Calculus Section 8.5 Partial Fractions (Linear and Quadratic Factors)

-Understand the concept of partial fraction decomposition

-Use partial faction decomposition with linear functions

The method of partial fractions is used to simplify certain integrations. Consider the integration of $\int \frac{1}{x^2-5x+6}dx$. In order to integrate this function, you would have to complete the square and use trigonometric substitution. Instead, if you notice that $\frac{1}{x^2-5x+6}=\frac{1}{x-3}-\frac{1}{x-2}$, then the integral will be two natural log functions.

Partial Fraction Steps:

1. Partial fractions can only be done if the degree of the numerator is less than the degree of the denominator
2. Factor the denominator as much as possible and get the form of the partial fraction decomposition(using A and B)
3. Multiply the equation through with the denominator
4. Set the numerators equal to each other
5. Expand the right side of the equation and group like terms (with x) together and constants together
6. Set the coefficients of terms with variables equal and coefficients of terms of constant together
7. Solve the system of equation and find the values of A and B

Alternate method: Carefully pick the x's so that the unknown constants will drop out.

Examples

1. $\int \frac{1}{x^2-5x+6}dx$

2. $\int \frac{5}{x^2-10x}dx$

3. Example: Repeated Linear Factors (include a fraction for each power of a factor)

$\int \frac{5x^2+20x+6}{x^3+2x^2+x}dx$

Use partial faction decomposition with Quadratic Functions

When using partial fractions with linear factors, making good choices for values of x can immediately yield values for your coefficients. However, with quadratic factors, a system of linear equations usually has to be solved, regardless of the choice of x.

Example: Distinct Linear and Quadratic Factors

4. Find $\int \frac{2x^3 - 4x - 8}{(x^2 - x)(x^2 + 4)} dx$

Example: Repeated Quadratic Factors

5. $\int \frac{8x^3 + 13x}{(x^2 + 2)^2} dx$

AP Calculus BC Ch. 8.5 Partial Fractions (Linear and Quadratic Functions)

-Understand the concept of partial fraction decomposition

I. Use partial faction decomposition with **linear functions** **Homework: page 559 #'s 7- 46 ETQ**

The method of partial fractions is used to simplify certain integrations by decomposing the expression into simpler rational expressions. Consider the integration of $\int \frac{1}{x^2-5x+6}dx$. In order to integrate this function, you would have to complete the square and use trigonometric substitution. Instead, if you notice that $\frac{1}{x^2-5x+6}=\frac{1}{x-3}-\frac{1}{x-2}$, then the integral will be two natural log functions.

Partial Fraction Steps:

1. Partial fractions can only be done if the **degree of the numerator is less than the degree of the denominator**
2. Factor the denominator as much as possible and get the form of the partial fraction decomposition (using A and B)
3. ~~Rewrite the right side as one fraction with common denominator~~ Multiply equation by LCD
4. Set the numerators equal to each other
5. Expand right side of equation and group like terms (with x) together and constants together
6. Set the coefficients of terms with variables equal and coefficients of terms of constants together
7. Solve the system of equation and find the values of A and B.

Alternate step: Carefully pick the *x*'s so that the unknown constants will drop out.

Class Examples

1. $\int \frac{1}{x^2-5x+6}dx = \int \frac{1}{(x-3)(x-2)} \quad = \frac{A}{x-3}+\frac{B}{x-2}$

$1 = A(x-2)+B(x-3)$

if $x=2$ $1=A(0)+B(2-3)$

$1=-B$ $\boxed{B=-1}$

if $x=3$ $1=A(3-2)+B(3-3)$

$\boxed{1=A}$

$\int \frac{1}{x-3}+\frac{-1}{x-2}dx = \boxed{\ln|x-3|-\ln|x-2|+C}$

2. $\int \frac{5}{x^2-10x}dx = \frac{A}{x}+\frac{B}{x-10}$

$x(x-10)$

$5 = A(x-10)+Bx$

If $x=10$, $5=0A+B(10)$

$\frac{1}{2}=B$

if $x=0$, $5=A(-10)$

$-\frac{1}{2}=A$

$=\int \frac{-1}{2x}+\frac{1}{2(x-10)}dx$

$-\frac{1}{2}\int\frac{1}{x}dx+\frac{1}{2}\int\frac{1}{x-10}dx$

$\boxed{-\frac{1}{2}\ln|x|+\frac{1}{2}\ln|x-10|+C}$

Example: Repeated Linear Factors (include a fraction for each power of a factor)

3. $\int \frac{5x^2+20x+6}{x^3+2x^2+x}dx$

$x(x^2+2x+1)$

$x(x+1)(x+1)$

$\int \frac{A}{x}+\frac{B}{x+1}+\frac{C}{(x+1)^2}dx$

$5x^2+20x+6 = A(x+1)^2+B(x)(x+1)+Cx$

If $x=0$, $5x^2+20x+6 = A(1)$

$0+0+6=A$ $A=6$

If $x=-1$, $5(-1)^2+20(-1)+6 = 0+0+C(-1)$

$5-20+6=-C$ $C=9$

If $x=1$, $5+20(1)+6=4A+2B+C$

$31=4(6)+2B+9$

$31=33+2B$

$-2=2B$

$-1=B$

$\int \frac{6}{x}-\frac{1}{x+1}+\frac{9}{(x+1)^2}dx$

$+9(x+1)^{-2}$

$6\ln|x|-\ln|x+1|+\frac{9(x+1)^{-1}}{-1}+C$

$\boxed{6\ln|x|-\ln|x+1|-\frac{9}{(x+1)}+C}$

II. Use partial fraction decomposition with **quadratic factors**

When using partial fractions with linear factors, making good choices for values of x can immediately yield values for your coefficients. However, with quadratic factors, a system of linear equations usually has to be solved, regardless of the choice of x.

Example: Distinct Linear and Quadratic Factors

4. Find $\int \frac{2x^3-4x-8}{(x^2-x)(x^2+4)}dx$ $\quad \frac{A}{x}+\frac{B}{x-1}+\frac{Cx+D}{x^2+4}$

$x(x-1)(x^2+4)$

$2x^3-4x-8 = A(x-1)(x^2+4)+Bx(x^2+4)+Cx(x-1)$

if $x=0$, $-8=A(-1)(4)+0+0$ $\quad -4A=-8,\ \underline{A=+2}$

if $x=1$, $2-4-8=0+B(1)(1+4)+C(0)$

$-10=5B \quad \underline{B=-2}$

let $x=-1$, $-2+4-8=2(-2)(5)-2(-1)(5)+(C(-1)+D)(-1)(-2)$

$-6=-20+10+2(-C)+2D$

$4=-2C+2D$

$\underline{2=-C+D}$

let $x=2$, $16-8-8=2(1)(8)+-2(2)(8)+(2C+D)(2)$

$0=16-32+4C+2D$, $\quad 16=4C+2D$, $\quad 8=2C+D$

$2=-C+D \qquad 2=-C+D$

$8=2C+D \qquad -8=-2C-D$

$-6=-3C$

$\underline{2=C}$

$2=-2+D$

$\underline{4=D}$

$=\int \frac{2}{x}+\frac{-2}{x-1}+\frac{2x+4}{x^2+4}dx$ $\quad u=x$, $a=2$

$\int \frac{2}{x}-\frac{2}{x-1}+\frac{2x}{x^2+4}+\frac{4}{x^2+4}dx$

$\boxed{2\ln|x|-2\ln|x-1|+\ln|x^2+4|+\frac{4}{2}\arctan\frac{x}{2}+C}$

Ex. Repeated Quadratic Factors

5. $\int \frac{8x^3+13x}{(x^2+2)^2}dx$ $\quad \frac{Ax+B}{x^2+2}+\frac{Cx+D}{(x^2+2)^2}$

$8x^3+13x=(Ax+B)(x^2+2)+Cx+D = Ax^3+2Ax+Bx^2+2B+Cx+D$

$\underline{8}x^3+13x = \underline{A}x^3+Bx^2+x(2A+C)+(2B+D)$

$\underline{8=A}$

$\underline{0=B}$

$13=2A+C$

$0=2B+D$

$13=2(8)+C$

$\underline{-3=C}$

$0=2(0)+D$

$\underline{0=D}$

$\int \frac{8x}{x^2+2}dx+\frac{-3x+0}{(x^2+2)^2}$

$u=x^2+2$, $\frac{du}{dx}=2x$, $dx=\frac{du}{2x}$

$u=x^2+2$, $\frac{du}{dx}=2x$, $dx=\frac{du}{2x}$

$\int \frac{-3x}{u^2}\cdot\frac{du}{2x} = \frac{-3}{2}\int u^{-2}du$

$=\frac{-3}{2}\left(\frac{u^{-1}}{-1}\right)$

$\boxed{4\ln|x^2+2| + \frac{3}{2(x^2+2)} + C}$