AP Calculus BC Ch. 8.4 Trig Substitution

The objective with trig substitution is to eliminate the radical in the integrand.

Recall the Arc Trig Integral Rules:

$$\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\frac{u}{a} + C \qquad \int \frac{du}{a^2+u^2} = \frac{1}{a}\arctan\frac{u}{a} + C \qquad \int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\text{arcsec}\frac{|u|}{a} + C$$

Consider the forms of these Pythagorean identities

a) $\cos^2\theta = 1 - \sin^2\theta$ b) $\sec^2\theta = 1 + \tan^2\theta$ c) $\tan^2\theta = \sec^2\theta - 1$

Trig Substitution (a > 0)

1. For $\sqrt{a^2-u^2}$, use $\sin\theta = \frac{u}{a}$ → Therefore, $\sqrt{a^2-u^2}$ = a $\cos\theta$

2. For $\sqrt{a^2+u^2}$, use $\tan\theta = \frac{u}{a}$ → Therefore, $\sqrt{a^2+u^2}$ = a $\sec\theta$

3. For $\sqrt{u^2-a^2}$, use $\sec\theta = \frac{u}{a}$ → Therefore, $\sqrt{u^2-a^2}$ = $\pm$ a $\tan\theta$

Ex. 1: Trig Substitution involving sin θ

Find $\int \frac{dx}{x^2\sqrt{9-x^2}}$ =

8.4 Continued:

Ex. 2: Trig Substitution involving $\tan\theta$

Find $\int \frac{dx}{\sqrt{4x^2+1}}$

Ex. 3: Trig Substitution: Rational Powers

Find $\int \frac{dx}{(x^2+1)^{\frac{3}{2}}}$

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Trig Substitution (a > 0)

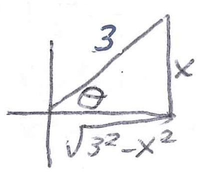
1. For $\sqrt{a^2-u^2}$, use $\sin\theta = \frac{u}{a}$ → Therefore, $\sqrt{a^2-u^2} = a\cos\theta$
2. For $\sqrt{a^2+u^2}$, use $\tan\theta = \frac{u}{a}$ → Therefore, $\sqrt{a^2+u^2} = a\sec\theta$
3. For $\sqrt{u^2-a^2}$, use $\sec\theta = \frac{u}{a}$ → Therefore, $\sqrt{u^2-a^2} = \pm a\tan\theta$

Ex. 1: Trig Substitution involving sin θ

Find $\int \frac{dx}{x^2\sqrt{9-x^2}} =$

$\sqrt{3^2-x^2}$

$\sin\theta = \frac{x}{3}$



$x = 3\sin\theta$ $x^2 = 9\sin^2\theta$

$\frac{dx}{d\theta} = 3\cos\theta$ $\cos\theta = \frac{\sqrt{9-x^2}}{3}$

$dx = 3\cos\theta\, d\theta$ $\sqrt{9-x^2} = 3\cos\theta$

$$\int \frac{3\cos\theta\, d\theta}{9\sin^2\theta \cdot 3\cos\theta}$$

$$\int \frac{d\theta}{9\sin^2\theta}$$

$$\frac{1}{9}\int \csc^2\theta\, d\theta$$

$$= \frac{-1}{9}\cot\theta + c$$

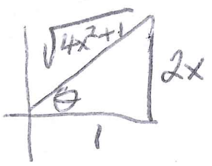
$$= \frac{-1}{9}\left(\frac{\sqrt{9-x^2}}{x}\right) + C$$

$$= \boxed{-\frac{\sqrt{9-x^2}}{9x} + C}$$

Ex. 2: Trig Substitution involving tan θ

Find $\int \frac{dx}{\sqrt{4x^2+1}}$

$\sqrt{(2x)^2+(1)^2}$



$\tan\theta = \frac{2x}{1}$

$x = \frac{1}{2}\tan\theta$

$\frac{dx}{d\theta} = \frac{1}{2}\sec^2\theta \quad dx = \frac{1}{2}\sec^2\theta\, d\theta$

$\sec\theta = \frac{\sqrt{4x^2+1}}{1}$

$\int \frac{\frac{1}{2}\sec^2\theta\, d\theta}{\sec\theta}$

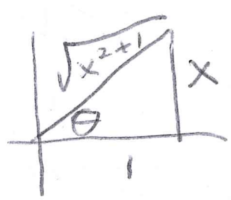
$\frac{1}{2}\int \sec\theta\, d\theta$

$= \frac{1}{2}\ln\left|\sec\theta + \tan\theta\right| + C$

$= \boxed{\frac{1}{2}\ln\left|\sqrt{4x^2+1} + 2x\right| + C}$

Ex. 3: Trig Substitution: Rational Powers

Find $\int \frac{dx}{(x^2+1)^{\frac{3}{2}}} = \int \frac{dx}{\left(\sqrt{x^2+1}\right)^3}$



$\tan\theta = \frac{u}{a}$

$\tan\theta = x$

$\sec^2\theta = \frac{dx}{d\theta}$

$dx = \sec^2\theta\, d\theta$

$\sec\theta = \sqrt{x^2+1}$

$\int \frac{\sec^2\theta\, d\theta}{(\sec\theta)^3}$

$\int \frac{1}{\sec\theta}\, d\theta$

$\int \cos\theta\, d\theta$

$= \sin\theta + C = \boxed{\frac{x}{\sqrt{x^2+1}} + C}$