**A.P. Calculus AB** **Trig** Curve Sketch WS #1

1. Determine whether or not the mean value theorem applies to the function $f(x) = 4 - \sec x$ on the interval $[-\pi, -\frac{\pi}{3}]$. If so, find the value(s) of $c$ as defined in the theorem. If not, state reason why.

2. Determine whether or not Rolle's theorem applies to the function, $f(x) = \sin(2x) + x$, on $[0, 2\pi]$. If so, find the value(s) of $c$ as defined in the theorem. If not, state reason why.

3. Determine whether or not Mean Value Theorem applies to the function, $f(x) = \sin(2x) + x$, on $[0, 2\pi]$. If so, find the value(s) of $c$ as defined in the theorem. If not, state reason why.

4. a) Given the function $y = x - 2\cos x$ on the interval $[-\pi, \pi]$, find intervals increasing, decreasing, relative extrema (ordered pairs!), Points of Inflection (ordered pairs!), intervals of concave up/down. Justify your answers!

   b) Create one sign line with all critical points from $f'(x)$ and $f''(x)$

   c) Sketch graph from information above.

**Justify your answers with "Because" statements!**

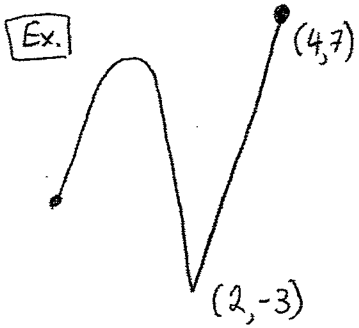
| | |
|---|---|
| Interval(s) increasing: | Interval(s) Decreasing: |
| Relative Maximum(s): | Relative Minimum(s) : |
| Interval(s) concave up: | Interval(s) concave down: |
| Point(s) of inflection: | |

| | |
|---|---|
| Sign Line: | Sketch Graph: |

Ch. 3.1

## Extreme Value Theorem (EVT) Ch. 3.1

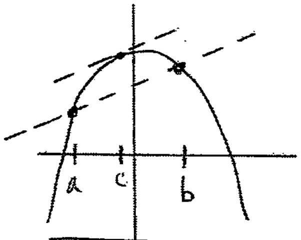
Ex.



Purpose: Find Abs max/min on closed interval

- $f(x)$ continuous $[a, b]$
- find critical points
  a) set $f'(x) = 0$
  b) set denominator of $f'(x) = 0$
- test critical points and endpoints into $f(x)$ to find absolute max/min
- Abs max is 7 at $x = 4$
  Abs min is -3 at $x = 2$

## 3.2a Mean Value Theorem (MVT)



purpose: find the location on the curve where the guaranteed slope occurs.

Conditions:

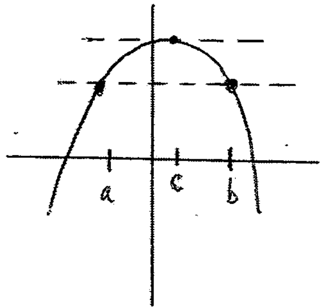
- $f(x)$ continuous on $[a, b]$
  (no VA, no holes on interval)
- $f(x)$ differentiable on $(a, b)$
  (no sharp turns, no slope undefined on $(a, b)$)

$$\boxed{\text{MVT: } f'(c) = \frac{f(b) - f(a)}{b - a}}$$

Steps:

1) find slope between endpoints $\left[\frac{f(b) - f(a)}{b - a}\right]$
2) find $f'(x)$
3) set $f'(x)$ = slope value, solve for $x$ (c-value)
4) Keep the c-values in interval $(a, b)$

## 3.2b Rolle's Theorem



$\boxed{\text{Rolle's Theorem: } f'(c)=0}$

Steps:

1) confirm endpts have same y-values
2) find $f'(x)$
3) set numerator of $f'(x)=0$, solve for $x$ (c-value)
4) Keep the c-values in interval $(a,b)$

Purpose: Find the location on the curve where the guaranteed slope of 0 occurs.

Conditions:

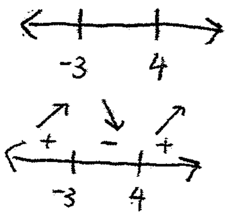
* $f(x)$ continuous $[a,b]$ (no breaks, no vertical asymptote, no holes)
* $f(x)$ differentiable $(a,b)$ (no sharp turns, no location with undefined slope)
* $f(a)=f(b)$ (endpoints with same y-value)

## Ch. 3.3 1st Derivative Test

Purpose: Use $f'(x)$ to determine slope behavior of graph and find relative max, relative mins of graph

1) Find critical points
   a) find $f'(x)$
   b) set numerator of $f'(x)=0$
   c) set denominator of $f'(x)=0$
2) Place critical values on $f'(x)$ sign line
3) Test intervals, plug in x-values into $f'(x)$
   a) Rel. max at $(-3,\_)$ b/c $f'(x)$ changes from + to −
   b) Rel. min at $(4,\_)$ b/c $f'(x)$ changes from − to +
   c) $f(x)$ increasing $(-\infty,3),(4,\infty)$ b/c $f'(x)>0$
   d) $f(x)$ decreasing $(-3,4)$ b/c $f'(x)<0$

$\boxed{\text{Ex:}}$



**A.P. Calculus AB** **Trig** WS KEY

1. Determine whether or not the mean value theorem applies to the function $f(x) = 4 - \sec x$ on the interval $[-\pi, -\frac{\pi}{3}]$. If so, find the value(s) of $c$ as defined in the theorem. If not, state reason why.

$f(x)$ not continuous at $x = -\pi/2$ since vertical asymptote at $x = -\pi/2$.
Mean Value Theorem does not apply.

2. Determine whether or not Rolle's theorem applies to the function, $f(x) = \sin(2x) + x$, on $[0, 2\pi]$. If so, find the value(s) of $c$ as defined in the theorem. If not, state reason why.

i) $f(x)$ continuous on $[0, 2\pi]$ ii) $f(x)$ differentiable on $(0, 2\pi)$

iii) $f(0) = \sin(0) + 0 = 0$
$f(2\pi) = \sin(2\pi) + 2\pi = 2\pi$

since $f(0) \neq f(2\pi)$, Rolle's theorem does not apply.

3. Determine whether or not Mean Value Theorem applies to the function, $f(x) = \sin(2x) + x$, on $[0, 2\pi]$. If so, find the value(s) of $c$ as defined in the theorem. If not, state reason why.

i) $f(x)$ continuous on $[0, 2\pi]$ ii) $f(x)$ differentiable on $(0, 2\pi)$

$f(0) = 0$
$f(2\pi) = 2\pi$

$$\boxed{\text{Set } f'(c) = \frac{f(b) - f(a)}{b - a}}$$

$$\frac{f(2\pi) - f(0)}{2\pi - 0} = \frac{2\pi - 0}{2\pi - 0} = \frac{2\pi}{2\pi} = \boxed{1}$$

$f'(x) = \cos(2x) \cdot 2 + 1$
$f'(x) = 2\cos(2x) + 1$

$2\cos(2x) + 1 = 1$
$2\cos(2x) = 0$
$\cos(2x) = 0$
$2\cos^2 x - 1 = 0$

$2\cos^2 x = 1$
$\cos^2 x = 1/2$
$\cos x = \pm\sqrt{1/2}$
$\cos x = \pm\frac{\sqrt{2}}{2}$

$x = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4$

$$\boxed{c = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4}$$

⑥

4. a) Given the function $\boxed{y = x - 2\cos x}$ on the interval $[-\pi,\pi]$, find intervals increasing, decreasing, relative extrema (ordered pairs!), Points of Inflection (ordered pairs!), intervals of concave up/down. Justify your answers!

b) Create one sign line with all critical points from f'(x) and f''(x)

c) Sketch graph from information above.

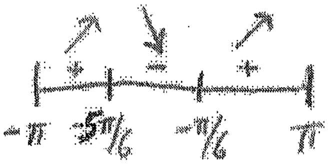
$y' = 1 - 2(-\sin x) = \boxed{1 + 2\sin x}$

$0 = 1 + 2\sin x$

$1 + 2\sin x = 0$

$\sin x = -1/2$

$x = -\pi/6, -5\pi/6$



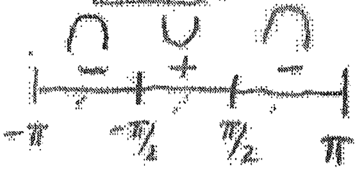
$y'' = 0 + 2\cos x = \boxed{2\cos x}$

$0 = 2\cos x$

$2\cos x = 0$

$\cos x = 0$

$x = -\pi/2, \pi/2$



$f(-5\pi/6) = -5\pi/6 - 2\cos(-5\pi/6) \quad -\frac{5\pi}{6} - 2\left(\frac{-\sqrt{3}}{2}\right) \approx -1$

$f(-\pi/6) = -\pi/6 - 2\cos(-\pi/6) \quad -\pi/6 - 2(\sqrt{3}/2) \approx -2$

$f(-\pi/2) = -\pi/2 - 2\cos(-\pi/2) = -\pi/2 - 2(0) \approx -1.5$

$f(\pi/2) = \pi/2 - 2\cos(\pi/2) = \pi/2 \quad \approx 1.5$

**Justify your answers with "Because" statements!**

Interval(s) increasing:
$(-\pi, -5\pi/6) \cup (-\pi/6, \pi)$ b/c $f'(x) > 0$

Interval(s) Decreasing:
$(-5\pi/6, -\pi/6)$ b/c $f'(x) < 0$

Relative Maximum(s): $(-5\pi/6, -5\pi/6 + \sqrt{3})$
b/c f'(x) changes from + to −

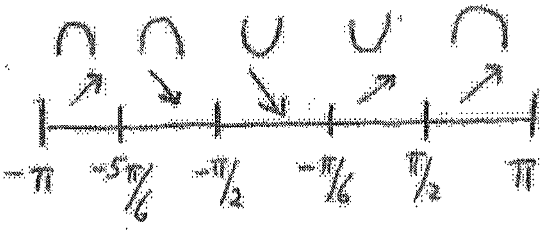
Relative Minimum(s): ~~at~~ $(-\pi/6, -\pi/6 - \sqrt{3})$
b/c f'(x) changes from − to +

Interval(s) concave up:
$(-\pi/2, \pi/2)$ b/c $f''(x) > 0$

Interval(s) concave down:
$(-\pi, -\pi/2) \cup (\pi/2, \pi)$ b/c $f''(x) < 0$

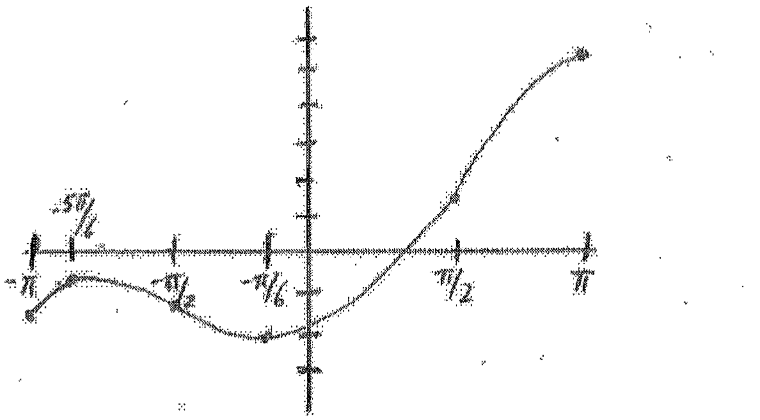
Point(s) of inflection:
at $(-\pi/2, -\pi/2)$ and $(\pi/2, \pi/2)$ b/c f''(x) changes signs.

Sign Line:



Sketch Graph: $f(-\pi) = -\pi - 2\cos(-\pi) = -\pi + 2 = -1.14$

$f(\pi) = \pi - 2\cos(\pi) = \pi + 2 = 5.14$



**A.P. Calculus AB** **Trig** Curve Sketch WS#2

NO CALCULATORS! All answers must be justified! Name ____________________

1) Determine whether or not the MVT applies to the function $f(x) = \sin x - x$ on $[-2\pi, -\pi]$.
If so, find the value(s) of $c$ as defined in the theorem.

2. Determine whether or not Rolle's theorem applies to the function $f(x) = \sec x$ on $\left[-\frac{7\pi}{6}, -\frac{5\pi}{6}\right]$
If so, find the value(s) of $c$ as defined in the theorem.

3. Determine whether or not the MVT applies to the function $f(x) = 5x + \cot\left(\frac{x}{2}\right)$ on $\left[\frac{\pi}{3}, \frac{5\pi}{3}\right]$.
If so, find the value(s) of $c$ as defined in the theorem, (leave answer(s) as inverse trig functions.)

4. Given the function $y = 2\sin x + 2\cos x$ on the interval $(-\pi, \pi)$, find:

a) the intervals of direction: increasing: ____________________

decreasing: ____________________

b) the relative extrema min: ____________________

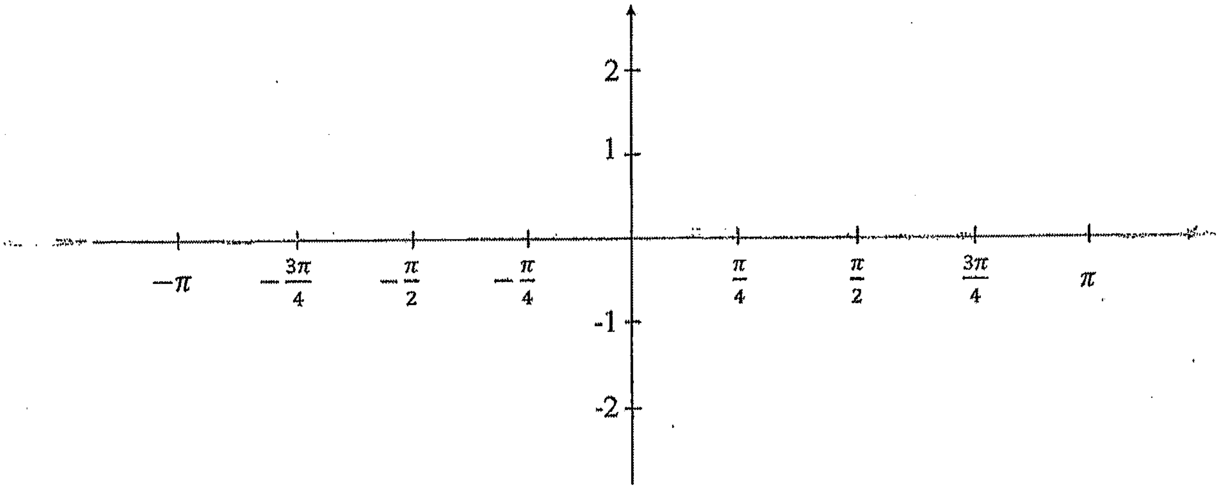
max: ____________________

c) the point(s) of inflection: ____________________

d) concavity intervals up: ____________________

down: ____________________

e) Sketch the graph based on your answers to parts a – d.



**A.P. Calculus AB** **Trig** - WS #2 Name Key

NO CALCULATORS! All answers must be justified!

1. Determine whether or not the MVT applies to the function $f(x) = \sin x - x$ on $[-2\pi, -\pi]$.
If so, find the value(s) of $c$ as defined in the theorem.

$f(x)$ continuous on $[-2\pi, -\pi]$ and differentiable on $(-2\pi, -\pi)$

* set $f'(x) = \frac{f(b)-f(a)}{b-a}$

derivative ↗ slope between endpts ↗

$f(-2\pi) = \sin(-2\pi) - (-2\pi) = 0 + 2\pi = 2\pi$

$f(-\pi) = \sin(-\pi) - (-\pi) = 0 + \pi = \pi$

$\frac{f(b)-f(a)}{b-a} = \frac{\pi - 2\pi}{-\pi-(-2\pi)} = \frac{-\pi}{\pi} = -1$

slope b/t endpts. ↗

$f'(x) = \cos x - 1 \rightarrow$

$\cos x - 1 = -1$

$\cos x = 0$

$x = -\frac{3\pi}{2}, \cancel{-\pi/2}$

$\boxed{c = -3\pi/2}$ on $[-2\pi, -\pi]$

2. Determine whether or not Rolle's theorem applies to the function $f(x) = \sec x$ on $\left[-\frac{7\pi}{6}, -\frac{5\pi}{6}\right]$
If so, find the value(s) of $c$ as defined in the theorem.

sec x has asymptote at $x = -3\pi/2, -\pi/2$, does not affect the interval

$f(x)$ continuous $\left[-\frac{7\pi}{6}, -\frac{5\pi}{6}\right]$, differentiable on $\left(-7\pi/6, -\frac{5\pi}{6}\right)$

$f\left(-\frac{7\pi}{6}\right) = \sec\left(-\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2}$

$f\left(-\frac{5\pi}{6}\right) = \sec\left(-\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}$

Since $f\left(-\frac{7\pi}{6}\right) = f\left(-\frac{5\pi}{6}\right)$, Rolle's theorem applies

*set $f'(x) = 0$

$f'(x) = \sec x \tan x$

$\sec x \tan x = 0$

$\frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} = 0$

$\frac{\sin x}{\cos^2 x} = 0 \quad x = -\pi, -2\pi$

$\boxed{c = -\pi \text{ on } \left[-\frac{7\pi}{6}, -\frac{5\pi}{6}\right]}$

3. Determine whether or not the MVT applies to the function $f(x) = 5x + \cot\left(\frac{x}{2}\right)$ on $\left[\frac{\pi}{3}, \frac{5\pi}{3}\right]$.
If so, find the value(s) of $c$ as defined in the theorem, (leave answer(s) as inverse trig functions.)

* $\cot x = \frac{\cos(x/2)}{\sin(x/2)}$: VA. when $\sin(x/2) = 0$, $x = 0, 2\pi, 4\pi \ldots$ Does not affect interval

$f(x)$ continuous $[\pi/3, 5\pi/3]$, differentiable on $(\pi/3, 5\pi/3)$

*set $f'(x) = \frac{f(b)-f(a)}{b-a}$

derivative ↗ slope b/t endpts

$f(\pi/3) = 5(\pi/3) + \cot(\pi/6) = \frac{5\pi}{3} + \sqrt{3}$

$f(5\pi/3) = 5(5\pi/3) + \cot(5\pi/6) = \frac{25\pi}{3} + -\sqrt{3}$

$\frac{f(5\pi/3) - f(\pi/3)}{5\pi/3 - \pi/3} = \frac{\left(\frac{25\pi}{3} - \sqrt{3}\right) - \left(\frac{5\pi}{3} + \sqrt{3}\right)}{\frac{5\pi}{3} - \pi/3} = \frac{\frac{20\pi}{3} - 2\sqrt{3}}{4\pi/3}$

$= \frac{\frac{20\pi}{3}}{\frac{4\pi}{3}} - \frac{2\sqrt{3}}{4\pi/3} = 5 - \frac{6\sqrt{3}}{4\pi}$

$f'(x) = 5 - \csc^2(x/2) \cdot \frac{1}{2}$

$5 - \frac{1}{2}\csc^2(x/2) = 5 - \frac{6\sqrt{3}}{4\pi}$

$(-2) \cdot -\frac{1}{2}\csc^2(x/2) = -\frac{6\sqrt{3}}{4\pi} \cdot (-2)$

$\csc^2(x/2) = \frac{3\sqrt{3}}{\pi}$

$\sqrt{\sin^2(x/2)} = \sqrt{\frac{\pi}{3\sqrt{3}}}$

$\sin(x/2) = \sqrt{\frac{\pi}{3\sqrt{3}}}$

$\frac{x}{2} = \sin^{-1}\left(\pm\sqrt{\frac{\pi}{3\sqrt{3}}}\right)$

$\boxed{x = 2\sin^{-1}\left(\pm\sqrt{\frac{\pi}{3\sqrt{3}}}\right)}$

⑩

4. Given the function $y = 2\sin x + 2\cos x$ on the interval $(-\pi, \pi)$, find:

a) the intervals of direction: increasing: $(-3\pi/4, \pi/4)$ b/c $y'(x) > 0$

decreasing: $(-\pi, -3\pi/4) \cup (\pi/4, \pi)$ b/c $y'(x) < 0$

$y'(x) = 2\cos x - 2\sin x$

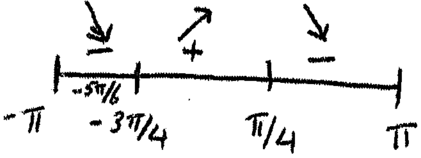
$0 = 2(\cos x - \sin x)$

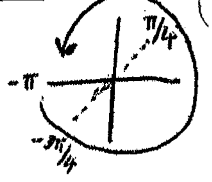
$0 = \cos x - \sin x$

$\sin x = \cos x$

$\frac{\sin x}{\cos x} = \frac{\cos x}{\cos x}$

$\tan x = 1 \qquad x = \frac{-3\pi}{4}, \pi/4$





b) the relative extrema

min: $(-3\pi/4, -2\sqrt{2})$ b/c $y'(x)$ changes from − to + at $x = -3\pi/4$

max: $(\pi/4, 2\sqrt{2})$ b/c $y'(x)$ changes from + to − at $x = \pi/4$

$f(-3\pi/4) = 2\sin(-3\pi/4) + 2\cos(-3\pi/4)$

$= -2\sqrt{2}$

c) the point(s) of inflection: POI at $(-\pi/4, 0)$ and $(3\pi/4, 0)$ b/c $f''(x)$ change signs

$y''(x) = -2\sin x - 2\cos x$

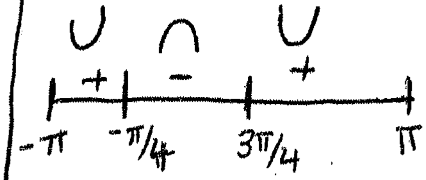
$0 = -2\sin x - 2\cos x$

$2\sin x = -2\cos x$

$\frac{2\sin x}{2\cos x} = -1$

$\tan x = -1$

$\tan x = -1 \qquad x = -\pi/4, 3\pi/4$

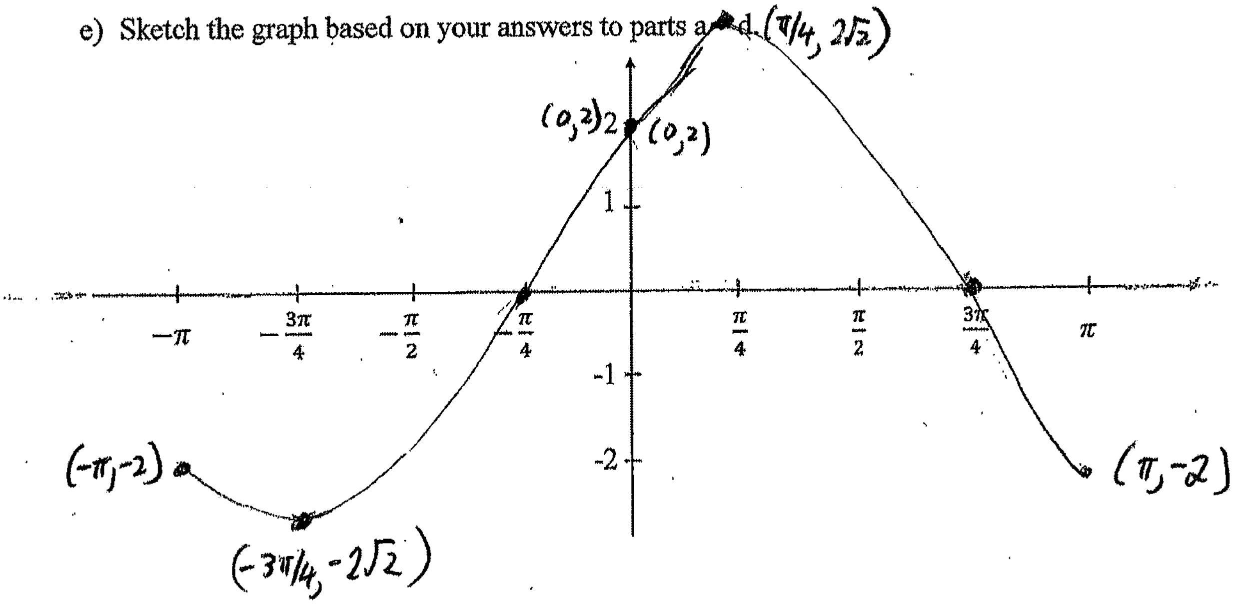


d) concavity intervals

up: $(-\pi, -\pi/4) \cup (3\pi/4, \pi)$ b/c $y''(x) > 0$

down: $(-\pi/4, 3\pi/4)$ b/c $y''(x) < 0$

e) Sketch the graph based on your answers to parts a-d.



## Curve Sketching Practice Problems

Use first derivative test & the test for concavity to find relative max, min, and POI. Sketch graph along with any vertical asymptotes (if they exist)

Pg. 216

41) $y = 2x - \tan x$

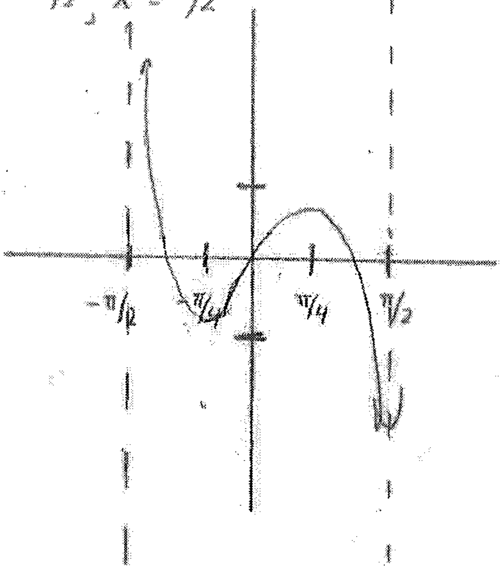
Pg. 216

42) $y = 2(x - 2) + \cot x$

/ p.216 41, 42

Given Domain: $-\pi/2 < x < \pi/2$

41) $y = 2x - \tan x$ $\longrightarrow$ V.A. $x = -\pi/2$, $x = \pi/2$

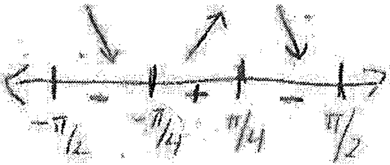


$y' = 2 - \sec^2 x$

$0 = 2 - \sec^2 x = 2 - \frac{1}{\cos^2 x}$

$2 = \sec^2 x$

$\sqrt{\frac{1}{2}} = \sqrt{\cos^2 x}$ $\cos x = \pm\frac{\sqrt{2}}{2}$ $x = -\pi/4, \pi/4$



$y'(-\pi/3) = 2 - \frac{1}{\cos^2(-\pi/3)}$

$= 2 - \frac{1}{(1/2)^2}$

$= 2 - 4 = -2$

Dec: $(-\pi/2, -\pi/4) \cup (\pi/4, \pi/2)$ b/c $f'(x) < 0$

Inc: $(-\pi/4, \pi/4)$ b/c $f'(x) > 0$

Max $(\pi/4, \underline{\pi/2 - 1})$ b/c $f'(x)$ changes from + to −

Min $(-\pi/4, \underline{-\pi/2 + 1})$ b/c $f'(x)$ changes from − to +

$f(\pi/4) = 2(\pi/4) - \tan(\pi/4)$

$f(\pi/4) = \frac{\pi}{2} - 1 \approx 0.5$

$f(-\pi/4) = 2(-\pi/4) - \tan(-\pi/4)$

$= -\pi/2 - (-1)$

$= -\pi/2 + 1 \approx -0.5$

$y'' = -2[\sec x] \cdot \sec x \tan x$ $y' = 2 - [\sec x]^2$

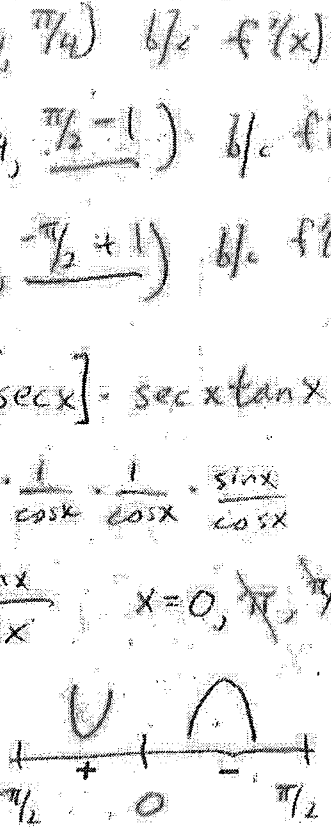
$= -2 \cdot \frac{1}{\cos x} \cdot \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x}$

$y'' = \frac{-2 \sin x}{\cos^3 x}$ $x = 0, \cancel{\pi}, \cancel{\pi/2}$

$-2\sin x = 0$ | $\cos^3 x = 0$

$\sin x = 0$ | $\cos x = 0$

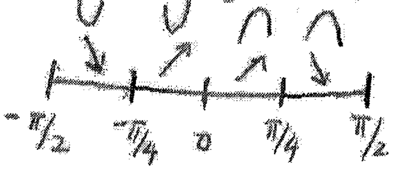
$x = 0, \pi, -\pi$ | $x = \pi/2, -\pi/2$



Concave up $(-\pi/2, 0)$ b/c $f''(x) > 0$

concave down $(0, \pi/2)$ b/c $f''(x) < 0$

POI $(0, 0)$ b/c $f''(x)$ change signs



p. 216

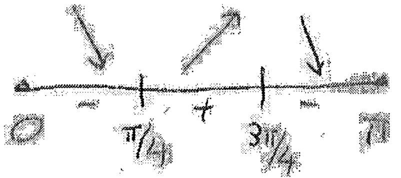
Given Domain: $0 < x < \pi$

42) $y = 2(x-2) + \cot x = 2x - 4 + \cot x \longrightarrow$ V.A. $x = 0,\ x = \pi$

$y' = 2 - \csc^2 x = 2 - \frac{1}{\sin^2 x}$

$0 = 2 - \frac{1}{\sin^2 x}$

$\frac{1}{\sin^2 x} = 2 \qquad \sin^2 x = \frac{1}{2} \qquad \sin x = \frac{\pm\sqrt{2}}{2} \qquad x = \pi/4,\ 3\pi/4$



Dec: $(0, \pi/4) \cup (3\pi/4, \pi)$ b/c $f'(x) < 0$ | $f(\pi/4) = 2(\pi/4) - 4 + \cot(\pi/4)$

Inc: $(\pi/4, 3\pi/4)$ b/c $f'(x) > 0$ | $f(\pi/4) = \pi/2 - 4 + 1$

Rel. max $(3\pi/4, \frac{3\pi}{2} - 5)$ b/c $f'(x)$ changes from + to -

Rel. min $(\pi/4, \pi/2 - 3)$ b/c $f'(x)$ changes from - to +

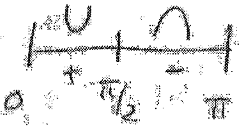
$y' = 2 - [\csc x]^2$

$y'' = 0 - 2[\csc x] \cdot (-\csc x \cot x)$

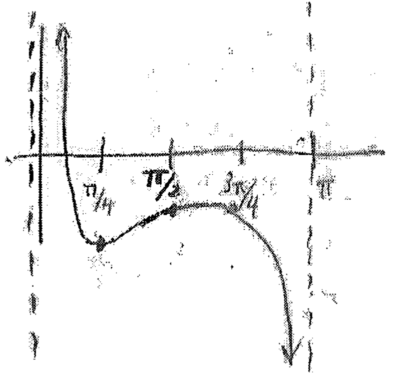
$0 = (2\csc^2 x)(\cot x)$

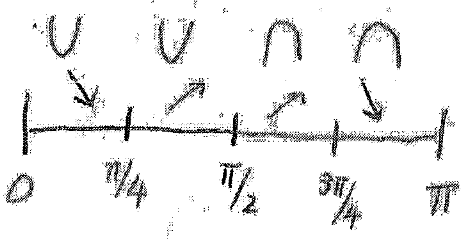
$2\csc^2 x = 0$ | $\cot x = 0$

none | $x = \pi/2$



POI $(\frac{\pi}{2}, \pi - 4)$ b/c $f''(x)$ change signs

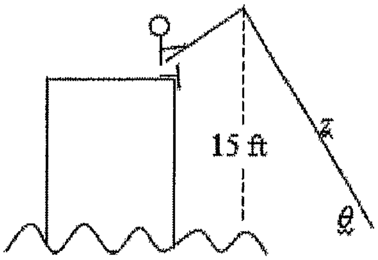




## AP Calculus Ch. 2.6 Trig Related Rates Notes

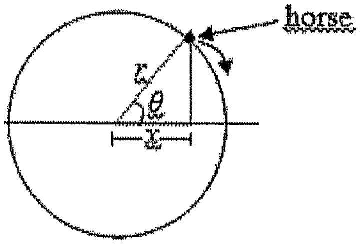
Steps for Related Rates:

**Example 1:** A fish is reeled in at a rate of 1 foot per second from a point 15 feet above the water (see figure below). At what rate is the angle between the line and the water changing when there is a total of 25 feet of line out?



**Example 2:** A carousel has a radius of 20 feet and completes one rotation every 30 seconds

a) Determine the angular velocity of the carousel in radians per second. Call this *dϑ/dt*.



b) If you were to ride on the carousel for 10 seconds, what angle would you be at compared to where you started?

c) Find *dx/dt* as a function of ϑ.

d) Find *dx/dt* when t = 10 seconds.

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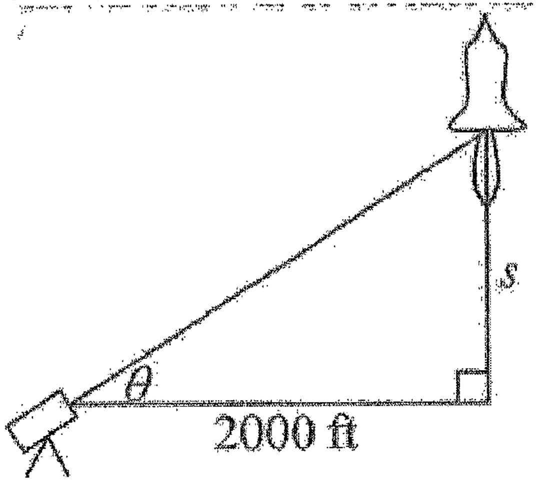
**Example 3:** A ladder, 50 ft long, is being pushed against the wall at a rate of 5 ft / sec. When the bottom of the ladder is 30 ft from the wall:

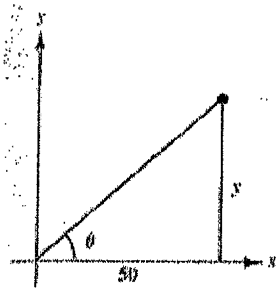
a) What is the velocity at the top of the ladder?
b) At what rate is the area of the triangle enclosed by the ladder, wall, and floor changing?
c) At what rate is the angle formed by the ladder and the floor changing at that time?

**Example 4:** A television camera at ground level is filming the lift-off of a space shuttle that is rising vertically according to $s(t) = 50t^2$, where $s$ is in feet and $t$ is in seconds. Find the rate of change in the angle of elevation of the camera shown below at 10 seconds after lift-off.

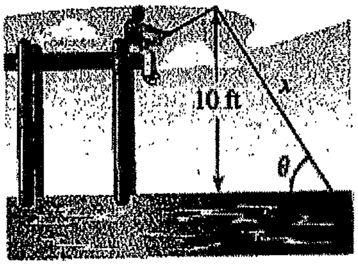


## 2.6 Related Rates (Trig) pg. 155-156 #38, 39, 40, 41

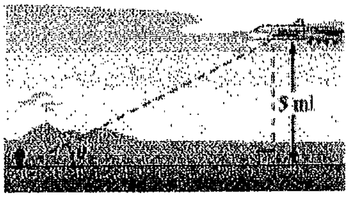
38. **Angle of Elevation** A balloon rises at a rate of 4 meters per second from a point on the ground 50 meters from an observer. Find the rate of change of the angle of elevation of the balloon from the observer when the balloon is 50 meters above the ground.



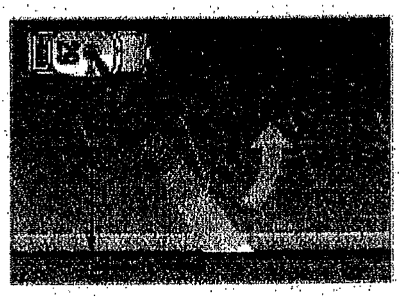
39. **Angle of Elevation** A fish is reeled in at a rate of 1 foot per second from a point 10 feet above the water (see figure). At what rate is the angle $\theta$ between the line and the water changing when there is a total of 25 feet of line from the end of the rod to the water?



**40. Angle of Elevation** An airplane flies at an altitude of 5 miles toward a point directly over an observer (see figure). The speed of the plane is 600 miles per hour. Find the rates at which the angle of elevation $\theta$ is changing when the angle is (a) $\theta = 30°$, (b) $\theta = 60°$, and (c) $\theta = 75°$.



**41. Linear vs. Angular Speed** A patrol car is parked 50 feet from a long warehouse (see figure). The revolving light on top of the car turns at a rate of 30 revolutions per minute. How fast is the light beam moving along the wall when the beam makes angles of (a) $\theta = 30°$, (b) $\theta = 60°$, and (c) $\theta = 70°$ with the perpendicular line from the light to the wall?



# 2.6 Related Rates (Trig) pg. 155-156 #38, 39, 40, 41

Key ⑲

**38. Angle of Elevation** A balloon rises at a rate of 4 meters per second from a point on the ground 50 meters from an observer. Find the rate of change of the angle of elevation of the balloon from the observer when the balloon is 50 meters above the ground.

$\frac{dy}{dt} = 4 m/s$ Find $\frac{d\theta}{dt} = \_\_$

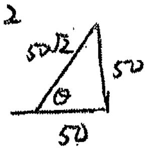
$y = 50$

$x^2 + y^2 = z^2$

$50^2 + 50^2 = z^2$

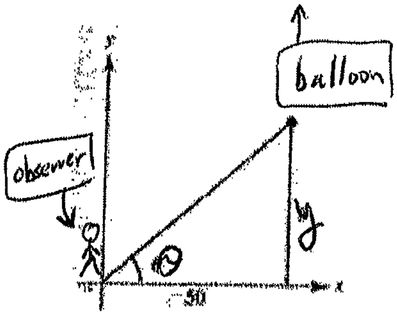
$5000 = z^2$

$z = 50\sqrt{2}$



$\sec\theta = \frac{50\sqrt{2}}{50}$

$\leftarrow \sec\theta = \sqrt{2}$



$\tan\theta = \frac{y}{50}$

$\tan\theta = \frac{1}{50}y$

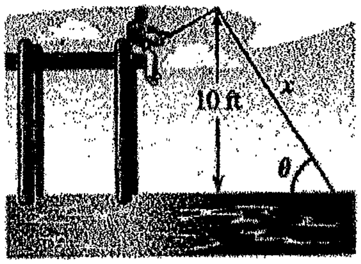
$\sec^2\theta\left(\frac{d\theta}{dt}\right) = \frac{1}{50}\left(\frac{dy}{dt}\right)$

$(\sqrt{2})^2\left(\frac{d\theta}{dt}\right) = \frac{1}{50}(4)$

$2\left(\frac{d\theta}{dt}\right) = \frac{1}{50}\cdot 4$

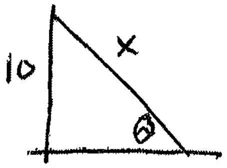
$\frac{d\theta}{dt} = \frac{1}{2}\cdot\frac{1}{50}\cdot 4 = \boxed{\frac{1}{25} \text{ rad/sec}}$

**39. Angle of Elevation** A fish is reeled in at a rate of 1 foot per second from a point 10 feet above the water (see figure). At what rate is the angle $\theta$ between the line and the water changing when there is a total of 25 feet of line from the end of the rod to the water?



$x = 25$ Find $\frac{d\theta}{dt} = \_\_$

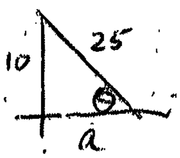
$\frac{dx}{dt} = -1$ ft/s



$\sin\theta = \frac{10}{x}$

$\sin\theta = 10x^{-1}$

$\cos\theta\left(\frac{d\theta}{dt}\right) = -10x^{-2}\left(\frac{dx}{dt}\right)$



$a^2 + 10^2 = 25^2$

$a = \sqrt{525} = 5\sqrt{21}$

$\cos\theta = \frac{5\sqrt{21}}{25} = \frac{\sqrt{21}}{5}$

$\left(\frac{\sqrt{21}}{5}\right)\left(\frac{d\theta}{dt}\right) = -10\left(\frac{1}{x^2}\right)\left(\frac{dx}{dt}\right)$

$\frac{d\theta}{dt} = \frac{5}{\sqrt{21}}(-10)\left(\frac{1}{25^2}\right)(-1)$

$= \frac{+5\cdot 10}{\sqrt{21}\cdot 25\cdot 25}$

$\boxed{\frac{d\theta}{dt} = \frac{2}{25\sqrt{21}} \approx 0.017 \text{ rad/sec}}$

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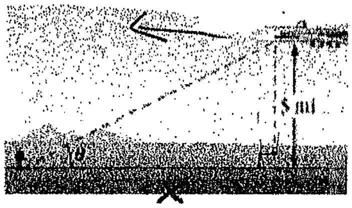
**40. Angle of Elevation** An airplane flies at an altitude of 5 miles toward a point directly over an observer (see figure). The speed of the plane is 600 miles per hour. Find the rates at which the angle of elevation $\theta$ is changing when the angle is (a) $\theta = 30°$, (b) $\theta = 60°$, and (c) $\theta = 75°$.



$\frac{dx}{dt} = -600$ mph

a) Find $\frac{d\theta}{dt} = \_$ when $\theta = 30°$

$\tan\theta = \frac{5}{x}$

$\tan\theta = 5x^{-1}$

$\sec^2\theta \frac{d\theta}{dt} = -5x^{-2}\left(\frac{dx}{dt}\right)$

$\frac{d\theta}{dt} = \cos^2\theta \cdot \frac{-5}{x^2}\left(\frac{dx}{dt}\right)$

$= (\cos 30)^2 \cdot \frac{-5}{(5\sqrt{3})^2} \cdot (-600)$

$= \left(\frac{\sqrt{3}}{2}\right)^2\left(\frac{-5}{25 \cdot 3}\right)(-600)$

a) $\frac{d\theta}{dt} = 30$ rad/hr $\cdot \frac{1hr}{60min} = \boxed{\frac{1}{2} \text{ rad/min}}$

$\frac{90rad}{hr} \cdot \frac{hr}{60min}$

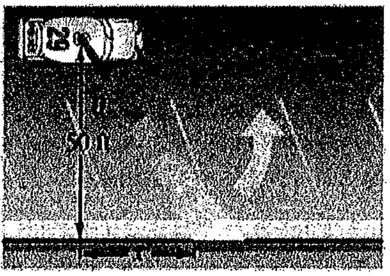
b) $\frac{d\theta}{dt} = (\cos 60)^2 \cdot \frac{-5}{(5/\sqrt{3})^2}(-600) = 90$ rad/hr $= \boxed{\frac{3}{2} \text{ rad/min}}$

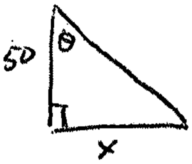
c) $\frac{d\theta}{dt} = (\cos 75)^2 \cdot \frac{-5}{5\tan 75} \cdot -600 \approx 111.96$ rad/hr $\approx \boxed{1.87 \text{ rad/min}}$

~~cos 75~~

$\tan\theta = \frac{5}{x}$

**41. Linear vs. Angular Speed** A patrol car is parked 50 feet from a long warehouse (see figure). The revolving light on top of the car turns at a rate of 30 revolutions per minute. How fast is the light beam moving along the wall when the beam makes angles of (a) $\theta = 30°$, (b) $\theta = 60°$, and (c) $\theta = 70°$ with the perpendicular line from the light to the wall?





$\tan\theta = \frac{x}{50}$

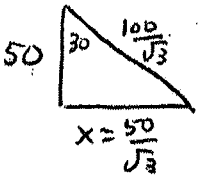
$\sec^2\theta\left(\frac{d\theta}{dt}\right) = \frac{1}{50}\left(\frac{dx}{dt}\right)$

$\frac{d\theta}{dt} = \frac{30rev}{min} \cdot \frac{2\pi rad}{revolution} \cdot \frac{1min}{60sec}$

$\frac{d\theta}{dt} = \pi$ rad/sec

a) $\theta = 30°$

Find $\frac{dx}{dt}$



$\sec\theta = \frac{\frac{100}{\sqrt{3}}}{50} = \frac{2}{\sqrt{3}}$

$\sec^2\theta = \frac{4}{3}$

$\frac{4}{3} \cdot \pi = \frac{1}{50}\frac{dx}{dt}$

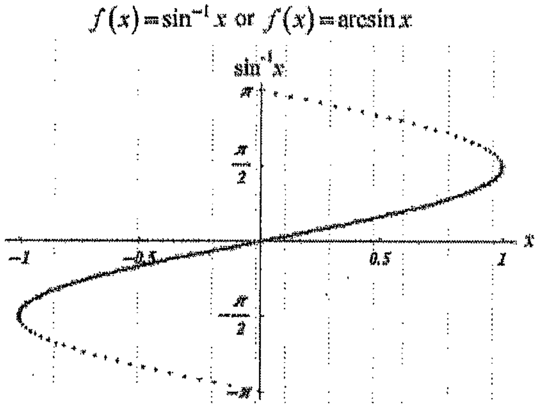
$\frac{200\pi}{3}$ ft/s $= \frac{dx}{dt}$

b) $\theta = 60°$, $\frac{dx}{dt} = 200\pi$ ft/s

c) $\theta = 70°$ $\frac{dx}{dt} = 427.43\pi$ ft/s

## **Calculus** **Ch. 5.6 Notes** **Derivatives of Inverse Trig Functions**

None of the six trig functions is one-to-one, so to create inverse functions, their domains must be restricted to a convenient one-to-one interval covering all the ratio values.



Arcsinx and sinx are inverse functions. This means that arcsin(sinx) = x and sin(arcsinx) = x.

Since $f[f^{-1}(x)] = x$, then sin(arcsin x) = x or $\sin[\sin^{-1} x] = x$

Inverse Trig Derivatives:

$$\frac{d}{dx}\arcsin u = \frac{u'}{\sqrt{1-u^2}} \qquad \frac{d}{dx}\arccos u = -\frac{u'}{\sqrt{1-u^2}}$$

$$\frac{d}{dx}\arctan u = \frac{u'}{1+u^2} \qquad \frac{d}{dx}arc\cot u = -\frac{u'}{1+u^2}$$

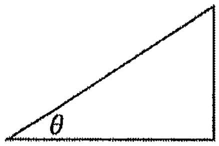
$$\frac{d}{dx}arc\sec u = \frac{u'}{|u|\sqrt{u^2-1}} \qquad \frac{d}{dx}arc\csc u = -\frac{u'}{|u|\sqrt{u^2-1}}$$

**Ex. 1**: Find $tan\left[arccos\left(\frac{\sqrt{2}}{2}\right)\right]$

This means "find the tangent of the angle whose cosine is $\frac{\sqrt{2}}{2}$"

Steps:

1. Draw the triangle and label angle $\theta$
2. Label the sides according to arccos given
3. Use Pythagorean theorem to find the third side
4. Find tan $\theta$



**Ex. 2**: Find $cos\left[arcsin\left(\frac{5}{13}\right)\right]$

Inverse Trig Derivatives:

$$\frac{d}{dx}\arcsin u = \frac{u'}{\sqrt{1-u^2}}$$

$$\frac{d}{dx}\arctan u = \frac{u'}{1+u^2}$$

$$\frac{d}{dx}\arccos u = -\frac{u'}{\sqrt{1-u^2}}$$

$$\frac{d}{dx}arc\cot u = -\frac{u'}{1+u^2}$$

$$\frac{d}{dx}arc\sec u = \frac{u'}{|u|\sqrt{u^2-1}}$$

$$\frac{d}{dx}arc\csc u = -\frac{u'}{|u|\sqrt{u^2-1}}$$

**Ex. 3**: Write the expression sec[arctan 3x] in algebraic form

**Ex. 4:** Find the derivative of y = arcsin x (using implicit method)

Steps:

1. Take the sine of both sides
2. Differentiate implicitly
3. Solve for dy/dx
4. Rewrite right side of the equation in terms of x

**Ex. 5**: Find y' for y = arctan(4x) (using implicit and derivative rule)

**Ex. 6**: Find y' for y = 2arcsec($3x^2$) (use derivative rules for ex.6 – ex.8)

**Ex. 7**: Find y' for y = xarccos(2x)

**Ex. 8**: Find y' for $\cos(\arctan x)$

# Calculus
# Ch. 5.6 Inverse Trig Derivatives
# Classwork Worksheet

**THEOREM 5.16 Derivatives of Inverse Trigonometric Functions**

Let $u$ be a differentiable function of $x$.

$$\frac{d}{dx}[\arcsin u] = \frac{u'}{\sqrt{1-u^2}} \qquad \frac{d}{dx}[\arccos u] = \frac{-u'}{\sqrt{1-u^2}}$$

$$\frac{d}{dx}[\arctan u] = \frac{u'}{1+u^2} \qquad \frac{d}{dx}[\text{arccot}\, u] = \frac{-u'}{1+u^2}$$

$$\frac{d}{dx}[\text{arcsec}\, u] = \frac{u'}{|u|\sqrt{u^2-1}} \qquad \frac{d}{dx}[\text{arccsc}\, u] = \frac{-u'}{|u|\sqrt{u^2-1}}$$

**Evaluating an Expression** In Exercises 21–24, evaluate each expression without using a calculator. (*Hint:* See Example 3.)

**21.** (a) $\sin\left(\arctan \frac{3}{4}\right)$

**22.** (a) $\tan\left(\arccos \frac{\sqrt{2}}{2}\right)$

**23.** (a) $\cot\left[\arcsin\left(-\frac{1}{2}\right)\right]$

**24.** (a) $\sec\left[\arctan\left(-\frac{3}{5}\right)\right]$

**Simplifying an Expression Using a Right Triangle** In Exercises 25–32, write the expression in algebraic form. (*Hint:* Sketch a right triangle, as demonstrated in Example 3.)

**25.** $\cos(\arcsin 2x)$

**26.** $\sec(\arctan 4x)$

**29.** $\tan\left(\text{arcsec}\, \frac{x}{3}\right)$

**31.** $\csc\left(\arctan \frac{x}{\sqrt{2}}\right)$

24

**THEOREM 5.16 Derivatives of Inverse Trigonometric Functions**

Let $u$ be a differentiable function of $x$.

$$\frac{d}{dx}[\arcsin u] = \frac{u'}{\sqrt{1-u^2}} \qquad \frac{d}{dx}[\arccos u] = \frac{-u'}{\sqrt{1-u^2}}$$

$$\frac{d}{dx}[\arctan u] = \frac{u'}{1+u^2} \qquad \frac{d}{dx}[\operatorname{arccot} u] = \frac{-u'}{1+u^2}$$

$$\frac{d}{dx}[\operatorname{arcsec} u] = \frac{u'}{|u|\sqrt{u^2-1}} \qquad \frac{d}{dx}[\operatorname{arccsc} u] = \frac{-u'}{|u|\sqrt{u^2-1}}$$

**Finding a Derivative** **In Exercises 39–58, find the derivative of the function.**

**39.** $f(x) = 2\arcsin(x-1)$

**44.** $f(x) = \arctan\sqrt{x}$

**46.** $h(x) = x^2 \arctan 5x$

**47.** $h(t) = \sin(\arccos t)$

**50.** $y = \ln(t^2+4) - \frac{1}{2}\arctan\frac{t}{2}$

Key

# Calculus
# Ch. 5.6 Inverse Trig Derivatives
# Classwork Worksheet

**THEOREM 5.16 Derivatives of Inverse Trigonometric Functions**

Let $u$ be a differentiable function of $x$.

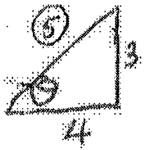
$$\frac{d}{dx}[\arcsin u] = \frac{u'}{\sqrt{1-u^2}} \qquad \frac{d}{dx}[\arccos u] = \frac{-u'}{\sqrt{1-u^2}}$$

$$\frac{d}{dx}[\arctan u] = \frac{u'}{1+u^2} \qquad \frac{d}{dx}[\operatorname{arccot} u] = \frac{-u'}{1+u^2}$$

$$\frac{d}{dx}[\operatorname{arcsec} u] = \frac{u'}{|u|\sqrt{u^2-1}} \qquad \frac{d}{dx}[\operatorname{arccsc} u] = \frac{-u'}{|u|\sqrt{u^2-1}}$$

**Evaluating an Expression** In Exercises 21–24, evaluate each expression without using a calculator. (*Hint:* See Example 3.)

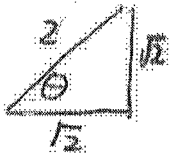
**21.** (a) $\sin\left(\arctan\frac{3}{4}\right)$



$\tan\theta = 3/4$

$= \boxed{\frac{3}{5}}$

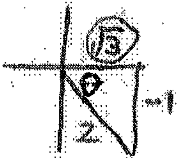
**22.** (a) $\tan\left(\arccos\frac{\sqrt{2}}{2}\right)$



$\cos\theta = \frac{\sqrt{2}}{2}$

$= \frac{\sqrt{2}}{\sqrt{2}} = \boxed{1}$

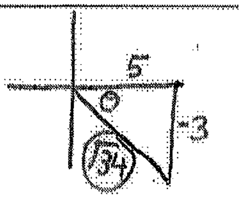
**23.** (a) $\cot\left[\arcsin\left(-\frac{1}{2}\right)\right]$



$= \frac{\sqrt{3}}{-1} = \boxed{-\sqrt{3}}$

$[-\pi/2, \pi/2]$

**24.** (a) $\sec\left[\arctan\left(-\frac{3}{5}\right)\right]$



$= \boxed{\frac{\sqrt{34}}{5}}$

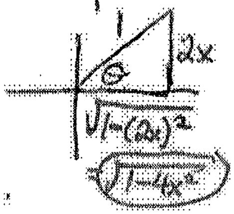
$\tan\theta = \frac{-3}{5}$

$[-\pi/2, \pi/2]$

**Simplifying an Expression Using a Right Triangle** In Exercises 25–32, write the expression in algebraic form. (*Hint:* Sketch a right triangle, as demonstrated in Example 3.)

**25.** $\cos(\arcsin 2x)$

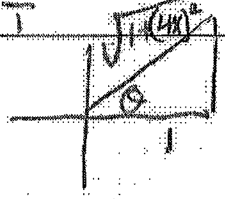
$\sin\theta = \frac{2x}{1}$



$= \frac{\sqrt{1-4x^2}}{1} = \boxed{\sqrt{1-4x^2}}$

**26.** $\sec(\arctan 4x)$

$\tan\theta = \frac{4x}{1}$

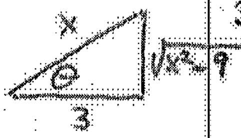


$= \frac{\sqrt{1+16x^2}}{1}$

$= \boxed{\sqrt{1+16x^2}}$

**29.** $\tan\left(\operatorname{arcsec}\frac{x}{3}\right)$

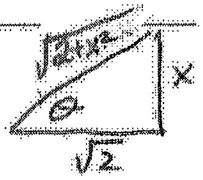
$\sec\theta = \frac{x}{3}$



$= \boxed{\frac{\sqrt{x^2-9}}{3}}$

**31.** $\csc\left(\arctan\frac{x}{\sqrt{2}}\right)$

$\tan\theta = \frac{x}{\sqrt{2}}$



$= \boxed{\frac{\sqrt{2+x^2}}{x}}$

**THEOREM 5.16 Derivatives of Inverse Trigonometric Functions**

Let $u$ be a differentiable function of $x$.

$$\frac{d}{dx}[\arcsin u] = \frac{u'}{\sqrt{1-u^2}} \qquad \frac{d}{dx}[\arccos u] = \frac{-u'}{\sqrt{1-u^2}}$$

$$\frac{d}{dx}[\arctan u] = \frac{u'}{1+u^2} \qquad \frac{d}{dx}[\operatorname{arccot} u] = \frac{-u'}{1+u^2}$$

$$\frac{d}{dx}[\operatorname{arcsec} u] = \frac{u'}{|u|\sqrt{u^2-1}} \qquad \frac{d}{dx}[\operatorname{arccsc} u] = \frac{-u'}{|u|\sqrt{u^2-1}}$$

**Finding a Derivative** In Exercises 39–58, find the derivative of the function.

**39.** $f(x) = 2\arcsin(x-1)$

$$f'(x) = 2\cdot\left(\frac{1}{\sqrt{1-(x-1)^2}}\right) = \boxed{\frac{2}{\sqrt{1-(x-1)^2}}}$$

**44.** $f(x) = \arctan\sqrt{x}$ $\quad \arctan\left(x^{1/2}\right)$

$$f'(x) = \frac{\frac{1}{2}x^{-1/2}}{1+(\sqrt{x})^2} = \frac{\frac{1}{2\sqrt{x}}}{1+x}$$

$$= \boxed{\frac{1}{2\sqrt{x}(1+x)}}$$

**46.** $h(x) = \underbrace{x^2}_{f}\,\underbrace{\arctan 5x}_{g}$

*product rule

$$h'(x) = \overbrace{2x}^{f'}\cdot\overbrace{\arctan(5x)}^{g} + \overbrace{x^2}^{f}\cdot\overbrace{\frac{5}{1+(5x)^2}}^{g'}$$

$$\boxed{h'(x) = 2x\arctan(5x) + \frac{5x^2}{1+25x^2}}$$

**47.** $h(t) = \sin(\arccos t)$

*chain rule

$$h'(t) = \underbrace{\cos(\arccos t)}_{t}\cdot\frac{-1}{\sqrt{1-t^2}}$$

out: sin( )

in: arccos t

$$h'(t) = t\cdot\left(\frac{-1}{\sqrt{1-t^2}}\right)$$

$$\boxed{h'(t) = \frac{-t}{\sqrt{1-t^2}}}$$

**50.** $y = \ln(t^2+4) - \frac{1}{2}\arctan\frac{t}{2}$ $\quad \leftarrow \frac{1}{2}\arctan\left(\frac{1}{2}t\right)$

*apply $\frac{d}{dx}\ln u = \frac{u'}{u}$

$$y' = \frac{2t}{t^2+4} - \frac{1}{2}\cdot\frac{\frac{1}{2}}{1+\left(\frac{t}{2}\right)^2}$$

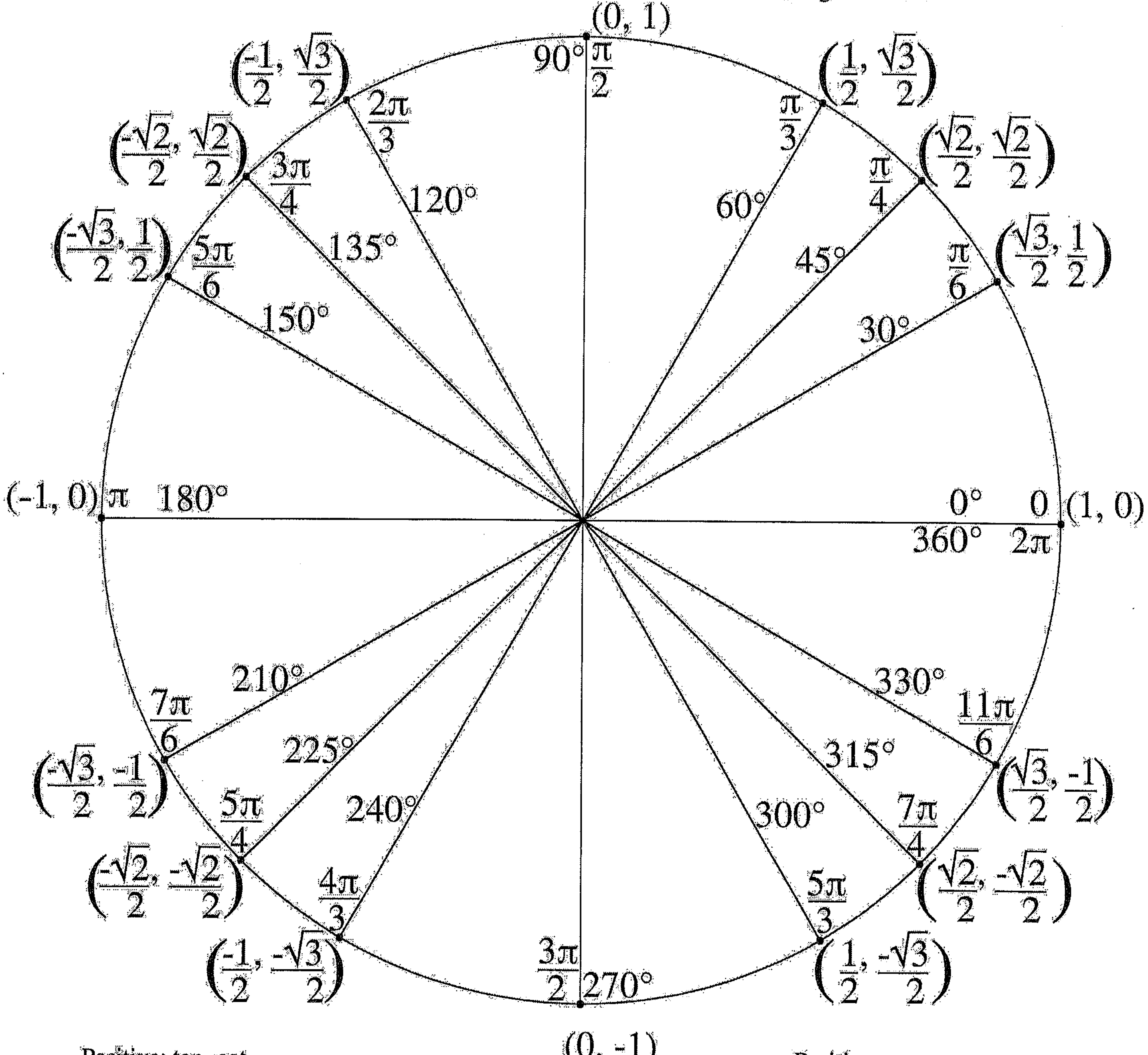
$$y' = \frac{2t}{t^2+4} - \frac{1}{4\left(1+\frac{t^2}{4}\right)}$$

$$y' = \frac{2t}{t^2+4} - \frac{1}{4+t^2} = \boxed{\frac{2t-1}{t^2+4}}$$

# The Unit Circle

Positive: sin, csc
Negative: cos, tan, sec, cot

Positive: sin, cos, tan, sec, csc, cot
Negative: none



Positive: tan, cot
Negative: sin, cos, sec, csc

Positive: cos, sec
Negative: sin, tan, csc, cot

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# The Unit Circle

**Practice filling in this unit circle until you can complete it in 5 minutes.**

Place the **degree** angle measure of each angle in the **dashed** blanks inside the circle, and the **radian** measure of each angle in the **solid** blanks inside the circle. Place the coordinates of each point in the ordered pairs outside the circle.