Calculus Ch. 8.7 Indeterminate Forms and L'Hôpital's Rule

-Recognize limits that produce indeterminate forms
-Apply L'Hopital's Rule to evaluate a limit

An indeterminate form of a limit occurs when you attempt to evaluate the limit and find that it produces the fraction 0/0 or ∞/∞. In order to evaluate a limit of that form, we have to simplify the limit some way. We did this in the first section on limits by factoring the polynomials and hopefully finding something that would divide out. For example:

$$\lim_{x\to 0}\frac{e^{2x}-1}{e^{x}-1}$$

However, some limits won't factor in a way that this would work. Consider the following limit:

$$\lim_{x\to 0}\frac{e^{2x}-1}{x}$$

There is nothing that can be done to re-write this function in a way that is not indeterminate.

L'Hôpital's Rule

Let *f* and *g* be functions that are differentiable on an open interval (a, b) containing *c*, except possibly at *c* itself. Assume that $g'(x) \neq 0$ for all x in (a, b), except possibly at c itself. If the limit of f(x)/g(x) as x approaches c produces an indeterminate form (0/0 or ∞/∞), then

$$\lim_{x\to c}\frac{f(x)}{g(x)}=\lim_{x\to c}\frac{f'(x)}{g'(x)}$$

provided that the limit of the derivatives exists (or is infinite). L'Hôpital's Rule also applies to one-sided limits.

Example) Indeterminate Form 0/0

Evaluate $\lim_{x\to 0}\frac{e^{2x}-1}{x}$

Example) Indeterminate Form ∞/∞

Evaluate $\lim_{x\to\infty}\frac{\ln x}{x}$

Example) Applying L'Hôpital's Rule More Than Once

Evaluate $\lim_{x \to -\infty} \frac{x^2}{e^{-x}}$

Example) Indeterminate Form $0 \cdot \infty$

Evaluate $\lim_{x \to \infty} e^{-x}\sqrt{x}$

Example) Indeterminate Form 1^∞

Evaluate $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$

Example) Indeterminate Form 0^0

Evaluate $\lim_{x \to 0^+} (\sin x)^x$

Example) Indeterminate Form $\infty - \infty$

Evaluate $\lim_{x \to 1^+} \left(\frac{1}{\ln x} - \frac{1}{x-1}\right)$

AP Calculus BC Ch. 8.7 Indeterminate Forms and L'Hôpital's Rule

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-Apply L'Hopital's Rule to evaluate a limit

Homework: page 574 #'s 5-54 D2S1

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$$\lim_{x\to 0}\frac{e^{2x}-1}{e^x-1} \quad \frac{(\cancel{e^x-1})(e^x+1)}{(\cancel{e^x-1})} \quad \lim_{x\to 0}(e^x+1) = e^0+1 = \boxed{2}$$

However, some limits won't factor in a way that this would work. Consider the following limit:

$$\lim_{x\to 0}\frac{e^{2x}-1}{x}$$

There is nothing that can be done to re-write this function in a way that is not indeterminate.

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Let *f* and *g* be functions that are differentiable on an open interval (a, b) containing *c*, except possibly at *c* itself. Assume that g'(x) ≠ 0 for all x in (a, b), except possibly at c itself. If the limit of f(x)/g(x) as x approaches c produces an indeterminate form (0/0 or ∞/∞), then

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provided that the limit of the derivatives exists (or is infinite). L'Hôpital's Rule also applies to one-sided limits.

Example) Indeterminate Form 0/0

Evaluate $\lim_{x\to 0}\frac{e^{2x}-1}{x}$ $\quad \frac{0}{0}$

$$L'H \to \lim_{x\to 0}\frac{2e^{2x}}{1} = 2e^0 = \boxed{2}$$

Example) Indeterminate Form ∞/∞

Evaluate $\lim_{x\to\infty}\frac{\ln x}{x}$ $\quad \frac{\infty}{\infty} = 0$

* or use comparative growth rate
L<R<P<E

$$L'H \to \lim_{x\to\infty}\frac{\frac{1}{x}}{1} = \frac{1}{\infty} = \boxed{0}$$

Example) Applying L'Hôpital's Rule More Than Once

Evaluate $\lim_{x\to-\infty}\frac{x^2}{e^{-x}}$ $= \frac{\infty}{\infty}$

$$L'H \to \lim_{x\to-\infty}\frac{2x}{-e^{-x}} \xrightarrow{L'H} \frac{2}{e^{-x}} =$$

$$\lim_{x\to-\infty} 2e^{x} = 2e^{-\infty} = \frac{2}{e^{\infty}} = \boxed{0}$$

Example) Indeterminate Form $0\cdot\infty$

comparative growth rate

Evaluate $\lim_{x\to\infty} e^{-x}\sqrt{x}$

$$\lim_{x\to\infty}\frac{\sqrt{x}}{e^{x}} = \boxed{0}$$

$$L'H \to \frac{\frac{1}{2}x^{-1/2}}{e^{x}} = \lim_{x\to\infty}\frac{1}{2\sqrt{x}e^{x}} = \frac{1}{\infty} = \boxed{0}$$

Example) Indeterminate Form 1^∞

Evaluate $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}$ $\quad y = \left(1+\frac{1}{x}\right)^{x}$

$$\ln y = \ln\left(1+\frac{1}{x}\right)^{x} = x\ln\left(1+\frac{1}{x}\right)$$

$$\ln y = x\ln\left(1+\frac{1}{x}\right) \qquad x^{-1}$$

$$= \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} \qquad \lim_{x\to\infty}\frac{\frac{1}{1+\frac{1}{x}}\cancel{(-x^{-2})}}{\cancel{-x^{-2}}} = 1$$

$$\ln y = 1$$

$$\log_e y = 1$$

$$\boxed{e^{1} = y} \qquad \boxed{\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x} = e}$$

Example) Indeterminate Form 0^0

Evaluate $\lim_{x\to0^+}(\sin x)^{x}$ $\quad y = (\sin x)^{x}$

$$\ln y = \ln(\sin x)^{x}$$

$$\ln y = x\ln(\sin x) \qquad \lim_{x\to0^+}\frac{\ln(\sin x)}{\frac{1}{x}} = \frac{-\infty}{\infty}$$

$$L'H \to \frac{\frac{1}{\sin x}\cos x}{-x^{-2}} \qquad \lim_{x\to0}\frac{\cot x}{-\frac{1}{x^{2}}} = \frac{-x^{2}}{\tan x}$$

$$L'H = \frac{-2x}{\sec^{2}x} \qquad \lim_{x\to0^+}\frac{0}{1} = 0$$

$$\ln y = 0$$

$$\log_e y = 0 \qquad e^{0} = y$$

$$\boxed{y = 1}$$

Example) Indeterminate Form $\infty-\infty$

Evaluate $\lim_{x\to1^+}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)$

$$\lim_{x\to1^+}\frac{x-1-\ln x}{(\ln x)(x-1)} = \frac{0}{0}$$

$$L'H \to \lim_{x\to1^+}\frac{1-\frac{1}{x}}{\left(\frac{1}{x}\right)(x-1)+\ln x(1)} = \frac{0}{0}$$

product rule

$$L'H \to \lim_{x\to1^+}\frac{x-1}{(x-1)+x\ln x} \quad \left(\frac{\cdot x}{x}\right)$$

$$L'H \to \lim_{x\to1^+}\frac{1}{1+1\ln x+x\left(\frac{1}{x}\right)} = \frac{1}{1+0+1} = \boxed{\frac{1}{2}}$$