of change other than motion

The tangent line of the function $f(x)$ at $x = a$ can give you an approximate value of $f(x)$ for points close to $x = a$.

Concave UP with a Tangent Line



Concave DOWN with a Tangent Line



1. f is concave up on its domain and $f(4) = 5$ and $f'(4) = 3$.
   a. What is the estimate for $f(3.8)$ using the local linear approximation for f at $x = 4$?

   b. Is it an underestimate or overestimate? Explain.

2. The function $f(x) = 5x - 2x^3 - 2$ is concave down at $x = 1$?.
   a. Find the tangent line of f at $x = 1$.

   b. What is the estimate for $f(1.1)$ using the local linear approximation for f at $x = 1$?

   c. Is it an underestimate or overestimate? Explain.

3. Consider the differential equation $\frac{dy}{dx} = e^y(2x^2 - 5x)$. Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(2) = 0$.
   a. Write an equation for the line tangent to the graph of f at the point (2,0).

   b. Use the tangent line to approximate $f(2.2)$.

Rates of Change other than Motion:

7

Increasing or Decreasing?

To know if something is increasing or decreasing, check the ______________________

Height is *increasing* if ____________________

Velocity is *decreasing* if ____________________

Recall: Derivative on a calculator.

Find $f'(3)$ if $f(x) = 5^{\sin x}$

Is the function already a rate of change?

- If $f(x)$ is the bunny population after x years, than what is $f'(x)$?

- If $f(x)$ is the rate at which a bunny population increases (bunnies per year), than what is $f'(x)$?

Rate of Change from a Table

t (years)	0	10	20	30
$P(t)$ (people)	100	120	150	200

Estimate $P'(15)$

Estimate $P'(20)$

Practice Problems:

1. A store is having a 12-hour sale. The total number of shoppers who have entered the store t hours after the sale begins is modeled by the function E defined by $E(t) = 0.3t^4 - 14t^3 + 110t^2$ for $0 \leq t \leq 12$. At what rate are shoppers entering the store 5 hours after the start of the sale?

2. The function $t = f(P)$ models the time, in days, for a small pond to evaporate as a function of the size P of the pond, measured in liters. What are the units for $f''(P)$?

NORMAL FLOAT AUTO REAL RADIAN MP

X	Y_1			
.5	.64411			
.64411	.63675			
.63675	.63673			
.63673	.63673			

X=

Figure 10

Then create a table (Figure 10) by entering the initial value 0.5 in the X column. The graphing calculator computes $Y_1(0.5)$ and displays 0.64411 in column Y_1 to the right of 0.5. The value Y_1 is the second approximation c_2 that we use in the iteration. Now enter $Y_1(0.5) = 0.64410789$ in the X column of the next row, and the entry in column Y_1 is the third approximation c_3. Repeat the process until the desired approximation is obtained. The fourth approximation to the zero of f is 0.63673.

NOW WORK Problem 47.

When Newton's Method Fails

You may wonder, does Newton's Method always work? The answer is no. The list below, while not exhaustive, gives some conditions under which Newton's Method fails.

- Newton's Method fails if the conditions of the theorem are not met
  - **(a)** $f'(c_n) = 0$: Algebraically, division by 0 is not defined. Geometrically, the tangent line is parallel to the x-axis and so has no x-intercept.
  - **(b)** $f'(c)$ is undefined: The process cannot be used.
- Newton's Method fails if the initial estimate c_1 is not "good enough":
  - **(a)** Choosing an initial estimate too far from the required zero could result in approximating a different zero of the function.
  - **(b)** The convergence could approach the zero so slowly that hundreds of iterations are necessary.
- Newton's Method fails if the terms oscillate between two values and so never get closer to the zero.

Problems 80–83 illustrate some of these possibilities.

4.2 Assess Your Understanding

Concepts and Vocabulary

1. *Multiple Choice* If $y = f(x)$ is a differentiable function, the differential $dy =$
(a) Δy (b) Δx (c) $f(x)dx$ (d) $f'(x)dx$
2. A linear approximation to a differentiable function f near x_0 is given by the function $L(x) =$ ________.
3. *True or False* The difference $|\Delta y - dy|$ measures the departure of the graph of $y = f(x)$ from the graph of the tangent line to f.
4. If Q is a quantity to be measured and ΔQ is the error made in measuring Q, then the relative error in the measurement at x_0 is given by the ratio ________.
5. *True or False* Newton's Method uses tangent lines to the graph of f to approximate the zeros of f.
6. *True or False* Before using Newton's Method, we need a first approximation for the zero.

Skill Building

In Problems 7–18, find the differential dy of each function.

7. $y = x^3 - 2x + 1$
8. $y = e^x + 2x - 1$
9. $y = 4(x^2+1)^{3/2}$ PAGE 272
10. $y = \sqrt{x^2 - 1}$
11. $y = 3\sin(2x) + x$
12. $y = \cos^2(3x) - x$
13. $y = e^{-x}$
14. $y = e^{\sin x}$
15. $y = xe^x$
16. $y = \dfrac{e^{-x}}{x}$
17. $y = \sin^{-1}(2x)$
18. $y = \tan^{-1} x^2$

In Problems 19–24:
(a) Find the differential dy for each function f.
(b) Evaluate dy and Δy at the given value of x when (i) $\Delta x = 0.5$, (ii) $\Delta x = 0.1$, and (iii) $\Delta x = 0.01$.
(c) Find the error $|\Delta y - dy|$ for each choice of $dx = \Delta x$.

19. $f(x) = e^x$ at $x = 1$ PAGE 272
20. $f(x) = e^{-x}$ at $x = 1$
21. $f(x) = x^{2/3}$ at $x = 2$
22. $f(x) = x^{-1/2}$ at $x = 1$
23. $f(x) = \cos x$ at $x = \pi$
24. $f(x) = \tan x$ at $x = 0$

In Problems 25–32:
(a) Find the linear approximation L(x) to f at x_0.
(b) Graph f and L on the same set of axes.

25. $f(x) = (x+1)^5$, $x_0 = 2$
26. $f(x) = x^3 - 1$, $x_0 = 0$
27. $f(x) = \sqrt{x}$, $x_0 = 4$ PAGE 274
28. $f(x) = x^{2/3}$, $x_0 = 1$
29. $f(x) = \ln x$, $x_0 = 1$
30. $f(x) = e^x$, $x_0 = 1$
31. $f(x) = \cos x$, $x_0 = \dfrac{\pi}{3}$
32. $f(x) = \sin x$, $x_0 = \dfrac{\pi}{6}$
33. Use differentials to approximate the change in:
(a) $y = f(x) = x^2$ as x changes from 3 to 3.001.
(b) $y = f(x) = \dfrac{1}{x+2}$ as x changes from 2 to 1.98.
34. Use differentials to approximate the change in:
(a) $y = x^3$ as x changes from 3 to 3.01.
(b) $y = \dfrac{1}{x-1}$ as x changes from 2 to 1.98.

In Problems 35–38, use the given information and a linear approximation to approximate the value of the function at c.

35. $f(2)=8;\ f'(2)=-3;\ c=2.06$

36. $f(-4)=3;\ f'(-4)=2;\ c=-3.6$

37. $f(-1)=0;\ f'(-1)=\frac{3}{2};\ c=-1.1$

38. $f(5)=\frac{1}{2};\ f'(5)=-3;\ c=5.2$

In Problems 39–46, for each function:

(a) Use the Intermediate Value Theorem to confirm that a zero exists in the given interval.

(b) Use Newton's Method with the first approximation c_1 to find c_3, the third approximation to the real zero.

39. $f(x)=x^3+3x-5$, interval: $(1, 2)$. Let $c_1=1.5$.

40. $f(x)=x^3-4x+2$, interval: $(1, 2)$. Let $c_1=1.5$.

41. $f(x)=2x^3+3x^2+4x-1$, interval: $(0, 1)$. Let $c_1=0.5$.

42. $f(x)=x^3-x^2-2x+1$, interval: $(0, 1)$. Let $c_1=0.5$.

43. $f(x)=x^3-6x-12$, interval: $(3, 4)$. Let $c_1=3.5$.

44. $f(x)=3x^3+5x-40$, interval: $(2, 3)$. Let $c_1=2.5$.

45. $f(x)=x^4-2x^3+21x-23$, interval: $(1, 2)$. Use a first approximation c_1 of your choice.

46. $f(x)=x^4-x^3+x-2$, interval: $(1, 2)$. Use a first approximation c_1 of your choice.

In Problems 47–52, for each function:

(a) Use the Intermediate Value Theorem to confirm that a zero exists in the given interval.

(b) Use technology with Newton's Method to find c_5, the fifth approximation to the real zero. Use the midpoint of the interval for the first approximation c_1.

47. $f(x)=x+e^x$, interval: $(-1, 0)$

48. $f(x)=x-e^{-x}$, interval: $(0, 1)$

49. $f(x)=x^3+\cos^2 x$, interval: $(-1, 0)$

50. $f(x)=x^2+2\sin x-0.5$, interval: $(0, 1)$

51. $f(x)=5-\sqrt{x^2+2}$, interval: $(4, 5)$

52. $f(x)=2x^2+x^{2/3}-4$, interval: $(1, 2)$

Applications and Extensions

53. **Area of a Disk** A circular plate is heated and expands. If the radius of the plate increases from $R=10$ cm to $R=10.1$ cm, use differentials to approximate the increase in the area of the top surface.

54. **Volume of a Cylinder** In a wooden block 3 cm thick, an existing circular hole with a radius of 2 cm is enlarged to a hole with a radius of 2.2 cm. Use differentials to approximate the volume of wood that is removed.

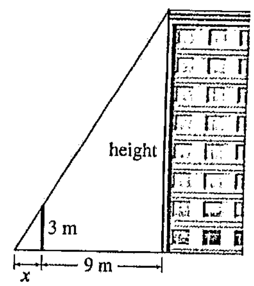
55. **Volume of a Balloon** Use differentials to approximate the change in volume of a spherical balloon of radius 3 m as the balloon swells to a radius of 3.1 m.

56. **Volume of a Paper Cup** A manufacturer produces paper cups in the shape of a right circular cone with a radius equal to one-fourth the height. Specifications call for the cups to have a top diameter of 4 cm. After production, it is discovered that the diameter measures only 3.8 cm. Use differentials to approximate the loss in capacity of the cup.

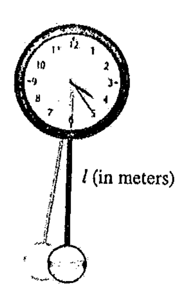
57. **Volume of a Sphere**
    (a) Use differentials to approximate the volume of material needed to manufacture a hollow sphere if its inner radius is 2 m and its outer radius is 2.1 m.
    (b) Is the approximation overestimating or underestimating the volume of material needed?
    (c) Discuss the importance of knowing the answer to (b) if the manufacturer receives an order for 10,000 spheres.

58. **Distance Traveled** A bee flies around a circle traced on an equator of a ball with a radius of 7 cm at a constant distance of 2 cm from the ball. An ant travels along the same circle but on the ball.
    (a) Use differentials to approximate how many more centimeters the bee travels than the ant in one round-trip journey.
    (b) Does the linear approximation overestimate or underestimate the difference in the distances the bugs travel? Explain.

59. **Estimating Height** To find the height of a building, the length of the shadow of a 3-m pole placed 9 m from the building is measured. See the figure. This measurement is found to be 1 m, with a percentage error of 1%. The height of the building is estimated to be 30 m. Use differentials to find the percentage error in the estimate.



60. **Pendulum Length** The period of the pendulum of a grandfather clock is $T=2\pi\sqrt{\frac{l}{g}}$, where l is the length (in meters) of the pendulum, T is the period (in seconds), and g is the acceleration due to gravity (9.8 m/s^2). Suppose an increase in temperature increases the length l of the pendulum, a thin wire, by 1%. What is the corresponding percentage error in the period? How much time will the clock lose (or gain) each day?



61. **Pendulum Length** Refer to Problem 60. If the pendulum of a grandfather clock is normally 1 m long and the length is increased by 10 cm, use differentials to approximate the number of minutes the clock will lose (or gain) each day.

62. **Luminosity of the Sun** The luminosity L of a star is the rate at which it radiates energy. This rate depends on the temperature T (in kelvin where 0 K is absolute zero) and the surface area A of the star's photosphere (the gaseous surface that emits the light). Luminosity at time t is given by the formula $L(t)=\sigma AT^4$, where σ is a constant, known as the **Stefan–Boltzmann constant.**

    As with most stars, the Sun's temperature has gradually increased over the 5 billion years of its existence, causing its luminosity to slowly increase. For this problem, we assume that increased luminosity L is due only to an increase in temperature T. That is, we treat A as a constant.

    (a) Find the rate of change of the temperature T of the Sun with respect to time t. Write the answer in terms of the rate of change of the Sun's luminosity L with respect to time t.

(10)

4.2 AP Practice Problems (p.281) – Linear Approximation

1. Let f be a function for which $f(2) = 6$ and $f'(2) = -3$. If the tangent line to the graph of f at 2 is used to approximate a zero of f, then the approximation is

   (A) 0 (B) 4 (C) 6 (D) 12

2. For small, positive values of h, $\sqrt[3]{8+h}$ is best approximated by

   (A) $4-\frac{h}{12}$ (B) $4+\frac{h}{12}$ (C) $2-\frac{h}{12}$ (D) $2+\frac{h}{12}$

3. A linear approximation to $f(x) = x\sin\left(\frac{\pi x}{2}\right)+x^2$ at $x = 3$ is

   (A) $y = 5x+6$ (B) $y = 5x-9$
   (C) $y = 7x-9$ (D) $y = 7x+9$

4. Using the tangent line to the graph of $f(x) = xe^x + 2$ at 0, the approximate value of $f(-0.3)$ is

   (A) 2.3 (B) 1.3 (C) 1.7 (D) -2.3

5. If $f'(x) = 2xe^{x^2-1} - 3\pi\sin(\pi x)$ and $f(1) = 4$, approximate $f(1.03)$ using a linear approximation.

   (A) 4.06 (B) 5.06 (C) 4 (D) 3.94

6. The tangent line to the graph of $f(x) = x^3 + 1$ at $x = 1$ is used to approximate $f(x)$ near 1. Which number below is the greatest value of x that results in an error less than or equal to 0.5?

(A) 1.30 (B) 1.35 (C) 1.40 (D) 1.45

7. A linear approximation L is used to approximate $f(x) = \sqrt{x}$, at c, $c > 0$. The approximation

(A) always underestimates the true value of f at c.
(B) always overestimates the true value of f at c.
(C) sometimes overestimates the true value of f at c.
(D) does not provide enough information to determine whether the true value of f at c is over- or underestimated.

8. Suppose $y = f(x)$ is a differentiable function. The table below gives values of f and f' for select numbers x in the domain of f. Use a linear approximation to approximate $f(3.1)$.

x	-3	0	1	3	5
$f(x)$	4	4	-1	-2	3
$f'(x)$	1	-1	-2	3	4

(A) 4.1 (B) -2.3 (C) 0.1 (D) -1.7

(12)

Calculus Notes Ch. 4.3 - Related Rates

Related Rates: Problems involving finding the rate of change for a variable with respect to time

This is also an application of implicit differentiation: Finding derivatives of variables with respect to time t.

Related Rates Steps:

1. Write what you are given
2. Write what you are trying to find
3. Write an algebraic or geometric equation relating the variables (needs to be in terms of the rates that you are either given or are trying to find)
4. Differentiate equation with respect to time t
5. Substitute and solve

*Important Note: Remember that when the item is getting bigger, the rate is positive
If the item is getting smaller, the rate is negative – regardless of direction

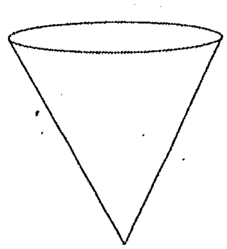
Example 1: The sides of a square are increasing at a rate of 5 cm/min. How fast is the area increasing when the sides measure 15 cm in length?

Example 2: A 25 foot ladder is leaning against a vertical wall. The bottom of the ladder is being pulled horizontally away from the wall at the constant rate of 3 ft / sec. a) How fast is the top of the ladder moving and in what direction when the bottom of the ladder is 15 ft from the wall? b) at what rate is the area changing when the bottom of the ladder is 15 ft from the wall?

Example 3: A spherical balloon is being inflated at a rate of 10 cubic centimeters per second. Find the rate of change of the surface area of the balloon at the moment when the surface area is 64π.

Example 4: Joe is standing 6 miles straight east of Moe. If Joe walks straight north at 3 mph while Moe walks straight south at 1 mph, at what rate is the distance between them changing after 2 hours?

Example 5: A conical tank (vertex down) is 10 feet across the top and 12 feet deep. If water is flowing into the tank at a rate of 10 cubic feet per minute, find the rate of change of the depth of the water when the water is 8 feet deep.



14

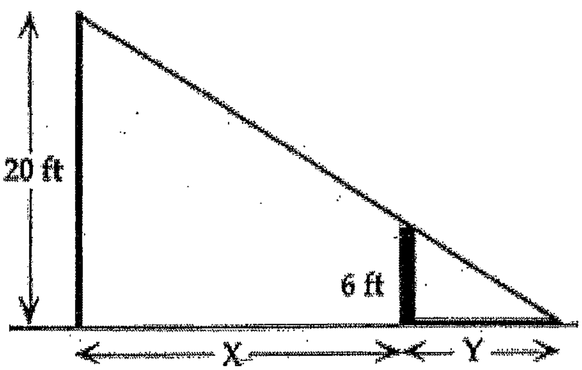
Related Rates Notes 2 - Similar Triangles and Shadow Problems

Example 1:

A man who is 6 feet tall is walking away from a lamp post at a rate of 5 feet per minute.

The lamp post is 20 feet tall. The person casts a shadow on the ground in front of them.

a) How fast is the shadow growing when the person is 30 feet from the lamp post?
b) How fast is the tip of the shadow moving when the person is 30 ft from the lamp post?



Notes:

1) $\frac{dx}{dt} = \text{rate of person walking}$

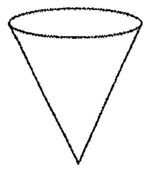
2) $\frac{dy}{dt} = \text{rate of change of shadow length}$

3) $\frac{dx}{dt} + \frac{dy}{dt} = \text{rate of change of tip of shadow}$

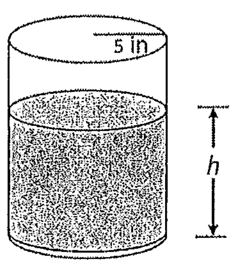
2. A street light is mounted at the top of a 15 ft pole. A man 5 ft tall walks towards the pole at a rate of 5ft per second. A) How fast is the tip of his shadow moving when he is 40 ft from the pole? B) How fast is the length of the shadow changing when he is 40 ft from the pole?

15

3. A conical tank (vertex down) is 40 feet across the top and 40 feet deep. If water is leaking out of the tank at a rate of 80 cubic feet per minute, find the rate of change of the radius of the water when the water is 8 feet deep. ($V = \frac{1}{3}\pi r^2 h$)



4. 2003 AB problem #5



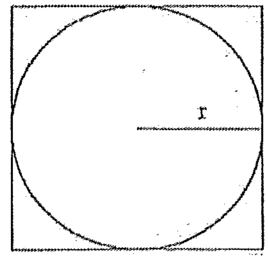
A coffeepot has the shape of a cylinder with radius 5 inches, as shown in the figure above. Let h be the depth of the coffee in the pot, measured in inches, where h is a function of time, t, measured in seconds. The volume, V, of coffee in the pot is changing at the rate of $-5\pi\sqrt{h}$ cubic inches per second. (The volume of a cylinder with radius r and height h is $V = \pi r^2 h$.) Find $\frac{dh}{dt}$ as a function of h. (This means your answer will contain the variable h)

(16) (16)

AP Calculus AB   Related Rates WS #3   Name______________________Date______

1994 AB5, BC2

1) A circle is inscribed in a square as shown in the figure. The circumference of the circle is increasing at a constant rate of 6 inches per minute. As the circle expands, the square expands to maintain the condition of tangency. (Note: A circle with radius r has circumference $C = 2\pi r$ and area $A = \pi r^2$.)



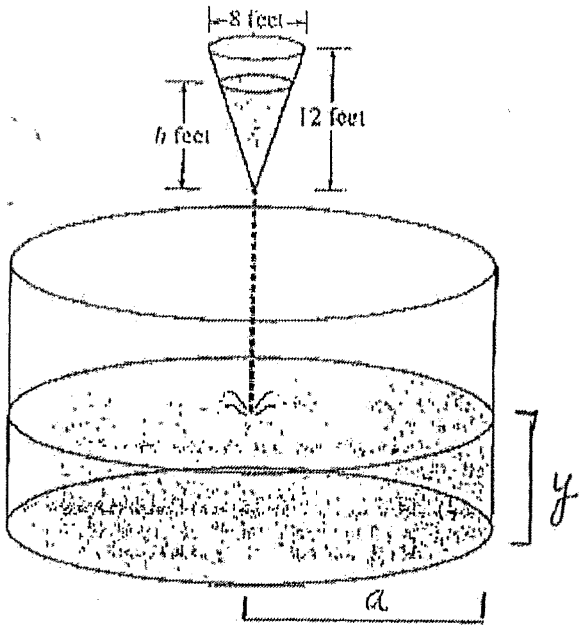
a) Find the rate at which the perimeter of the square is increasing. Indicate units of measure.

b) At the instant when the area of the circle is 25π square inches, find the rate of increase in the area enclosed between the circle and the square. Indicate units of measure.

2. Suppose that a spherical balloon grows in such a way that after t seconds, $V = 4\sqrt{t}$ in^3. How fast is the radius changing after 64 seconds? ($V = \frac{4}{3}\pi r^3$)

(17) (17)

3. 1995 AB 5



As shown in the figure above, water is draining from a conical tank with height 12 feet and diameter 8 feet into a cylindrical tank that has a base with area 400π square feet. The depth h, in feet, of the water in the conical tank is changing at the rate of $(h-12)$ feet per minute. (The volume V of a cone with radius r and height h is $V=\frac{1}{3}\pi r^2 h$.)

(a) Write an expression for the volume of water in the conical tank as a function of h.

(b) At what rate is the volume of water in the conical tank changing when $h=3$? Indicate units of measure.

(c) Let y be the depth, in feet, of the water in the cylindrical tank. At what rate is y changing when $h=3$? Indicate units of measure.

$V=\pi a^2 y$

Ch. 4.3 Related Rates Exercise Problems (Day 1)

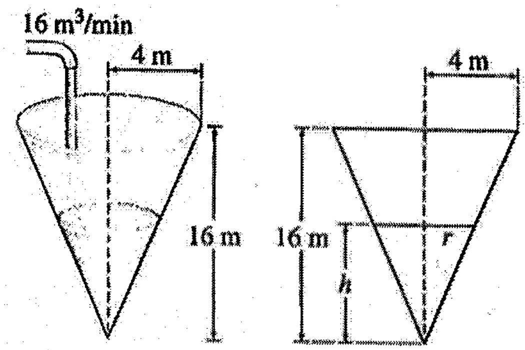
Pg. 286-291 #9, 10, 22, 23, 35, 38

9. Volume of a Cube If each edge of a cube is increasing at the constant rate of 3 cm/s, how fast is the volume of the cube increasing when the length x of an edge is 10 cm?

10. Volume of a Sphere If the radius of a sphere is increasing at 1 cm/s, find the rate of change of its volume when the radius is 6 cm.

22. Filling a Tank Water is flowing into a vertical cylindrical tank of diameter 6 m at the rate of 5 m^3/min. Find the rate at which the depth of the water is rising.

23. **Fill Rate** A container in the form of a right circular cone (vertex down) has radius 4 m and height 16 m. See the figure. If water is poured into the container at the constant rate of 16 m^3/min, how fast is the water level rising when the water is 8 m deep? *Hint:* The volume V of a cone of radius r and height h is $V = \frac{1}{3}\pi r^2 h$.



35. **Falling Ladder** An 8-m ladder is leaning against a vertical wall. If a person pulls the base of the ladder away from the wall at the rate of 0.5 m/s, how fast is the top of the ladder moving down the wall when the base of the ladder is

(a) 3 m from the wall?

(b) 4 m from the wall?

(c) 6 m from the wall?

38. **Tracking a Rocket** When a rocket is launched, it is tracked by a tracking dish on the ground located a distance D from the point of launch. The dish points toward the rocket and adjusts its angle of elevation θ to the horizontal (ground level) as the rocket rises. Suppose a rocket rises vertically at a constant speed of 2.0 m/s, with the tracking dish located 150 m from the launch point. Find the rate of change of the angle θ of elevation of the tracking dish with respect to time t (tracking rate) for each of the following:

    (a) Just after launch.

    (b) When the rocket is 100 m above the ground.

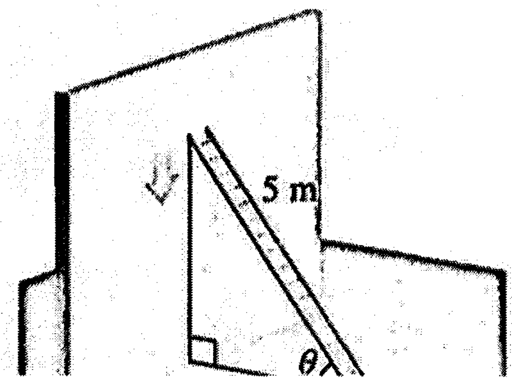
    (c) *When the rocket is 1.0 km above the ground.*

    (d) *Use the results in (a)–(c) to describe the behavior of the tracking rate as the rocket climbs higher and higher. What limit does the tracking rate approach as the rocket gets extremely high?*

Ch. 4.3 Related Rates Exercise Problems (Day 2)

Pg. 286-291 #19, 39, 40, 53, 54

19. **Change in Inclination** A ladder 5 m long is leaning against a wall. If the lower end of the ladder slides away from the wall at the rate of 0.5 m/s, at what rate is the inclination θ of the ladder with respect to the ground changing when the



39. **Lengthening Shadow** *A child, 1 m tall, is walking directly under a street lamp that is 6 m above the ground. If the child walks away from the light at the rate of 20 m/min, how fast is the child's shadow lengthening?*

40. **Approaching a Pole** A boy is walking toward the base of a pole 20 m high at the rate of 4 km/h. At what rate (in meters per second) is the distance from his feet to the top of the pole changing when he is 5 m from the pole?

53. **Change in Volume** The height h and width x of an open box with a square base are related to its volume by the formula $V = hx^2$. Discuss how the volume changes

(a) if h decreases with time, but x remains constant.

(b) if both h and x change with time.

54. **Rate of Change** Let $y = 2e^{\cos x}$. If both x and y vary with time in such a way that y increases at a steady rate of 5 units per second, at what rate is x changing when $x = \dfrac{\pi}{2}$?

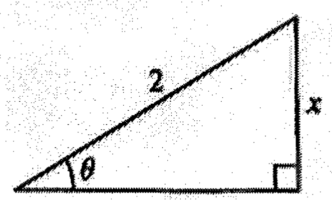
4.3 AP Practice Problems (p. 290-291) – Related Rates

1. A spherical balloon is inflated at the rate of 50 m^3/min. Find the rate at which the radius of the balloon is increasing when the diameter is 20 m.

   (A) $\frac{1}{2\pi}$ m/min (B) $\frac{5}{8\pi}$ m/min

   (C) $\frac{1}{8\pi}$ m/min (D) $\frac{5}{4\pi}$ m/min

2. In the right triangle below, θ is changing at the rate of 2 radians per second. At what rate is x changing at the instant when $x = 1$ cm?



   (A) 2 cm/s (B) $2\sqrt{3}$ cm/s (C) $\sqrt{3}$ cm/s (D) $4\sqrt{3}$ cm/s

3. The radius of a circle is decreasing at a constant rate of 2 in./min. What is the rate of change in the area of the circle when its area is 25π in.^{2}?

   (A) -20π in.^{2}/min (B) -25π in.^{2}/min

   (C) 20π in.^{2}/min (D) $20\pi^2$ in.^{2}/min

24

4. The radius r of a sphere is increasing at a rate of 2 cm/s. At the instant when $r = 12$ cm, what is the rate of change in the surface area S of the sphere? (The surface area S of a sphere with radius r is $S = 4\pi r^2$.)

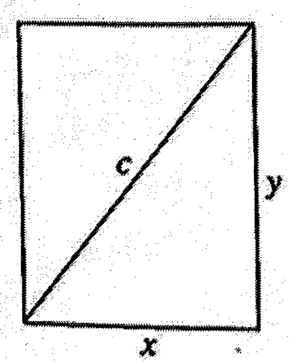
(A) $96\pi\ \text{cm}^2/\text{s}$ (B) $1152\pi\ \text{cm}^2/\text{s}$

(C) $576\pi\ \text{cm}^2/\text{s}$ (D) $192\pi\ \text{cm}^2/\text{s}$

5. The sides of the rectangle shown below are increasing so that the rate of change of y with respect to time t is three times the rate of change of x with respect to t. If $\frac{dc}{dt} = 1$, what is the rate of change of x when $x = 6$ and $y = 8$?

(A) 3 (B) $\frac{1}{3}$

(C) 1 (D) $\frac{1}{6}$



6. The area of a circle is increasing at a rate of $48\pi\ \text{ft}^2/\text{h}$. How fast is the radius of the circle increasing when its area is $36\pi\ \text{ft}^2$?

(A) 4 ft/h (B) 6 ft/h (C) $4\sqrt{3}$ ft/h (D) $\frac{4}{\pi}$ ft/h

7. The radius r and height h of a right circular cone are both increasing at a constant rate of 2 cm/h. At what rate in centimeters cubed per hour is the volume V of the cone increasing when $r = 6$ cm and $h = 15$ cm? (The volume V of a right circular cone of height h and radius r is $V = \frac{1}{3}\pi r^2 h$.)

(A) 24π cm^3/h (B) 96π cm^3/h

(C) 144π cm^3/h (D) 180π cm^3/h

8. Two roads cross at right angles. A police officer sits in a car 65 m east of the crossing and observes a car speeding northbound at 84 m/s. At what speed (in meters per second) is the car distancing itself from the police officer 5 seconds after it passes the crossing?

(A) 166.024 m/s (B) 83.012 m/s

(C) 84 m/s (D) 95.859 m/s

26

9. A roofer's 13-meter ladder is placed against the wall of a building with its base on level ground. The top of the ladder slips down the wall as the bottom of the ladder slips away from the building at a constant rate of 5 m/s.

(a) At what rate is the top of the ladder moving when it is 5 meters from the ground?

(b) At what rate is the area of the triangle formed by the ladder, the wall, and the ground changing when the top of the ladder is 5 m from the ground?

(c) If θ is the angle formed by the ladder and the ground, what is the rate of change in θ when the top of the ladder is 5 m from the ground?

AP Calculus – 4.4 Notes - L'Hopital's Rule and Indeterminate Form

Recall: When evaluating limits, first try direct substitution! $\lim_{x \to 3} \frac{2x-5}{x} =$

1. $\lim_{x \to 2} \frac{x^2-7x+10}{x-2} =$

L'Hospital's Rule:

Suppose $f(a) = 0$ and $g(a) = 0$ and $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0}$ or $\frac{\infty}{\infty}$. L'Hopital's Rule allows you to apply the following:

Evaluate each limit. Use L'Hospital's when possible.

2. $\lim_{x \to 2} \frac{x-2}{3x^3-6x^2+x-2}$

3. $\lim_{x \to 0} \frac{\sin(6x)}{x}$

4. $\lim_{x \to 0} \frac{1-\cos(x)}{x^2}$

5. $\lim_{x \to \infty} \frac{2x^2}{e^{2x}}$

L'HOSPITAL'S IS NOT THE QUOTIENT RULE!!

6. $\frac{d}{dx} \frac{\sin(6x)}{x}$

28

Practice Problems:

Find the following. Use L'Hôpital's when possible.

1. $\lim\limits_{x\to 1} \frac{x-1}{x^2-3x+2}$	2. $\lim\limits_{x\to -5} \frac{x^2-2x-35}{x+5}$	3. $\lim\limits_{x\to 0} \frac{4x}{\ln(x+1)}$
4. $\lim\limits_{x\to 0} \frac{x-1}{x^2-3x+2}$	5. $\lim\limits_{x\to 1} \frac{2(x^2-1)}{\ln x^2}$	6. $\frac{d}{dx} \frac{6x^2+x}{\sin(x)}$

16. If $f(x) = 2x^3 + 5$, then $\lim\limits_{x\to 0} \frac{f(x)-f(0)}{x^3}$ is

(A) 0 (B) 1 (C) 2 (D) 3 (E) The limit does not exist.

17. Functions f, g, and h are twice-differentiable functions with $g(3) = h(3) = 5$. The line $y = 5 + \frac{1}{2}(x-3)$ is tangent to both the graph of g at $x = 3$ and the graph of h at $x = 3$.

a. Find $h'(3)$.

b. Let a be the function given by $a(x) = 2x^3h(x)$. Write an expression for $a'(x)$. Find $a'(3)$.

c. The function h satisfies $h(x) = \frac{x^2-9}{1-(f(x))^3}$ for $x \neq 3$. It is known that $\lim\limits_{x\to 3} h(x)$ can be evaluated using L'Hospital's Rule. Use $\lim\limits_{x\to 3} h(x) = 5$ to find $f(3)$ and $f'(3)$. Show the work that leads to your answers.

EXAMPLE 10 **Finding the Limit of an Indeterminate Form of the Type 1^∞**

Find $\lim\limits_{x\to 0^+}(1+x)^{1/x}$.

Solution

The expression $(1+x)^{1/x}$ is an indeterminate form at 0^+ of the type 1^∞.

Step 1 Let $y=(1+x)^{1/x}$. Then $\ln y = \dfrac{1}{x}\ln(1+x)$.

Step 2
$$\lim_{x\to 0^+}\ln y = \lim_{x\to 0^+}\frac{\ln(1+x)}{x} \underset{\uparrow}{=} \lim_{x\to 0^+}\frac{\dfrac{d}{dx}\ln(1+x)}{\dfrac{d}{dx}x} = \lim_{x\to 0^+}\frac{\dfrac{1}{1+x}}{1}=1$$

Type $\dfrac{0}{0}$; use L'Hôpital's Rule

Step 3 Since $\lim\limits_{x\to 0^+}\ln y = 1$, $\lim\limits_{x\to 0^+} y = e^1 = e$. ■

NOW WORK Problem 85.

4.4 Assess Your Understanding

Concepts and Vocabulary

1. *True or False* $\dfrac{f(x)}{g(x)}$ is an indeterminate form at c of the type $\dfrac{0}{0}$ if $\lim\limits_{x\to c}\dfrac{f(x)}{g(x)}$ does not exist.
2. *True or False* If $\dfrac{f(x)}{g(x)}$ is an indeterminate form at c of the type $\dfrac{0}{0}$, then L'Hôpital's Rule states that $\lim\limits_{x\to c}\dfrac{f(x)}{g(x)} = \lim\limits_{x\to c}\left[\dfrac{d}{dx}\dfrac{f(x)}{g(x)}\right]$.
3. *True or False* $\dfrac{1}{x}$ is an indeterminate form at 0.
4. *True or False* $x\ln x$ is not an indeterminate form at 0^+ because $\lim\limits_{x\to 0^+} x = 0$ and $\lim\limits_{x\to 0^+}\ln x = -\infty$, and $0\cdot -\infty = 0$.
5. In your own words, explain why $\infty-\infty$ is an indeterminate form, but $\infty+\infty$ is not an indeterminate form.
6. In your own words, explain why $0\cdot\infty \neq 0$.

Skill Building

In Problems 7–26:
(a) Determine whether each expression is an indeterminate form at c.
(b) If it is, identify the type. If it is not an indeterminate form, state why.

(PAGE 292) 7. $\dfrac{1-e^x}{x}$, $c=0$

8. $\dfrac{1-e^x}{x-1}$, $c=0$

9. $\dfrac{e^x}{x}$, $c=0$

10. $\dfrac{e^x}{x}$, $c=\infty$

11. $\dfrac{\ln x}{x^2}$, $c=\infty$

12. $\dfrac{\ln(x+1)}{e^x-1}$, $c=0$

13. $\dfrac{\sec x}{x}$, $c=0$

14. $\dfrac{x}{\sec x - 1}$, $c=0$

15. $\dfrac{\sin x(1-\cos x)}{x^2}$, $c=0$

16. $\dfrac{\sin x - 1}{\cos x}$, $c=\dfrac{\pi}{2}$

17. $\dfrac{\tan x - 1}{\sin(4x-\pi)}$, $c=\dfrac{\pi}{4}$

18. $\dfrac{e^x-e^{-x}}{1-\cos x}$, $c=0$

19. x^2e^{-x}, $c=\infty$

20. $x\cot x$, $c=0$

21. $\csc\dfrac{x}{2}-\cot\dfrac{x}{2}$, $c=0$

22. $\dfrac{x}{x-1}+\dfrac{1}{\ln x}$, $c=1$

23. $\left(\dfrac{1}{x^2}\right)^{\sin x}$, $c=0$

24. $(e^x+x)^{1/x}$, $c=0$

25. $(x^2-1)^x$, $c=0$

26. $(\sin x)^x$, $c=0$

In Problems 27–42, identify each quotient as an indeterminate form of the type $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$. Then find the limit.

27. $\lim\limits_{x\to 2}\dfrac{x^2+x-6}{x^2-3x+2}$

28. $\lim\limits_{x\to 1}\dfrac{2x^3+5x^2-4x-3}{x^3+x^2-10x+8}$

(PAGE 294) 29. $\lim\limits_{x\to 1}\dfrac{\ln x}{x^2-1}$

30. $\lim\limits_{x\to 0}\dfrac{\ln(1-x)}{e^x-1}$

31. $\lim\limits_{x\to 0}\dfrac{e^x-e^{-x}}{\sin x}$

32. $\lim\limits_{x\to 0}\dfrac{\tan(2x)}{\ln(1+x)}$

33. $\lim\limits_{x\to 1}\dfrac{\sin(\pi x)}{x-1}$

34. $\lim\limits_{x\to \pi}\dfrac{1+\cos x}{\sin(2x)}$

(PAGE 295) 35. $\lim\limits_{x\to \infty}\dfrac{x^2}{e^x}$

36. $\lim\limits_{x\to \infty}\dfrac{e^x}{x^4}$

37. $\lim\limits_{x\to \infty}\dfrac{\ln x}{e^x}$

38. $\lim\limits_{x\to \infty}\dfrac{x+\ln x}{x\ln x}$

(PAGE 294) 39. $\lim\limits_{x\to 0}\dfrac{e^x-1-\sin x}{1-\cos x}$

40. $\lim\limits_{x\to 0}\dfrac{e^x-e^{-x}-2\sin x}{3x^3}$

41. $\lim\limits_{x\to 0}\dfrac{\sin x - x}{x^3}$

42. $\lim\limits_{x\to 0}\dfrac{x^3}{\cos x - 1}$

In Problems 43–58, identify each expression as an indeterminate form of the type $0 \cdot \infty$, $\infty - \infty$, 0^0, 1^∞, or ∞^0. Then find the limit.

43. $\lim_{x \to 0^+} (x^2 \ln x)$

44. $\lim_{x \to \infty} (xe^{-x})$

PAGE 297 45. $\lim_{x \to \infty} [x(e^{1/x} - 1)]$

46. $\lim_{x \to \pi/2} [(1 - \sin x) \tan x]$

PAGE 298 47. $\lim_{x \to \pi/2} (\sec x - \tan x)$

48. $\lim_{x \to 0} \left(\cot x - \dfrac{1}{x} \right)$

49. $\lim_{x \to 1} \left(\dfrac{1}{\ln x} - \dfrac{x}{\ln x} \right)$

50. $\lim_{x \to 0} \left(\dfrac{1}{x} - \dfrac{1}{e^x - 1} \right)$

PAGE 298 51. $\lim_{x \to 0^+} (2x)^{3x}$

52. $\lim_{x \to 0^+} x^{x^2}$

53. $\lim_{x \to \infty} (x + 1)^{e^{-x}}$

54. $\lim_{x \to \infty} (1 + x^2)^{1/x}$

55. $\lim_{x \to 0^+} (\csc x)^{\sin x}$

56. $\lim_{x \to \infty} x^{1/x}$

57. $\lim_{x \to \pi/2^-} (\sin x)^{\tan x}$

58. $\lim_{x \to 0} (\cos x)^{1/x}$

In Problems 59–88, find each limit.

59. $\lim_{x \to 0^+} \dfrac{\cot x}{\cot(2x)}$

60. $\lim_{x \to \infty} \dfrac{\ln(\ln x)}{\ln x}$

PAGE 296 61. $\lim_{x \to 1/2^-} \dfrac{\ln(1 - 2x)}{\tan(\pi x)}$

62. $\lim_{x \to 1^-} \dfrac{\ln(1 - x)}{\cot(\pi x)}$

63. $\lim_{x \to \infty} \dfrac{x^4 + x^3}{e^x + 1}$

64. $\lim_{x \to \infty} \dfrac{x^2 + x - 1}{e^x + e^{-x}}$

65. $\lim_{x \to 0} \dfrac{xe^{4x} - x}{1 - \cos(2x)}$

66. $\lim_{x \to 0} \dfrac{x \tan x}{1 - \cos x}$

67. $\lim_{x \to 0} \dfrac{\tan^{-1} x}{x}$

68. $\lim_{x \to 0} \dfrac{\tan^{-1} x}{\sin^{-1} x}$

69. $\lim_{x \to 0} \dfrac{\cos x - 1}{\cos(2x) - 1}$

70. $\lim_{x \to 0} \dfrac{\tan x - \sin x}{x^3}$

71. $\lim_{x \to 0^+} (x^{1/2} \ln x)$

72. $\lim_{x \to \infty} [(x - 1)e^{-x^2}]$

73. $\lim_{x \to \pi/2} [\tan x \ln(\sin x)]$

74. $\lim_{x \to 0^+} [\sin x \ln(\sin x)]$

75. $\lim_{x \to 0} [\csc x \ln(x + 1)]$

76. $\lim_{x \to \pi/4} [(1 - \tan x) \sec(2x)]$

77. $\lim_{x \to a} \left[(a^2 - x^2) \tan\left(\dfrac{\pi x}{2a} \right) \right]$

78. $\lim_{x \to 1^+} \left[(1 - x) \tan\left(\dfrac{1}{2} \pi x \right) \right]$

79. $\lim_{x \to 1} \left(\dfrac{1}{\ln x} - \dfrac{1}{x - 1} \right)$

80. $\lim_{x \to 1} \left(\dfrac{x}{x - 1} - \dfrac{1}{\ln x} \right)$

81. $\lim_{x \to \pi/2} \left(x \tan x - \dfrac{\pi}{2} \sec x \right)$

82. $\lim_{x \to \pi} (\cot x - x \csc x)$

83. $\lim_{x \to 1^-} (1 - x)^{\tan(\pi x)}$

84. $\lim_{x \to 0^+} x^{\sqrt{x}}$

PAGE 299 85. $\lim_{x \to 0} \left(\dfrac{\sin x}{x} \right)^{1/x}$

86. $\lim_{x \to \infty} \left(1 + \dfrac{5}{x} + \dfrac{3}{x^2} \right)^x$

87. $\lim_{x \to (\pi/2)^-} (\tan x)^{\cos x}$

88. $\lim_{x \to 0^+} (x^2 + x)^{-\ln x}$

Applications and Extensions

89. **Wolf Population** In 2014 there were 229 wolves in Wyoming outside of Yellowstone National Park. Suppose the population of wolves in the region at time t follows the logistic growth curve

$$w = w(t) = \frac{Ke^{rt}}{\frac{K}{40} + e^{rt} - 1}$$

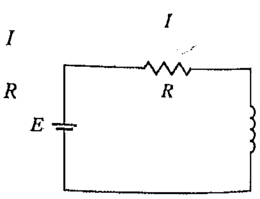
where $K = 252$, $r = 0.283$, and $t = 0$ represents the population in the year 2000.
Source: Federal Wildlife Service.
(a) Find $\lim_{t \to \infty} w(t)$.
(b) Interpret the answer found in (a) in the context of the problem.
(c) Use technology to graph $w = w(t)$.

90. **Skydiving** The downward velocity v of a skydiver with nonlinear air resistance can be modeled by

$$v = v(t) = -A + RA \frac{e^{Bt + C} - 1}{e^{Bt + C} + 1}$$

where t is the time in seconds, and A, B, C, and R are positive constants with $R > 1$.
(a) Find $\lim_{t \to \infty} v(t)$.
(b) Interpret the limit found in (a).
(c) If the velocity v is measured in feet per second, reasonable values of the constants are $A = 108.6$, $B = 0.554$, $C = 0.804$, and $R = 2.62$. Graph the velocity of the skydiver with respect to time.

91. **Electricity** The equation governing the amount of current I (in amperes) in a simple RL circuit consisting of a resistance R (in ohms), an inductance L (in henrys), and an electromotive force E (in volts) is $I = \dfrac{E}{R}(1 - e^{-Rt/L})$.



(a) Find $\lim_{t \to \infty} I(t)$ and $\lim_{R \to 0^+} I(t)$.
(b) Interpret these limits.

92. Find $\lim_{x \to 0} \dfrac{a^x - b^x}{x}$, where $a \neq 1$ and $b \neq 1$ are positive real numbers.

93. Show that $\lim_{x \to \infty} \dfrac{\ln x}{x^n} = 0$, for $n \geq 1$ an integer.

94. Show that $\lim_{x \to \infty} \dfrac{x^n}{e^x} = 0$ for $n \geq 1$ an integer.

95. Show that $\lim_{x \to 0^+} (\cos x + 2 \sin x)^{\cot x} = e^2$.

96. Find $\lim_{x \to \infty} \dfrac{P(x)}{e^x}$, where P is a polynomial function.

97. Find $\lim_{x \to \infty} [\ln(x + 1) - \ln(x - 1)]$.

98. Show that $\lim_{x \to 0^+} \dfrac{e^{-1/x^2}}{x} = 0$. *Hint:* Write $\dfrac{e^{-1/x^2}}{x} = \dfrac{\frac{1}{x}}{e^{1/x^2}}$.

99. If n is an integer, show that $\lim_{x \to 0^+} \dfrac{e^{-1/x^2}}{x^n} = 0$.

100. Show that $\lim_{x \to \infty} \sqrt[x]{x} = 1$.

4.4 AP Practice Problems (p. 301) – L'Hopital's Rule

1. $\lim_{x\to 0} \frac{e^{4x}-1}{\sin(2x)} =$

(A) 0 (B) 2 (C) 4 (D) does not exist

2. $\lim_{x\to 0} \frac{1-\cos^2(3x)}{x^2} =$

(A) 18 (B) 9 (C) 0 (D) 3

3. Find $\lim_{x\to\infty} \frac{x^{-3/2}}{\sin\frac{1}{x}}$.

(A) $\frac{3}{2}$ (B) 1 (C) 0 (D) ∞

4. $\lim_{x\to 1} \frac{\ln x^3}{x^2-1} =$

(A) 0 (B) 1 (C) $\frac{3}{2}$ (D) 3

32

5. For any positive integer k, $\lim_{x \to \infty} \frac{\ln x}{x^k} =$

(A) 0 (B) 1 (C) $k+1$ (D) ∞

6. $\lim_{\theta \to 0} \frac{1 - \cos(2\theta)}{3 \sin \theta} =$

(A) -2 (B) $\frac{2}{3}$ (C) 0 (D) $-\frac{1}{3}$

7. $\lim_{x \to \frac{\pi}{2}^-} \frac{\ln(\cos x)}{\tan x} =$

(A) $-\infty$ (B) 0 (C) 1 (D) ∞

8. Which of the following are indeterminate forms at 0?

I $\frac{x}{\ln(x+1)}$ II $\frac{e^x}{x^2 - 2x}$ III $\frac{x}{1 - \cos(\pi x)}$

(A) I only (B) I and III only

(C) II and III only (D) I, II, and III

Ch. 4 Review AP Practice Problems (p. 304)

1. If $y = \tan(3x + 2y)$, find the rate of change of y with respect to x at the origin.

   (A) -3 (B) 1 (C) 3 (D) 5

2. The linear approximation for $f(x) = xe^x$ near $x = 2$ is

   (A) $L(x) = 3e^2(x-2)$ (B) $L(x) = xe^x + xe^x(x-2)$

   (C) $L(x) = 2e^2 + 3e^2(x-2)$ (D) $L(x) = 2e^2 + 2e^2(x-2)$

3. $\lim\limits_{x \to \infty} \dfrac{x}{\ln x} =$

   (A) 0 (B) 1 (C) e (D) ∞

34

4. The radius of a circle is increasing at a constant positive rate with respect to time. What is the radius of the circle when the rate of change of the area with respect to time is equal to twice the rate of change of the circumference with respect to time?

(A) 2 (B) $\frac{1}{2}$ (C) 1 (D) 4

5. $\lim_{x \to 0} \frac{xe^x}{\sin x} =$

(A) 0 (B) 1 (C) e (D) ∞

6. Use the linear approximation to $f(x) = \tan x$ at $x = 0$ to approximate $f(0.2)$.

(A) -0.2 (B) 0 (C) 0.2 (D) 0.8

Free Response Questions

7. Water is being pumped into a cylindrical tank that measures 20 m in height and 4 m in radius at a constant rate of 5 m^3/h. Ten meters from the bottom of the tank there is a hole in the tank. Water leaks from that hole at a rate of 1 m^3/h.

   (a) Find the rate at which the water is rising in the tank until it reaches the leak.

   (b) Find the rate at which the water is rising in the tank after it passes the leak.

   (c) What is the total time it will take for the tank to begin to overflow?

36

8. The position function of an object moving along the x-axis is $s(t) = t \sin t + 3$, where s in meters and $t \geq 0$ in minutes.

   (a) Find the initial position of the object. Find its position at $t = \frac{2\pi}{3}$ min.

   (b) Find and interpret $s'\left(\frac{2\pi}{3}\right)$ in the context of this problem.

   (c) Find the average velocity of the object from $t = 0$ to $t = \frac{2\pi}{3}$.