# Calculus Section 8.3 Trig Functions with Powers

-Solve trig integrals involving powers of sine and cosine **Homework: page 540 # 5 –71 EOO**

In this section we will evaluate integrals of the form $\int \sin^m x \cos^n x dx$ and $\int \sec^m x \tan^n x dx$ where either m or n is a positive integer. In order to find these integrals, we have to write the integrand as a combination of trig functions that we can use the Power Rule on. For example, we can integrate $\int \sin^5 x \cos x dx$ by letting u = sinx and du = cosxdx.

To break up the integral into manageable parts, use the following identities:

$\sin^2 x + \cos^2 x = 1$ Pythagorean identity

$\sin^2 x = \dfrac{1-\cos 2x}{2}$ Half-angle identity for $\sin^2 x$

$\cos^2 x = \dfrac{1+\cos 2x}{2}$ Half-angle identity for $\cos^2 x$

**Guidelines for Evaluating Integrals Involving Sine and Cosine**

1) If the power of sine is odd and positive, save one sine and convert the rest to cosines.
2) If the power of cosine is odd and positive, save one cosine and convert the rest to sines.
3) If the powers of both the sine and cosine are even and nonnegative, use the half-angle identities to convert the integrand to odd powers of the cosine.

**Example) Power of Sine is Odd and Positive**

Find $\int \sin^3 x \cos^4 x dx$

**Example) Power of Cosine is Odd and Positive**

Find $\int_0^{\pi/2} \dfrac{\cos^3 x}{\sqrt{\sin x}} dx$

**Example) Power is Even and Nonnegative**

Find $\int \cos^4 x dx$

**Guidelines for Evaluating Integrals Involving Secant and Tangent** (Note: $1+\tan^2 x = \sec^2 x$)

1) If the power of secant is even and positive, save a secant-squared factor and convert the rest to tangents.
2) If the power of the tangent is odd and positive, save a secant-tangent and convert the rest to secants.
3) If there are no secants and the power of tangent is even and positive, convert a tangent-squared to a (secant-squared − 1). Expand and repeat as necessary.
4) If the integral is only secant with an odd positive power, use integration by parts.
5) If none of the first four guidelines apply, try to convert to sines and cosines.

**Example) Power of Tangent is Odd and Positive**

Find $\int \frac{\tan^3 x}{\sqrt{\sec x}} dx$

**Example) Power of Secant is Even and Positive**

Find $\int \sec^4 3x \tan^3 3x dx$

## AP Calculus BC     Ch. 8.3 Trig Functions with Powers

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3) If the powers of both the sine and cosine are even and nonnegative, use the half-angle identities to convert the integrand to odd powers of the cosine.

**Ex: Power of Sine is Odd and Positive**

Find $\int \sin^3 x \cos^4 x dx$

$$\int \sin^2 x \cdot \cos^4 x \cdot \sin x \, dx$$

$$\int (1-\cos^2 x)\cos^4 x \cdot \sin x \, dx$$

$$\int \cos^4 x - \cos^6 x \cdot \sin x \, dx$$

$u = \cos x$
$\frac{du}{dx} = -\sin x$
$dx = \frac{du}{-\sin x}$

$$\int u^4 - u^6 \cdot \cancel{\sin x} \cdot \frac{du}{-\cancel{\sin x}}$$

$$\int -u^4 + u^6 \, du$$

$$-\frac{u^5}{5} + \frac{u^7}{7} + C$$

$$\boxed{-\frac{1}{5}\cos^5 x + \frac{1}{7}\cos^7 x + C}$$

**Ex: Power of Cosine is Odd and Positive**

Find $\int_0^{\pi/2} \frac{\cos^3 x}{\sqrt{\sin x}} dx$

$$\int \frac{\cos^2 x}{(\sin x)^{1/2}} \cdot \cos x \, dx$$

$$\int \frac{(1-\sin^2 x)}{(\sin x)^{1/2}} \cdot \cos x \, dx$$

$$\int \frac{1}{(\sin x)^{1/2}} - (\sin x)^{3/2} \cdot \cos x \, dx$$

$u = \sin x$
$\frac{du}{dx} = \cos x$
$dx = \frac{du}{\cos x}$

$$\int u^{-1/2} - u^{3/2} \cdot \cancel{\cos x} \cdot \frac{du}{\cancel{\cos x}}$$

$$\frac{u^{1/2}}{1/2} - \frac{u^{5/2}}{5/2}$$

$$2(\sin x)^{1/2} - \frac{2}{5}(\sin x)^{5/2} \Big]_0^{\pi/2}$$

$$2(1) - \frac{2}{5}(1) = \frac{10}{5} - \frac{2}{5} = \boxed{\frac{8}{5}}$$

**Example) Power is Even and Nonnegative** *convert to in terms of $\cos(2x)$

Find $\int \cos^4 x dx$

*$\cos 2x = 2\cos^2 x - 1$

$\cos 2x + 1 = 2\cos^2 x$

$\frac{1}{2}(\cos 2x + 1) = \cos^2 x$

$$\int \left[\frac{1}{2}(\cos 2x + 1)\right]^2 dx$$

$$\int \frac{1}{4}(\cos^2 2x + 2\cos 2x + 1) dx$$

$$\int \frac{1}{4}\left[\frac{1}{2}[\cos 4x + 1] + 2\cos 2x + 1\right] dx$$

$$\int \frac{1}{8}\cos 4x \, dx + \int \frac{1}{8} + \frac{1}{4} \, dx + \int \frac{1}{2}\cos 2x \, dx$$

$u = 4x$

$\frac{du}{dx} = 4 \quad dx = \frac{du}{4}$

$$\boxed{\frac{1}{32}\sin 4x + \frac{3}{8}x + \frac{1}{4}\sin 2x + C}$$

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1) If the power of secant is even and positive, save a secant-squared factor and convert the rest to tangents.
2) If the power of the tangent is odd and positive, save a secant-tangent and convert the rest to secants.
3) If there are no secants and the power of tangent is even and positive, convert a tangent-squared to a (secant-squared – 1). Expand and repeat as necessary.
4) If the integral is only secant with an odd positive power, use integration by parts.
5) If none of the first four guidelines apply, try to convert to sines and cosines.

**Example) Power of Tangent is Odd and Positive**

Find $\int \frac{\tan^3 x}{\sqrt{\sec x}} dx$

$$\int \frac{\tan^2 x}{(\sec x)^{1/2}} \cdot \tan x \, dx$$

$$\int \frac{\sec^2 x - 1}{[\sec x]^{1/2}} \cdot \tan x \, dx$$

$$\int \frac{(\sec^2 x - 1) \cdot \sec x \cdot \tan x \, dx}{(\sec x)^{1/2}(\sec x)}$$

$$\int \frac{\sec^2 x - 1}{(\sec x)^{3/2}} \cdot \sec x \tan x \, dx$$

$$\int (\sec x)^{1/2} - (\sec x)^{-3/2} \cdot \sec x \tan x \, dx$$

$u = \sec x \qquad dx = \frac{du}{\sec x \tan x}$

$\frac{du}{dx} = \sec x \tan x$

$$\int u^{1/2} - u^{-3/2} \cdot \cancel{\sec x \tan x} \cdot \frac{du}{\cancel{\sec x \tan x}}$$

$$\frac{u^{3/2}}{3/2} - \frac{u^{-1/2}}{-1/2} + C$$

$$\boxed{\frac{2}{3}(\sec x)^{3/2} + 2(\sec x)^{-1/2} + C}$$

**Example) Power of Secant is Even and Positive**

Find $\int \sec^4 3x \tan^3 3x dx$ $\qquad u = \tan x \quad \frac{du}{dx} = \sec^2 x$

$$\int \sec^2(3x) \cdot \tan^3(3x) \cdot \sec^2(3x) \, dx$$

$$\int (1 + \tan^2(3x)) \cdot \tan^3(3x) \cdot \sec^2(3x) \, dx$$

$$\int \tan^3(3x) + \tan^5(3x) \cdot \sec^2(3x) \, dx$$

$u = \tan(3x)$

$\frac{du}{dx} = \sec^2(3x) \cdot 3$

$dx = \frac{du}{3\sec^2(3x)}$

$$\int u^3 + u^5 \cdot \cancel{\sec^2(3x)} \frac{du}{3\cancel{\sec^2(3x)}}$$

$$\frac{1}{3}\int u^3 + u^5 \, du$$

$$\frac{1}{3}\left(\frac{u^4}{4} + \frac{u^6}{6}\right) + C = \boxed{\frac{1}{12}(\tan^4(3x)) + \frac{1}{18}(\tan^6(3x)) + C}$$