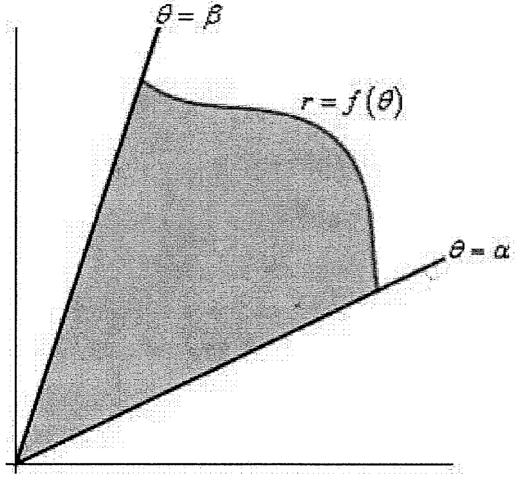
BC Calculus   Ch. 10.5 Notes   Polar Area

We are going to look at areas enclosed by polar curves, that's ***enclosed***, not ***under*** as we typically have in these problems. These problems work a little differently in polar coordinates. Here is a sketch of what the area that we'll be finding in this section looks like.



The formula for polar area is different from all previous area formulas, because it is not based on rectangles. Instead, polar area uses and infinite number of sectors to find the area. Remember that a sector is a hunk of a circle, a slice of pizza from the whole pizza.

The area of a sector of a circle is given by $A = \frac{1}{2}\theta r^2$, where θ is in radians. Our area is bounded by the radial lines from $\theta = \alpha$ to $\theta = \beta$

$$Area = \frac{1}{2}\int_{\alpha}^{\beta} r^2 d\theta$$

Example 1:
Find the area bounded by the graph of $r = 2 + 2\sin\theta$.

Example 2:
Find the area of one petal of $r = 2\sin 3\theta$.

Example 3:
Find the area of one petal of $r = 4\cos 2\theta$.

Example 4:
Find the area inside one loop of $r^2 = 4\cos 2\theta$.

Example 5:
Find the area inside $r = 3\sin\theta$ and outside $r = 2 - \sin\theta$.

Example 6:
Find the area of the common interior of $r = 3\cos\theta$ and $r = 1 + \cos\theta$.

Example 7:

A polar curve is defined by the equation $r = \theta + \sin 2\theta$ for $0 \le \theta \le \pi$.

(a) Find the area bounded by the curve and the x-axis.

(b) Find the angle θ that corresponds to the point on the curve where $x = -2$.

(c) For $\frac{\pi}{3} < \theta < \frac{2\pi}{3}$, $\frac{dr}{d\theta}$ is negative. What does this say about the graph on this interval?

(d) At what angle θ in the interval $0 \le \theta \le \frac{\pi}{2}$ is the curve farthest away from the origin. Justify your answer.

Key

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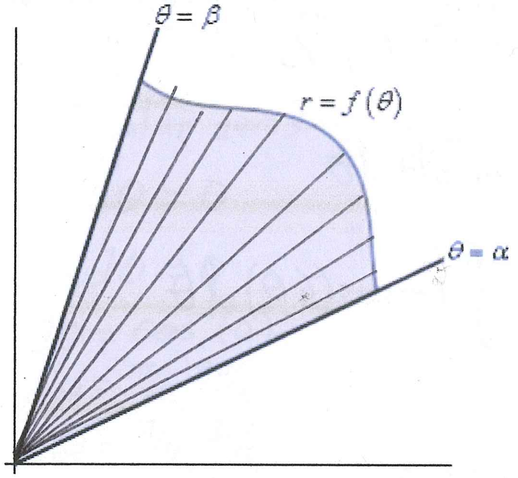

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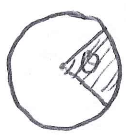


$A = \frac{1}{2} r^2 \theta$

Recall power-reducing indentities (half-angle)

1) $\sin^2 u = \frac{1}{2}(1 - \cos 2u)$

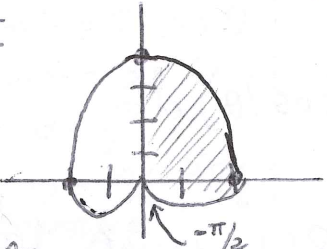
2) $\cos^2 u = \frac{1}{2}(1 + \cos 2u)$

$$Area = \frac{1}{2}\int_{\alpha}^{\beta} r^2 \, d\theta$$

$d\theta$: θ are infinitely thin.

Example 1:
Find the area bounded by the graph of $r = 2 + 2\sin\theta$. *cardioid traces out once from $[0, 2\pi]$

θ	r
0	2
$\pi/2$	4
π	2
$3\pi/2$	0
2π	2



$\frac{1}{2}\int_0^{2\pi} (2 + 2\sin\theta)^2 \, d\theta$

$\frac{1}{2}\int 4 + 8\sin\theta + 4\sin^2\theta \, d\theta = \int 4 + 8\sin\theta + 4\left(\frac{1}{2}(1 - \cos 2\theta)\right)$

$\frac{1}{2}\int 4 + 8\sin\theta + 2 - 2\cos 2\theta \, d\theta = \int 6 + 8\sin\theta - 2\cos 2\theta \, d\theta$

$\frac{1}{2}\left[6\theta - 8\cos\theta - 2\left(\frac{1}{2}\right)\sin 2\theta\right]_0^{2\pi}$

$\frac{1}{2}\left[6(2\pi) - \cancel{8(1)} - 0 - (6(0) - \cancel{8} - 0)\right]$

$\frac{1}{2}(12\pi) = \boxed{6\pi}$

or use symmetry to minimize distance b/t bounds

$2\left[\frac{1}{2}\int_{-\pi/2}^{\pi/2} (2 + 2\sin\theta)^2 \, d\theta\right]$

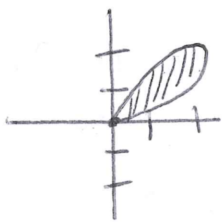
or

$2\left[\frac{1}{2}\int_{\pi/2}^{3\pi/2} (2 + 2\sin\theta)^2 \, d\theta\right]$

$$* \sin^2 u = \tfrac{1}{2}(1-\cos 2u)$$
$$\cos^2 u = \tfrac{1}{2}(1+\cos 2u)$$

Example 2:
Find the area of one petal of $r = 2\sin 3\theta$.

*rose curve, 3 petals $[0,\pi]$
*use polar zeros and symmetry to create integral bounds.



*one petal is traced out from 0 to $\pi/3$

$$A = \frac{1}{2}\int_0^{\pi/3} (2\sin 3\theta)^2\, d\theta = \frac{1}{2}\int 4\sin^2 3\theta\, d\theta$$

find polar zeros

$2\sin 3\theta = 0$

$\sin 3\theta = 0$

$3\theta = \sin^{-1}(0)$

$3\theta = 0, \pi, 2\pi, 3\pi$

$\underline{\theta = 0, \pi/3}, 2\pi/3, \pi$

$$= \frac{1}{2}\int 4\left[\tfrac{1}{2}(1-\cos 2(3\theta))\right] d\theta = \int 1-\cos(6\theta)\, d\theta$$

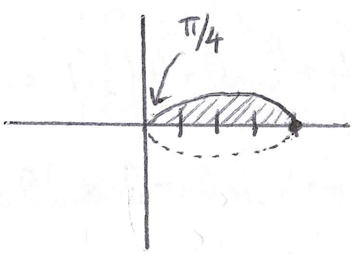
$$= \theta - \frac{1}{6}\sin(6\theta)\Big]_0^{\pi/3}$$

$$= \pi/3 - \tfrac{1}{6}\sin(2\pi) - \left(0 - \tfrac{1}{6}\sin(0)\right)$$

$$= \pi/3 - 0 - 0 = \boxed{\pi/3}$$

Example 3:
Find the area of one petal of $r = 4\cos 2\theta$.

*use polar zeros and symmetry to create integral bounds.
* rose curve, 4 petals $[0, 2\pi]$



$$A = 2\cdot\left[\frac{1}{2}\int_0^{\pi/4} (4\cos 2\theta)^2\, d\theta\right] = \int 16(\cos 2\theta)^2\, d\theta$$

$$= \int 16\cdot\left[\tfrac{1}{2}(1+\cos 2(2\theta))\right] = \int 8(1+\cos 4\theta)\, d\theta$$

$$= 8\cdot\left[\theta + \tfrac{1}{4}\sin 4\theta\right]_0^{\pi/4}$$

$$= 8\cdot\left[\pi/4 + \tfrac{1}{4}\sin(\pi) - \left(0 + \tfrac{1}{4}(0)\right)\right]$$

$$= 8(\pi/4) = \boxed{2\pi} \rightarrow \text{Area of one petal}$$

<u>polar zeros:</u>

$4\cos 2\theta = 0$

$\cos 2\theta = 0$

$2\theta = \cos^{-1}(0)$

$2\theta = \pi/2, 3\pi/2$

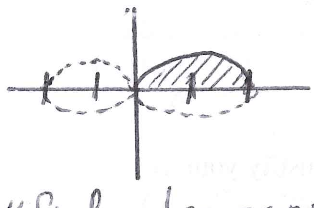
$\theta = \pi/4, \frac{3\pi}{4}$

Example 4:

*lemniscate (figure 8) → ∞

Find the area inside one loop of $\boxed{r^2 = 4\cos 2\theta}$.

this is already r^2



$$A = 2\left[\frac{1}{2}\int_0^{\pi/4} [4\cos 2\theta]\, d\theta\right]$$

*find polar zeros

$4\cos 2\theta = 0$

$\cos 2\theta = 0$

$2\theta = \cos^{-1}(0)$

$2\theta = \pi/2,\ 3\pi/2$

$\theta = \pi/4,\ 3\pi/4$

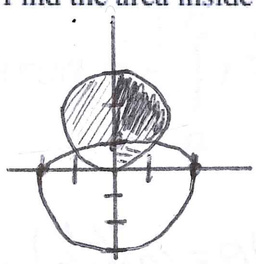
$$A = 4\left[\tfrac{1}{2}\sin 2\theta\right]_0^{\pi/4} = 4\left[\tfrac{1}{2}\sin(\pi/2) - \tfrac{1}{2}(0)\right]$$

$$A = 4\left(\tfrac{1}{2}(1)\right) = \boxed{2}$$

Example 5:

Find the area inside $r = 3\sin\theta$ (circle with diameter of 3) and outside $r = 2 - \sin\theta$ (dimpled limaçon).

θ	r
0	2
$\pi/2$	1
π	2
$3\pi/2$	3
2π	2



*use symmetry

*Find Area of half circle and subtract out the bottom portion of half circle

*Find intersection

$3\sin\theta = 2 - \sin\theta$

$4\sin\theta = 2$

$\sin\theta = \frac{1}{2}$

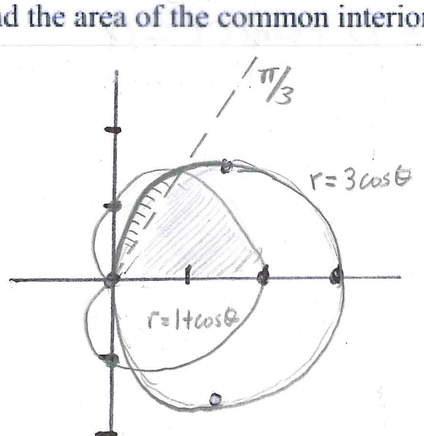
$\theta = \pi/6,\ 5\pi/6$

$$2\cdot\left[\frac{1}{2}\int_{\pi/6}^{\pi/2} [3\sin\theta]^2 - \frac{1}{2}\int_{\pi/6}^{\pi/2} [2-\sin\theta]^2\, d\theta\right]$$

$$\int_{\pi/6}^{\pi/2} (3\sin\theta)^2 - (2-\sin\theta)^2\, d\theta = \boxed{5.196}$$

Example 6:

Find the area of the common interior of $r = 3\cos\theta$ (circle) and $r = 1 + \cos\theta$ (cardioid).



*Find intersection:

$3\cos\theta = 1 + \cos\theta$

$2\cos\theta = 1$

$\cos\theta = 1/2$

$\theta = \pi/3$

$$A = 2\left[\frac{1}{2}\int_0^{\pi/3} (1+\cos\theta)^2\, d\theta + \frac{1}{2}\int_{\pi/3}^{\pi/2} (3\cos\theta)^2\, d\theta\right]$$

$$\boxed{A = 3.926}$$

*we need 2 separate integrals b/c we have separate intervals using separate curves

Example 7:

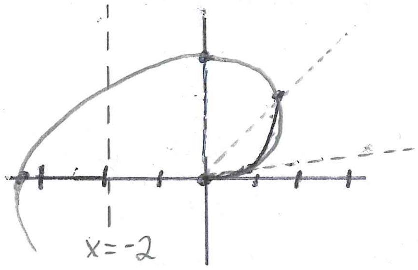
A polar curve is defined by the equation $r = \theta + \sin 2\theta$ for $0 \le \theta \le \pi$.

(a) Find the area bounded by the curve and the x-axis.

(b) Find the angle θ that corresponds to the point on the curve where $x = -2$.

(c) For $\frac{\pi}{3} < \theta < \frac{2\pi}{3}$, $\frac{dr}{d\theta}$ is negative. What does this say about the graph on this interval?

(d) At what angle θ in the interval $0 \le \theta \le \frac{\pi}{2}$ is the curve farthest away from the origin. Justify your answer.



θ	r
0	0
$\pi/12$	$\pi/12 + \frac{1}{2} \approx 0.75$
$\pi/4$	$\pi/4 + 1 \approx 1.75$
$\pi/2$	$\pi/2 + 0 \approx 1.5$
π	$\pi + 0 = 3.14$

* Graph has spiral shape

(a) $A = \frac{1}{2}\int_0^{\pi} (\theta + \sin 2\theta)^2 \, d\theta = \boxed{4.382}$

(b) $x = -2$ and $x = r\cos\theta$

* set up equation to solve for θ

$r\cos\theta = -2$ * $r = \theta + \sin 2\theta$

$(\theta + \sin 2\theta)\cos\theta = -2$

$(x + \sin 2x)\cos x + 2 = 0$

* Graph in calculator to find x-intercept (solve for θ)

$\boxed{\theta = 2.786}$

(c) If $\frac{dr}{d\theta} < 0$, then the radius is decreasing, and therefore the graph is moving towards the pole. (approaching pole in the interval $\pi/3 < \theta < 2\pi/3$)

(d) To find the point furthest away, we test critical points and endpoints (EVT) * Finding Absolute max.

$r(\theta) = \theta + \sin 2\theta$

$r'(\theta) = 1 + 2\cos 2\theta$

$0 = 1 + 2\cos 2\theta$

$\cos 2\theta = -1/2$ $2\theta = \cos^{-1}(-1/2)$

$2\theta = \frac{2\pi}{3}, 4\pi/3$

$\theta = \pi/3, \cancel{2\pi/3}$

critical point: $\theta = \pi/3$

endpoints: $\theta = 0, \pi/2$

Test points:

$r(0) = 0 + \sin(0) = 0$

$r(\pi/2) = \pi/2 + \sin\pi = \pi/2 \approx 1.571$

$r(\pi/3) = \pi/3 + \sin(2\pi/3) = \frac{\pi}{3} + \frac{\sqrt{3}}{2} \approx 1.913$

* Graph is farthest away from origin at $\theta = \pi/3$ where $r \approx 1.913$