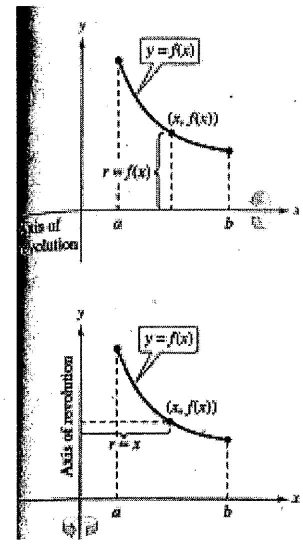
# 10.3 Additional Notes    Area of Surface of Revolution (Parametric Form)

(Recall Ch. 7.4 Surface of Revolution formula adapted from S = $2\pi r L$)




**Definition of the Area of a Surface of Revolution**

Let $y = f(x)$ have a continuous derivative on the interval $[a, b]$. The area $S$ of the surface of revolution formed by revolving the graph of $f$ about a horizontal or vertical axis is

$$S = 2\pi \int_a^b r(x)\sqrt{1 + [f'(x)]^2}\, dx \qquad y \text{ is a function of } x.$$

where $r(x)$ is the distance between the graph of $f$ and the axis of revolution. If $x = g(y)$ on the interval $[c, d]$, then the surface area is

$$S = 2\pi \int_c^d r(y)\sqrt{1 + [g'(y)]^2}\, dy \qquad x \text{ is a function of } y.$$

where $r(y)$ is the distance between the graph of $g$ and the axis of revolution.

## Parametric Form for Area of Surface of Revolution

**THEOREM 10.9 Area of a Surface of Revolution**

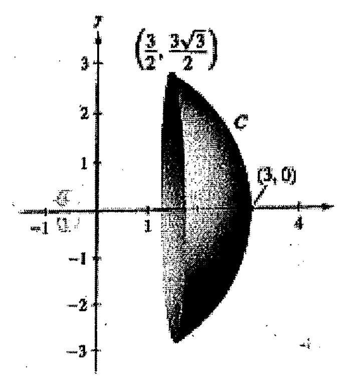
If a smooth curve $C$ given by $x = f(t)$ and $y = g(t)$ does not cross itself on an interval $a \le t \le b$, then the area $S$ of the surface of revolution formed by revolving $C$ about the coordinate axes is given by the following.

1. $S = 2\pi \displaystyle\int_a^b g(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$ Revolution about the $x$-axis: $g(t) \ge 0$

2. $S = 2\pi \displaystyle\int_a^b f(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$ Revolution about the $y$-axis: $f(t) \ge 0$

**Example 1:**

Let $C$ be the arc of the circle $x^2 + y^2 = 9$ from $(3, 0)$ to $\left(\frac{3}{2}, \frac{3\sqrt{3}}{2}\right)$




Find the area of the surface formed by revolving $C$ about the x-axis

## 10.5 Additional Notes Polar Arc Length and Polar Form for Area of Surface of Revolution

**THEOREM 10.14 Arc Length of a Polar Curve**

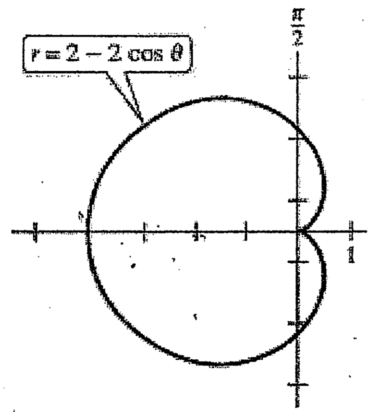
Let $f$ be a function whose derivative is continuous on an interval $\alpha \le \theta \le \beta$. The length of the graph of $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ is

$$s = \int_{\alpha}^{\beta} \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

**Example 2:**

Find the length of the arc from $\theta = 0$ to $\theta = 2\pi$ for the cardioid

$r = f(\theta) = 2 - 2\cos\theta$




## Polar Form for Area of Surface of Revolution

**THEOREM 10.15 Area of a Surface of Revolution**

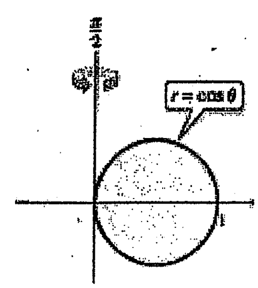
Let $f$ be a function whose derivative is continuous on an interval $\alpha \le \theta \le \beta$. The area of the surface formed by revolving the graph of $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ about the indicated line is as follows.

1. $S = 2\pi \int_{\alpha}^{\beta} f(\theta)\sin\theta\sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta$ About the polar axis

2. $S = 2\pi \int_{\alpha}^{\beta} f(\theta)\cos\theta\sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta$ About the line $\theta = \frac{\pi}{2}$

**Example 3:**

Find the area of the surface formed by revolving the circle $r = f(\theta) = \cos\theta$ about the line $\theta = \pi/2$, as shown



# 10.3 Additional Notes    Area of Surface of Revolution (Parametric Form)

Key

(Recall Ch. 7.4 Surface of Revolution formula adapted from S = $2\pi rL$)

**Definition of the Area of a Surface of Revolution**

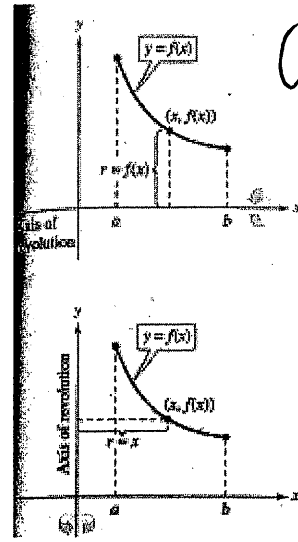
Let $y = f(x)$ have a continuous derivative on the interval $[a, b]$. The area $S$ of the surface of revolution formed by revolving the graph of $f$ about a horizontal or vertical axis is

$$S = 2\pi \int_a^b r(x)\sqrt{1 + [f'(x)]^2}\, dx \qquad y \text{ is a function of } x.$$

where $r(x)$ is the distance between the graph of $f$ and the axis of revolution. If $x = g(y)$ on the interval $[c, d]$, then the surface area is

$$S = 2\pi \int_c^d r(y)\sqrt{1 + [g'(y)]^2}\, dy \qquad x \text{ is a function of } y.$$

where $r(y)$ is the distance between the graph of $g$ and the axis of revolution.




## Parametric Form for Area of Surface of Revolution

**THEOREM 10.9 Area of a Surface of Revolution**

If a smooth curve $C$ given by $x = f(t)$ and $y = g(t)$ does not cross itself on an interval $a \le t \le b$, then the area $S$ of the surface of revolution formed by revolving $C$ about the coordinate axes is given by the following.

1. $S = 2\pi \int_a^b \underset{y(t)}{g(t)}\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$   Revolution about the $x$-axis: $g(t) \ge 0$

2. $S = 2\pi \int_a^b \underset{x(t)}{f(t)}\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$   Revolution about the $y$-axis: $f(t) \ge 0$

**Example 1:**

Let $C$ be the arc of the circle $x^2 + y^2 = 9$ from $(3, 0)$ to $\left(\frac{3}{2}, \frac{3\sqrt{3}}{2}\right)$   $r = 3$

Find the area of the surface formed by revolving $C$ about the $\boxed{\text{x-axis}}$

$x = 3\cos t$ | $y = 3\sin t$

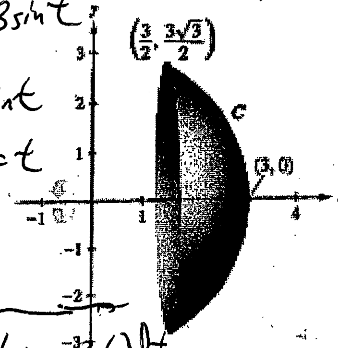
$\frac{3}{2} = 3\cos t$ | $\frac{3\sqrt{3}}{2} = 3\sin t$

$\frac{1}{2} = \cos t$ | $\frac{\sqrt{3}}{2} = \sin t$

$\cos^{-1}\left(\frac{1}{2}\right) = t$ | $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = t$

$t = \pi/3$ | $t = \pi/3$

$t = \pi/3$   $\left(\frac{3}{2}, \frac{3\sqrt{3}}{2}\right)$




$t = 0$

$x = r\cos t \qquad y = r\sin t$

$x = 3\cos t \qquad y = 3\sin t$

$\frac{dx}{dt} = -3\sin t \qquad \frac{dy}{dt} = 3\cos t$

$$S = 2\pi \int_0^{\pi/3} (3\sin t)\sqrt{(-3\sin t)^2 + (3\cos t)^2}\, dt$$

$$6\pi \int_0^{\pi/3} \sin t \sqrt{9(\underbrace{\sin^2 t + \cos^2 t}_{1})}\, dt$$

$$6\pi \int 3\sin t\, dt$$

$$6\pi \cdot -3\cos t \Big]_0^{\pi/3} = -18\pi\left(\tfrac{1}{2}\right) + 18\pi(1)$$

$$= -9\pi + 18\pi = \boxed{9\pi}$$

## 10.5 Additional Notes Polar Arc Length and Polar Form for Area of Surface of Revolution

**THEOREM 10.14 Arc Length of a Polar Curve**

Let $f$ be a function whose derivative is continuous on an interval $\alpha \le \theta \le \beta$. The length of the graph of $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ is

$$s = \int_{\alpha}^{\beta} \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta = \boxed{\int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.}$$

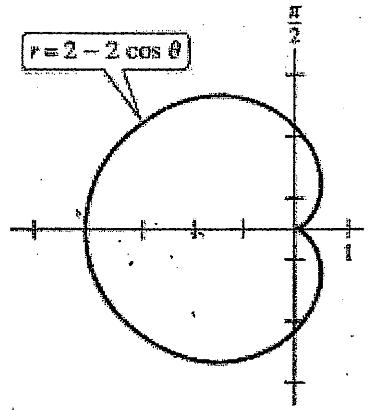
**Example 2:**

Find the length of the arc from $\theta = 0$ to $\theta = 2\pi$ for the cardioid

$r = f(\theta) = 2 - 2\cos\theta$ $\qquad r'(\theta) = 2\sin\theta$

$$S = \int_0^{2\pi} \sqrt{(2-2\cos\theta)^2 + (2\sin\theta)^2}\, d\theta = 16$$

plug in Calculator




## Polar Form for Area of Surface of Revolution

**THEOREM 10.15 Area of a Surface of Revolution**

Let $f$ be a function whose derivative is continuous on an interval $\alpha \le \theta \le \beta$. The area of the surface formed by revolving the graph of $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ about the indicated line is as follows.

1. $S = 2\pi \int_{\alpha}^{\beta} \underbrace{f(\theta)}_{r} \sin\theta \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta$ About the polar axis

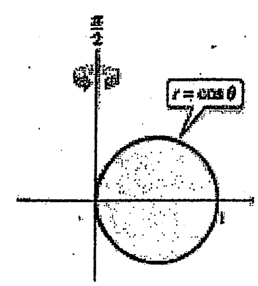
2. $S = 2\pi \int_{\alpha}^{\beta} \underbrace{f(\theta)}_{r} \cos\theta \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta$ About the line $\theta = \frac{\pi}{2}$

* circle traces out from $0$ to $\pi$

**Example 3:**

Find the area of the surface formed by revolving the circle $r = f(\theta) = \cos\theta$ about the line $\theta = \pi/2$, as shown

$r'(\theta) = -\sin\theta$




$$S = 2\pi \int_0^{\pi} \cos\theta \cdot \cos\theta \sqrt{\cos^2\theta + (-\sin\theta)^2}\, d\theta$$

$\cos 2\theta = 2\cos^2\theta - 1$

$\cos^2\theta = \frac{1}{2}(\cos 2\theta + 1)$

$$2\pi \int_0^{\pi} \cos^2\theta \cdot \sqrt{1}\, d\theta$$

$$\cancel{2}\pi \int_0^{\pi} \frac{1}{\cancel{2}}(1 + \cos 2\theta)\, d\theta$$

$$\pi \cdot \left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi}$$

$$\pi^2 + 0 - 0$$

$$\boxed{\pi^2}$$