AP Calculus Ch. 8.8 Improper Integrals

-Evaluate an improper integral that has an infinite limit of integration
-Evaluate an improper integral that has an infinite discontinuity

The definition of a definite integral $\int_a^b f(x)dx$ requires that the interval [a, b] be finite. Furthermore, the Fundamental Theorem of Calculus requires that *f* be continuous on [a, b]. In this section, we will learn a way to evaluate integrals that do not meet these requirements. We'll see problems where one or both of the limits of integration are infinity and integrate functions that have infinite discontinuities in the interval [a, b]. Integrals that possess either property are called **improper integrals**. A function is said to have an **infinite discontinuity** at c if $\lim_{x\to c} f(x)=\infty$ or $\lim_{x\to c} f(x)=-\infty$.

The trick to solving an improper integral is to consider the integration in terms of a limit. For instance,

$$\int_1^b \frac{dx}{x^2}=\left[\frac{-1}{x}\right]_1^b=\frac{-1}{b}-(-1)=1-\frac{1}{b}$$

This integral is interpreted as the area of the shaded region under the graph on the interval from [a, b]. Taking the limit as $b\to\infty$ produces

$$\int_1^\infty \frac{dx}{x^2}=\lim_{b\to\infty}\left(\int_1^b \frac{dx}{x^2}\right)=\lim_{b\to\infty}\left(1-\frac{1}{b}\right)=1$$

Ex. 1: An Improper Integral that Diverges

Evaluate $\int_1^\infty \frac{dx}{x}$

Ex. 2: An Improper Integral that Converges

Evaluate $\int_0^\infty e^{-x}dx$

Ex. 3

Evaluate $\int_0^\infty \frac{1}{x^2+1}dx$

Ex. 4: Using L'Hôpital's Rule

Evaluate $\int_1^\infty (1-x)e^{-x}dx$

Ex. 5: Infinite Upper and Lower Limits of Integration

Evaluate $\int_{-\infty}^{\infty} \frac{e^x}{1+e^{2x}}\,dx$

Now we will look at improper integrals with an infinite discontinuity at or between the limits of integration.

Ex. 6 Improper Integral with an Infinite Discontinuity

Evaluate $\int_{0}^{1} \frac{dx}{\sqrt[3]{x}}$

Ex. 7 Improper Integrals that Diverge

Evaluate $\int_{0}^{3} \frac{dx}{x^3}$

Ex. 8 Improper Integral with Interior Discontinuity

Evaluate $\int_{-1}^{2} \frac{dx}{x^3}$

Ex. 9 Doubly Improper Integral

Evaluate $\int_{0}^{\infty} \frac{dx}{\sqrt{x}(x+1)}$

AP Calculus BC Ch. 8.8 Improper Integrals

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Homework: page 585 #'s 19-45 odd

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Ex. 1: An Improper Integral that Diverges

Evaluate $\int_1^\infty \frac{dx}{x}$ $\quad \ln x \Big]_1^\infty$

$\lim_{b\to\infty} \ln x \Big]_1^b = \ln b - \ln 1$

$\lim_{b\to\infty} \ln b - \ln 1 = \boxed{\infty}$

Ex. 2: An Improper Integral that Converges

Evaluate $\int_0^\infty e^{-x}dx$ $\quad u=-x \quad dx=-du \quad \frac{du}{dx}=-1$

$-e^{-x}\Big]_0^b$

$\lim_{b\to\infty} -e^{-b}+e^{-0}$

$\lim_{b\to\infty} -\frac{1}{e^b}+1 = 0+1 = \boxed{1}$

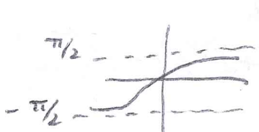
Ex. 3

Evaluate $\int_0^\infty \frac{1}{x^2+1}dx$

$\lim_{b\to\infty} \frac{1}{1}\arctan\left(\frac{x}{1}\right)\Big]_0^b$

$\lim_{b\to\infty} \arctan b - \arctan(0)$

$\frac{\pi}{2} - 0 = \boxed{\frac{\pi}{2}}$



Ex. 4: Using L'Hôpital's Rule

Evaluate $\int_1^\infty (1-x)e^{-x}dx$

	u	dv
+	1-x	e^{-x}
-	-1	$-e^{-x}$
+	0	e^{-x}

$(1-x)(-e^{-x}) + 1e^{-x}$

$-e^{-x}+xe^{-x}+e^{-x}$

$=xe^{-x}\Big]_1^b$

$\lim_{b\to\infty} be^{-b} - 1e^{-1}$

$\lim_{b\to\infty} \frac{b}{e^b} - \frac{1}{e}$

$\lim_{b\to\infty} \frac{1}{e^b} - \frac{1}{e} = 0 - \frac{1}{e}$

$= \boxed{-\frac{1}{e}}$

Ex. 5: Infinite Upper and Lower Limits of Integration

Evaluate $\int_{-\infty}^{\infty} \frac{e^x}{1+e^{2x}}\,dx$

$\int_{-\infty}^{0} \frac{e^x}{1+e^{2x}}\,dx + \int_{0}^{\infty} \frac{e^x}{1+e^{2x}}\,dx$

$u = e^x$ $\int \frac{du}{1+u^2}$

$\frac{du}{dx} = e^x$

$\arctan(e^x)\Big]_b^0$ $\arctan(e^x)\Big]_0^b = \arctan(e^b) - \arctan(1)$

$\lim_{b\to-\infty} \arctan 1 - \arctan(e^b)$

$\pi/4 - 0 + \pi/2 - \pi/4 = \boxed{\pi/2}$

Next, we now look at improper integrals with an infinite discontinuity at or between the limits of integration.

Ex. 6 Improper Integral with an Infinite Discontinuity

Evaluate $\int_0^1 \frac{dx}{\sqrt[3]{x}}$ discontinuity at $x=0$

$\int x^{-1/3}\,dx = \frac{3}{2}x^{2/3}\Big]_b^1 = \frac{3}{2}(1)^{2/3} - \frac{3}{2}(b)^{2/3}$

$\lim_{b\to 0^+} \frac{3}{2} - \frac{3}{2}(0)^{2/3} = \boxed{\frac{3}{2}}$

Ex. 7 Improper Integrals that Diverge

Evaluate $\int_0^3 \frac{dx}{x^3}$ Discontinuity at $x=0$

$\int x^{-3}\,dx$ $\frac{x^{-2}}{-2} = \frac{-1}{2x^2}\Big]_0^3$

$\lim_{x\to 0^+} \frac{-1}{2(3)^2} + \frac{1}{2(b)^2} = \frac{-1}{18} + \infty$

$= \infty$ $\boxed{\text{diverges}}$

Discontinuity at $x=0$

Ex. 8 Improper Integral with Interior Discontinuity

Evaluate $\int_{-1}^{2} \frac{dx}{x^3}$

$\int_{-1}^{0} x^{-3}\,dx + \int_{0}^{2} x^{-3}\,dx$

$\frac{x^{-2}}{-2}\Big]_{-1}^{b} =$ $\frac{x^{-2}}{-2}\Big]_b^2$

$\lim_{x\to 0^-} \frac{-1}{2(b)^2} + \frac{1}{2(1)^2}$ $\lim_{b\to 0^+} -\frac{1}{2(2)^2} + \frac{1}{2b^2}$

$\frac{-1}{2(0)^2} + \frac{1}{2}$

$\infty + \frac{1}{2}$ $-\frac{1}{8} + \infty = \infty$ (diverges)

Ex. 9 Doubly Improper Integral

Evaluate $\int_0^{\infty} \frac{dx}{\sqrt{x}(x+1)}$

$\int_0^1 \frac{dx}{\sqrt{x}(x+1)} + \int_1^{\infty} \frac{dx}{\sqrt{x}(x+1)}$

$u = x^{1/2}$

$\frac{du}{dx} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$

$dx = 2\sqrt{x}\,du$

$2\arctan\sqrt{x}\Big]_b^1$

$\lim_{b\to 0^+} 2\arctan 1 - 2\arctan b$

$2\arctan\Big]_1^b$

$\lim_{b\to\infty} 2\arctan b - 2\arctan 1$

$2(\pi/4) - 2(0) + 2(\pi/2) - 2(\pi/4) = \boxed{\pi}$