The discussion in Example 8 forms the basis of the definition of a limit. We state the definition here but postpone the details until Section 1.6. It is customary to use the Greek letters ε (epsilon) and δ (delta) in the definition, so we call it the *ε-δ definition of a limit.*

DEFINITION ε-δ Definition of a Limit

Suppose f is a function defined everywhere in an open interval containing c, except possibly at c. Then the **limit of the function f as x approaches c** is the number L, written

$$\boxed{\lim_{x\to c} f(x) = L}$$

if, given any number $\varepsilon > 0$, there is a number $\delta > 0$ so that

$$\boxed{\text{whenever } 0 < |x - c| < \delta \quad \text{then } |f(x) - L| < \varepsilon}$$

Notice in the definition that f is defined everywhere in an open interval containing c except possibly at c. If f is defined at c and there is an open interval containing c that contains no other numbers in the domain of f, then $\lim_{x\to c} f(x)$ does not exist.

1.1 Assess Your Understanding

Concepts and Vocabulary

PAGE 79 1. *Multiple Choice* The limit as x approaches c of a function f is written symbolically as [(a) $\lim f(x)$ (b) $\lim_{c\to x} f(x)$ (c) $\lim_{x\to c} f(x)$]

PAGE 79 2. *True or False* The tangent line to the graph of f at a point $P = (c, f(c))$ is the limiting position of the secant lines passing through P and a point $(x, f(x))$, $x \neq c$, as x moves closer to c.

3. *True or False* If f is not defined at $x = c$, then $\lim_{x\to c} f(x)$ does not exist.

4. *True or False* The limit L of a function $y = f(x)$ as x approaches the number c depends on the value of f at c.

5. *True or False* If $f(c)$ is defined, this suggests that $\lim_{x\to c} f(x)$ exists.

6. *True or False* The limit of a function $y = f(x)$ as x approaches a number c equals L if at least one of the one-sided limits as x approaches c equals L.

Skill Building

In Problems 7–12, complete each table and investigate the limit.

7. $\lim_{x\to 1} 2x$

	x approaches 1 from the left →				← x approaches 1 from the right		
x	0.9	0.99	0.999 →	1	← 1.001	1.01	1.1
$f(x) = 2x$							

8. $\lim_{x\to 2}(x+3)$

	x approaches 2 from the left →				← x approaches 2 from the right		
x	1.9	1.99	1.999 →	2	← 2.001	2.01	2.1
$f(x) = x+3$							

PAGE 80 9. $\lim_{x\to 0}(x^2+2)$

	x approaches 0 from the left →				← x approaches 0 from the right		
x	−0.1	−0.01	−0.001 →	0	← 0.001	0.01	0.1
$f(x) = x^2+2$							

10. $\lim_{x\to -1}(x^2-2)$

	x approaches −1 from the left →				← x approaches −1 from the right		
x	−1.1	−1.01	−1.001 →	−1	← −0.999	−0.99	−0.9
$f(x) = x^2-2$							

11. $\lim_{x\to -3}\dfrac{x^2-9}{x+3}$

	x approaches −3 from the left →				← x approaches −3 from the right		
x	−3.5	−3.1	−3.01 →	−3	← −2.99	−2.9	−2.5
$f(x) = \dfrac{x^2-9}{x+3}$							

12. $\lim_{x\to -1}\dfrac{x^3+1}{x+1}$

	x approaches −1 from the left →				← x approaches −1 from the right		
x	−1.1	−1.01	−1.001 →	−1	← −0.999	−0.99	−0.9
$f(x) = \dfrac{x^3+1}{x+1}$							

In Problems 13–16, use technology to complete the table and investigate the limit.

13. $\lim_{x\to 0} \frac{2-2e^x}{x}$

	x approaches 0 from the left →				← x approaches 0 from the right		
x	−0.2	−0.1	−0.01	→ 0 ←	0.01	0.1	0.2
$f(x)=\frac{2-2e^x}{x}$							

14. $\lim_{x\to 1} \frac{\ln x}{x-1}$

	x approaches 1 from the left →				← x approaches 1 from the right		
x	0.9	0.99	0.999	→ 1 ←	1.001	1.01	1.1
$f(x)=\frac{\ln x}{x-1}$							

15. $\lim_{x\to 0} \frac{1-\cos x}{x}$, where x is measured in radians

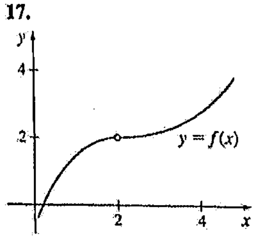
	x approaches 0 from the left →				← x approaches 0 from the right		
x (in radians)	−0.2	−0.1	−0.01	→ 0 ←	0.01	0.1	0.2
$f(x)=\frac{1-\cos x}{x}$							

16. $\lim_{x\to 0} \frac{\sin x}{1+\tan x}$, where x is measured in radians

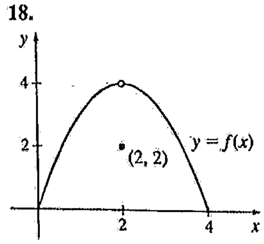
	x approaches 0 from the left →				← x approaches 0 from the right		
x (in radians)	−0.2	−0.1	−0.01	→ 0 ←	0.01	0.1	0.2
$f(x)=\frac{\sin x}{1+\tan x}$							

In Problems 17–20, use the graph to investigate (a) $\lim_{x\to 2^-} f(x)$, *(b)* $\lim_{x\to 2^+} f(x)$, *(c)* $\lim_{x\to 2} f(x)$.

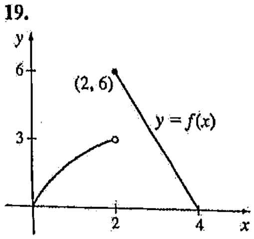
17.



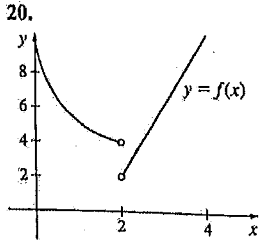
18.



19.

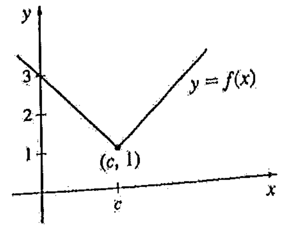


20.

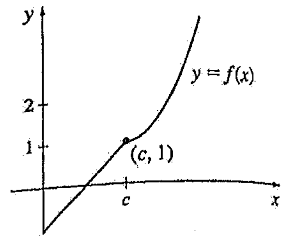


In Problems 21–28, use the graph to investigate $\lim_{x\to c} f(x)$. *If the limit does not exist, explain why.*

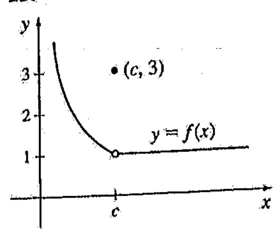
21.



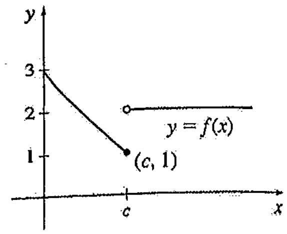
22.



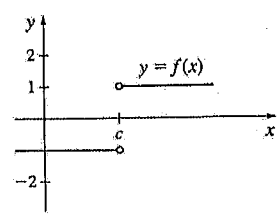
23.



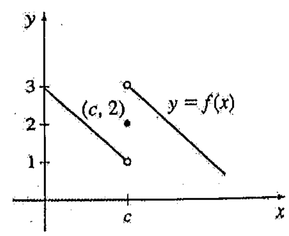
24.



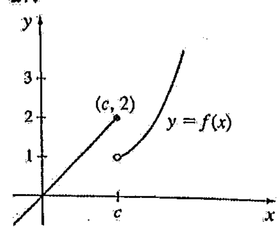
25.



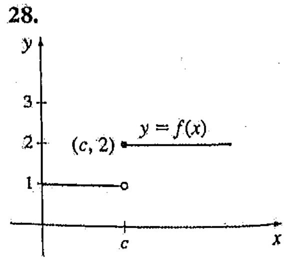
26.



27.



28.



In Problems 29–36, use a graph to investigate $\lim_{x\to c} f(x)$ *at the number c.*

29. $f(x)=\begin{cases} 2x+5 & \text{if } x\le 2 \\ 4x+1 & \text{if } x>2 \end{cases}$ at $c=2$

30. $f(x)=\begin{cases} 2x+1 & \text{if } x\le 0 \\ 2x & \text{if } x>0 \end{cases}$ at $c=0$

31. $f(x)=\begin{cases} 3x-1 & \text{if } x<1 \\ 4 & \text{if } x=1 \\ 4x & \text{if } x>1 \end{cases}$ at $c=1$

32. $f(x)=\begin{cases} x+2 & \text{if } x<2 \\ 4 & \text{if } x=2 \\ x^2 & \text{if } x>2 \end{cases}$ at $c=2$

33. $f(x)=\begin{cases} 2x^2 & \text{if } x<1 \\ 3x^2-1 & \text{if } x>1 \end{cases}$ at $c=1$

34. $f(x)=\begin{cases} x^3 & \text{if } x<-1 \\ x^2-1 & \text{if } x>-1 \end{cases}$ at $c=-1$

35. $f(x)=\begin{cases} x^2 & \text{if } x\le 0 \\ 2x+1 & \text{if } x>0 \end{cases}$ at $c=0$

36. $f(x)=\begin{cases} x^2 & \text{if } x<1 \\ 2 & \text{if } x=1 \\ -3x+2 & \text{if } x>1 \end{cases}$ at $c=1$

Applications and Extensions

In Problems 37–40, sketch a graph of a function with the given properties. Answers will vary.

37. $\lim_{x\to 2} f(x)=3$; $\lim_{x\to 3^-} f(x)=3$; $\lim_{x\to 3^+} f(x)=1$; $f(2)=3$; $f(3)=1$

38. $\lim_{x\to -1} f(x)=0$; $\lim_{x\to 2^-} f(x)=-2$; $\lim_{x\to 2^+} f(x)=-2$; $f(-1)$ is not defined; $f(2)=-2$

39. $\lim_{x\to 1} f(x)=4$; $\lim_{x\to 0^-} f(x)=-1$; $\lim_{x\to 0^+} f(x)=0$; $f(0)=-1$; $f(1)=2$

40. $\lim_{x\to 2} f(x)=2$; $\lim_{x\to -1} f(x)=0$; $\lim_{x\to 1} f(x)=1$; $f(-1)=1$; $f(2)=3$

In Problems 41–50, use either a graph or a table to investigate each limit.

41. $\lim_{x\to 5^+} \dfrac{|x-5|}{x-5}$

42. $\lim_{x\to 5^-} \dfrac{|x-5|}{x-5}$

43. $\lim_{x\to \left(\frac{1}{2}\right)^-} \lfloor 2x \rfloor$

44. $\lim_{x\to \left(\frac{1}{2}\right)^+} \lfloor 2x \rfloor$

45. $\lim_{x\to \left(\frac{2}{3}\right)^-} \lfloor 2x \rfloor$

46. $\lim_{x\to \left(\frac{2}{3}\right)^+} \lfloor 2x \rfloor$

47. $\lim_{x\to 2^+} \sqrt{|x|-x}$

48. $\lim_{x\to 2^-} \sqrt{|x|-x}$

49. $\lim_{x\to 2^+} \sqrt[3]{\lfloor x \rfloor - x}$

50. $\lim_{x\to 2^-} \sqrt[3]{\lfloor x \rfloor - x}$

51. **Slope of a Tangent Line** For $f(x)=3x^2$:
 (a) Find the slope of the secant line containing the points (2, 12) and (3, 27).
 (b) Find the slope of the secant line containing the points (2, 12) and $(x, f(x))$, $x\neq 2$.
 (c) Create a table to investigate the slope of the tangent line to the graph of f at 2 using the result from (b).
 (d) On the same set of axes, graph f, the tangent line to the graph of f at the point (2, 12), and the secant line from (a).

52. **Slope of a Tangent Line** For $f(x)=x^3$:
 (a) Find the slope of the secant line containing the points (2, 8) and (3, 27).
 (b) Find the slope of the secant line containing the points (2, 8) and $(x, f(x))$, $x\neq 2$.
 (c) Create a table to investigate the slope of the tangent line to the graph of f at 2 using the result from (b).
 (d) On the same set of axes, graph f, the tangent line to the graph of f at the point (2, 8), and the secant line from (a).

53. **Slope of a Tangent Line** For $f(x)=\frac{1}{2}x^2-1$:
 (a) Find the slope m_{sec} of the secant line containing the points $P=(2, f(2))$ and $Q=(2+h, f(2+h))$.
 (b) Use the result from (a) to complete the following table:

h	−0.5	−0.1	−0.001	0.001	0.1	0.5
m_{sec}						

 (c) Investigate the limit of the slope of the secant line found in (a) as $h\to 0$.
 (d) What is the slope of the tangent line to the graph of f at the point $P=(2, f(2))$?
 (e) On the same set of axes, graph f and the tangent line to f at $P=(2, f(2))$.

54. **Slope of a Tangent Line** For $f(x)=x^2-1$:
 (a) Find the slope m_{sec} of the secant line containing the points $P=(-1, f(-1))$ and $Q=(-1+h, f(-1+h))$.
 (b) Use the result from (a) to complete the following table:

h	−0.1	−0.01	−0.001	−0.0001	0.0001	0.001	0.01	0.1
m_{sec}								

 (c) Investigate the limit of the slope of the secant line found in (a) as $h\to 0$.
 (d) What is the slope of the tangent line to the graph of f at the point $P=(-1, f(-1))$?
 (e) On the same set of axes, graph f and the tangent line to f at $P=(-1, f(-1))$.

PAGE 84 55. (a) Investigate $\lim_{x\to 0} \cos\dfrac{\pi}{x}$ by using a table and evaluating the function $f(x)=\cos\dfrac{\pi}{x}$ at $x=-\frac{1}{2}, -\frac{1}{4}, -\frac{1}{8}, -\frac{1}{10}, -\frac{1}{12}, \ldots, \frac{1}{12}, \frac{1}{10}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}$.
 (b) Investigate $\lim_{x\to 0} \cos\dfrac{\pi}{x}$ by using a table and evaluating the function $f(x)=\cos\dfrac{\pi}{x}$ at $x=-1, -\frac{1}{3}, -\frac{1}{5}, -\frac{1}{7}, -\frac{1}{9}, \ldots, \frac{1}{9}, \frac{1}{7}, \frac{1}{5}, \frac{1}{3}, 1$.
 (c) Compare the results from (a) and (b). What do you conclude about the limit? Why do you think this happens? What is your view about using a table to draw a conclusion about limits?
 (d) Use technology to graph f. Begin with the x-window $[-2\pi, 2\pi]$ and the y-window $[-1, 1]$. If you were finding $\lim_{x\to 0} f(x)$ using a graph, what would you conclude? Zoom in on the graph. Describe what you see. *Hint*: Be sure your calculator is set to the radian mode.

56. (a) Investigate $\lim_{x\to 0} \cos\dfrac{\pi}{x^2}$ by using a table and evaluating the function $f(x)=\cos\dfrac{\pi}{x^2}$ at $x=-0.1, -0.01, -0.001, -0.0001, 0.0001, 0.001, 0.01, 0.1$.

(b) Investigate $\lim_{x \to 0} \cos \frac{\pi}{x^2}$ by using a table and evaluating the function $f(x) = \cos \frac{\pi}{x^2}$ at

$$x = -\frac{2}{3}, -\frac{2}{5}, -\frac{2}{7}, -\frac{2}{9}, \ldots, \frac{2}{9}, \frac{2}{7}, \frac{2}{5}, \frac{2}{3}.$$

(c) Compare the results from (a) and (b). What do you conclude about the limit? Why do you think this happens? What is your view about using a table to draw a conclusion about limits?

(d) Use technology to graph f. Begin with the x-window $[-2\pi, 2\pi]$ and the y-window $[-1, 1]$. If you were finding $\lim_{x \to 0} f(x)$ using a graph, what would you conclude? Zoom in on the graph. Describe what you see. *Hint*: Be sure your calculator is set to the radian mode.

PAGE 84 **57.** (a) Use a table to investigate $\lim_{x \to 2} \frac{x-8}{2}$.

(b) How close must x be to 2, so that $f(x)$ is within 0.1 of the limit?

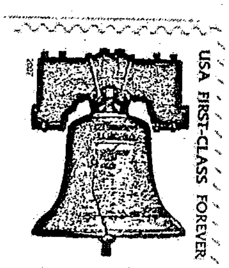
(c) How close must x be to 2, so that $f(x)$ is within 0.01 of the limit?

58. (a) Use a table to investigate $\lim_{x \to 2}(5 - 2x)$.

(b) How close must x be to 2, so that $f(x)$ is within 0.1 of the limit?

(c) How close must x be to 2, so that $f(x)$ is within 0.01 of the limit?

59. First-Class Mail As of January 2019, the U.S. Postal Service charged \$0.55 postage for first-class letters weighing up to and including 1 ounce, plus a flat fee of \$0.15 for each additional or partial ounce up to and including 3.5 ounces. First-class letter rates do not apply to letters weighing more than 3.5 ounces.



Kathryn Sidenstricker/Dreamstime.com

Source: U.S. Postal Service Notice 123.

(a) Find a function C that models the first-class postage charged, in dollars, for a letter weighing w ounces. Assume $w > 0$.

(b) What is the domain of C?

(c) Graph the function C.

(d) Use the graph to investigate $\lim_{w \to 2^-} C(w)$ and $\lim_{w \to 2^+} C(w)$. Do these suggest that $\lim_{w \to 2} C(w)$ exists?

(e) Use the graph to investigate $\lim_{w \to 0^+} C(w)$.

(f) Use the graph to investigate $\lim_{w \to 3.5^-} C(w)$.

60. First-Class Mail As of January 2019, the U.S. Postal Service charged \$1.00 postage for first-class large envelope weighing up to and including 1 ounce, plus a flat fee of \$0.15 for each additional or partial ounce up to and including 13 ounces. First-class rates do not apply to large envelopes weighing more than 13 ounces.

Source: U.S. Postal Service Notice 123.

(a) Find a function C that models the first-class postage charged, in dollars, for a large envelope weighing w ounces. Assume $w > 0$.

(b) What is the domain of C?

(c) Graph the function C.

(d) Use the graph to investigate $\lim_{w \to 1^-} C(w)$ and $\lim_{w \to 1^+} C(w)$. Do these suggest that $\lim_{w \to 1} C(w)$ exists?

(e) Use the graph to investigate $\lim_{w \to 12^-} C(w)$ and $\lim_{w \to 12^+} C(w)$. Do these suggest that $\lim_{w \to 12} C(w)$ exists?

(f) Use the graph to investigate $\lim_{w \to 0^+} C(w)$.

(g) Use the graph to investigate $\lim_{w \to 13^-} C(w)$.

61. Correlating Student Success to Study Time Professor Smith claims that a student's final exam score is a function of the time t (in hours) that the student studies. He claims that the closer to seven hours one studies, the closer to 100% the student scores on the final. He claims that studying significantly less than seven hours may cause one to be underprepared for the test, while studying significantly more than seven hours may cause "burnout."

(a) Write Professor Smith's claim symbolically as a limit.

(b) Write Professor Smith's claim using the ε-δ definition of limit.

Source: Submitted by the students of Millikin University.

62. The definition of the slope of the tangent line to the graph of $y = f(x)$ at the point $(c, f(c))$ is $m_{\tan} = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$.

Another way to express this slope is to define a new variable $h = x - c$. Rewrite the slope of the tangent line $m_{\tan}$ using h and c.

63. If $f(2) = 6$, can you conclude anything about $\lim_{x \to 2} f(x)$? Explain your reasoning.

64. If $\lim_{x \to 2} f(x) = 6$, can you conclude anything about $f(2)$? Explain your reasoning.

65. The graph of $f(x) = \frac{x-3}{3-x}$ is a straight line with a point punched out.

(a) What straight line and what point?

(b) Use the graph of f to investigate the one-sided limits of f as x approaches 3.

(c) Does the graph suggest that $\lim_{x \to 3} f(x)$ exists? If so, what is it?

66. (a) Use a table to investigate $\lim_{x \to 0}(1 + x)^{1/x}$.

(b) Use graphing technology to graph $g(x) = (1 + x)^{1/x}$.

(c) What do (a) and (b) suggest about $\lim_{x \to 0}(1 + x)^{1/x}$?

CAS (d) Find $\lim_{x \to 0}(1 + x)^{1/x}$.

Challenge Problems

For Problems 67–70, investigate each of the following limits.

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is an integer} \\ 0 & \text{if } x \text{ is not an integer} \end{cases}$$

67. $\lim_{x \to 2} f(x)$ **68.** $\lim_{x \to 1/2} f(x)$ **69.** $\lim_{x \to 3} f(x)$ **70.** $\lim_{x \to 0} f(x)$

Preparing for the AP® Exam

AP® Practice Problems

PAGE 83 1. To investigate $\lim_{x\to 1^-} \sqrt{1-x}$ using a table, evaluate $f(x)=\sqrt{1-x}$, by choosing

(A) numbers close to 0, some slightly less than 0 and some slightly greater than 0.

(B) only numbers slightly less than 1.

(C) only numbers slightly greater than 1.

(D) numbers close to 1, some slightly less than 1 and some slightly greater than 1.

PAGE 81 2. $\lim_{x\to -2^+} x^3 = -8$ is called a

(A) lower limit (B) negative limit

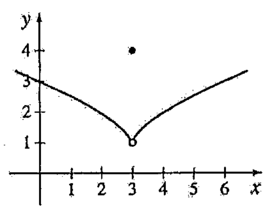
(C) positive limit (D) right-hand limit

PAGE 79 3. "The limit as x approaches 0 of the function $f(x) = \cos x$ is equal to the number 1," is written symbolically as

(A) $\lim_{\cos x\to 0} \cos x = 1$ (B) $\lim_{x\to \cos x} \cos x = 1$

(C) $\lim_{x\to 0} \cos x = 1$ (D) $\lim_{x\to 1} \cos x = 0$

PAGE 82 4. The graph of a function f is given below.

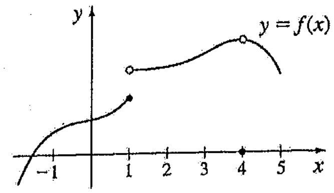


Which statement best describes $\lim_{x\to 3} f(x)$?

(A) $\lim_{x\to 3} f(x) = 1$ (B) $\lim_{x\to 3} f(x) = 3$

(C) $\lim_{x\to 3} f(x) = 4$ (D) $\lim_{x\to 3} f(x)$ doesn't exist.

PAGE 83 5. The graph of a piecewise function f is shown.



Use the graph to determine which of the following statements is true.

(A) $\lim_{x\to -1^+} f(x)$ does not exist. (B) $\lim_{x\to 1^-} f(x) = f(1)$.

(C) $\lim_{x\to 1} f(x)$ exists. (D) $\lim_{x\to 4} f(x) = f(4)$.

PAGE 81 6. The table below gives values of three functions near 1.

x	0.7	0.8	0.9	0.95	1	1.05	1.1	1.2	1.3
$f(x)$	0	0	0	0	0	0.9	0.9	0.9	0.9
$g(x)$	−0.9	−0.95	−0.095	−0.009	undefined	0.009	0.095	0.95	0.995
$h(x)$	1.0	0.08	0.008	0.0008	1	−0.005	−0.025	−0.05	−0.25

For which of these functions does the table suggest that the limit as x approaches 1 is 0?

(A) f only (B) h only (C) f and g only (D) g and h only

PAGE 79 7. Interpret $\lim_{x\to 2}(x^3+3x-4) = 10$.

PAGE 81 8. Use a calculator to create a table to investigate $\lim_{x\to 0} \frac{e^x-1}{x}$.

1.2 Analytic Techniques for Finding Limits of Functions

OBJECTIVES ***When you finish this section, you should be able to:***

1. **Find the limit of a sum, a difference, and a product (p. 90)**
2. **Find the limit of a power and the limit of a root (p. 93)**
3. **Find the limit of a polynomial (p. 95)**
4. **Find the limit of a quotient (p. 95)**
5. **Find the limit of an average rate of change (p. 98)**
6. **Find the limit of a difference quotient (p. 98)**

In Section 1.1, we used numerical techniques (tables) and graphical techniques to investigate limits. We saw that these techniques are not always reliable. The only way to be sure a limit is correct is to use the ε-δ definition of a limit. In this section, we state without proof analytic techniques based on the ε-δ definition. Some of the results are proved in Section 1.6 and others in Appendix B. As we will see, these analytic techniques involve using algebraic properties of limits.

We begin with two basic limits.

Summary

Two Basic Limits

- $\lim_{x\to c} A = A$, where A is a constant
- $\lim_{x\to c} x = c$, c a real number

Properties of Limits

If f and g are functions for which $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ both exist, and k is a constant, then

- **Limit of a Sum or a Difference:** $\lim_{x\to c}[f(x) \pm g(x)] = \lim_{x\to c} f(x) \pm \lim_{x\to c} g(x)$
- **Limit of a Product:** $\lim_{x\to c}[f(x)\cdot g(x)] = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x)$
- **Limit of a Constant Times a Function:** $\lim_{x\to c}[kg(x)] = k\lim_{x\to c} g(x)$
- **Limit of a Power:** $\lim_{x\to c}[f(x)]^n = \left[\lim_{x\to c} f(x)\right]^n$ where $n \ge 2$ is an integer
- **Limit of a Root:** $\lim_{x\to c}\sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to c} f(x)}$ provided $f(x) > 0$ if $n \ge 2$ is even
- **Limit of $[f(x)]^{m/n}$:** $\lim_{x\to c}[f(x)]^{m/n} = \left[\lim_{x\to c} f(x)\right]^{m/n}$ provided $[f(x)]^{m/n}$ is defined for positive integers m and n
- **Limit of a Quotient:** $\lim_{x\to c}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$ provided $\lim_{x\to c} g(x) \ne 0$
- **Limit of a Polynomial Function:** $\lim_{x\to c} P(x) = P(c)$
- **Limit of a Rational Function:** $\lim_{x\to c} R(x) = R(c)$ if c is in the domain of R

1.2 Assess Your Understanding

Concepts and Vocabulary

1. (a) $\lim_{x\to 4}(-3) =$ ________; (b) $\lim_{x\to 0}\pi =$ ________
2. If $\lim_{x\to c} f(x) = 3$, then $\lim_{x\to c}[f(x)]^5 =$ ________.
3. If $\lim_{x\to c} f(x) = 64$, then $\lim_{x\to c}\sqrt[3]{f(x)} =$ ________.
4. (a) $\lim_{x\to -1} x =$ ________; (b) $\lim_{x\to e} x =$ ________
5. (a) $\lim_{x\to 0}(x-2) =$ ____; (b) $\lim_{x\to 1/2}(3+x) =$ ________
6. (a) $\lim_{x\to 2}(-3x) =$ ________; (b) $\lim_{x\to 0}(3x) =$ ________
7. *True or False* If p is a polynomial function, then $\lim_{x\to 5} p(x) = p(5)$.
8. If the domain of a rational function R is $\{x \mid x \ne 0\}$, then $\lim_{x\to 2} R(x) = R($____$)$.
9. *True or False* Properties of limits cannot be used for one-sided limits.
10. *True or False* If $f(x) = \frac{(x+1)(x+2)}{x+1}$ and $g(x) = x+2$, then $\lim_{x\to -1} f(x) = \lim_{x\to -1} g(x)$.

Skill Building

In Problems 11–44, find each limit using algebraic properties of limits.

11. $\lim_{x\to 3}[2(x+4)]$
12. $\lim_{x\to -2}[3(x+1)]$
13. $\lim_{x\to -2}[x(3x-1)(x+2)]$ PAGE 92
14. $\lim_{x\to -1}[x(x-1)(x+10)]$
15. $\lim_{t\to 1}(3t-2)^3$ PAGE 93
16. $\lim_{x\to 0}(-3x+1)^2$
17. $\lim_{x\to 4}(3\sqrt{x})$
18. $\lim_{x\to 8}\left(\frac{1}{4}\sqrt[3]{x}\right)$
19. $\lim_{x\to 3}\sqrt{5x-4}$ PAGE 94
20. $\lim_{t\to 2}\sqrt{3t+4}$
21. $\lim_{t\to 2}[t\sqrt{(5t+3)(t+4)}]$
22. $\lim_{t\to -1}[t\sqrt[3]{(t+1)(2t-1)}]$
23. $\lim_{x\to 3}(\sqrt{x}+x+4)^{1/2}$ PAGE 94
24. $\lim_{t\to 2}(t\sqrt{2t}+4)^{1/3}$
25. $\lim_{t\to -1}[4t(t+1)]^{2/3}$
26. $\lim_{x\to 0}(x^2-2x)^{3/5}$
27. $\lim_{t\to 1}(3t^2-2t+4)$
28. $\lim_{x\to 0}(-3x^4+2x+1)$
29. $\lim_{x\to \frac{1}{2}}(2x^4-8x^3+4x-5)$ PAGE 95
30. $\lim_{x\to -\frac{1}{3}}(27x^3+9x+1)$
31. $\lim_{x\to 4}\frac{x^2+4}{\sqrt{x}}$ PAGE 96
32. $\lim_{x\to 3}\frac{x^2+5}{\sqrt{3x}}$
33. $\lim_{x\to -2}\frac{2x^3+5x}{3x-2}$ PAGE 96
34. $\lim_{x\to 1}\frac{2x^4-1}{3x^3+2}$
35. $\lim_{x\to 2}\frac{x^2-4}{x-2}$ PAGE 97
36. $\lim_{x\to -2}\frac{x+2}{x^2-4}$
37. $\lim_{x\to -1}\frac{x^3-x}{x+1}$
38. $\lim_{x\to -1}\frac{x^3+x^2}{x^2-1}$
39. $\lim_{x\to -8}\left(\frac{2x}{x+8}+\frac{16}{x+8}\right)$
40. $\lim_{x\to 2}\left(\frac{3x}{x-2}-\frac{6}{x-2}\right)$
41. $\lim_{x\to 2}\frac{\sqrt{x}-\sqrt{2}}{x-2}$ PAGE 97
42. $\lim_{x\to 3}\frac{\sqrt{x}-\sqrt{3}}{x-3}$
43. $\lim_{x\to 4}\frac{\sqrt{x+5}-3}{(x-4)(x+1)}$
44. $\lim_{x\to 3}\frac{\sqrt{x+1}-2}{x(x-3)}$

In Problems 45–50, find each one-sided limit using properties of limits.

45. $\lim_{x\to 3^-}(x^2-4)$

46. $\lim_{x\to 2^+}(3x^2+x)$

47. $\lim_{x\to 3^-}\dfrac{x^2-9}{x-3}$

48. $\lim_{x\to 3^+}\dfrac{x^2-9}{x-3}$

49. $\lim_{x\to 3^-}(\sqrt{9-x^2}+x)^2$

50. $\lim_{x\to 2^+}(2\sqrt{x^2-4}+3x)$

In Problems 51–58, use the information below to find each limit.

$$\lim_{x\to c} f(x)=5 \qquad \lim_{x\to c} g(x)=2 \qquad \lim_{x\to c} h(x)=0$$

51. $\lim_{x\to c}[f(x)-3g(x)]$

52. $\lim_{x\to c}[5f(x)]$

53. $\lim_{x\to c}[g(x)]^3$

54. $\lim_{x\to c}\dfrac{f(x)}{g(x)-h(x)}$

55. $\lim_{x\to c}\dfrac{h(x)}{g(x)}$

56. $\lim_{x\to c}[4f(x)\cdot g(x)]$

57. $\lim_{x\to c}\left[\dfrac{1}{g(x)}\right]^2$

58. $\lim_{x\to c}\sqrt[3]{5g(x)-3}$

In Problems 59 and 60, use the graphs of the functions and properties of limits to find each limit, if it exists. If the limit does not exist, write, "the limit does not exist," and explain why.

59. (a) $\lim_{x\to 4}[f(x)+g(x)]$

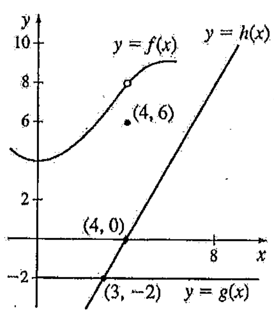
(b) $\lim_{x\to 4}\{f(x)[g(x)-h(x)]\}$

(c) $\lim_{x\to 4}[f(x)\cdot g(x)]$

(d) $\lim_{x\to 4}[2h(x)]$

(e) $\lim_{x\to 4}\dfrac{g(x)}{f(x)}$

(f) $\lim_{x\to 4}\dfrac{h(x)}{f(x)}$



60. (a) $\lim_{x\to 3}\{2[f(x)+h(x)]\}$

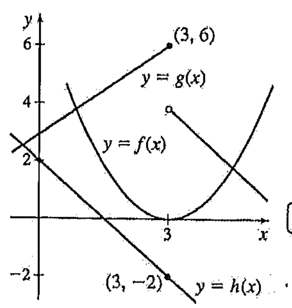
(b) $\lim_{x\to 3^-}[g(x)+h(x)]$

(c) $\lim_{x\to 3}\sqrt[3]{h(x)}$

(d) $\lim_{x\to 3}\dfrac{f(x)}{h(x)}$

(e) $\lim_{x\to 3}[h(x)]^3$

(f) $\lim_{x\to 3}[f(x)-2h(x)]^{3/2}$



In Problems 61–66, for each function f, find the limit as x approaches c of the average rate of change of f from c to x. That is, find

$$\lim_{x\to c}\frac{f(x)-f(c)}{x-c}$$

61. $f(x)=3x^2, \quad c=1$

62. $f(x)=8x^3, \quad c=2$

PAGE 98 **63.** $f(x)=-2x^2+4, \quad c=1$

64. $f(x)=20-0.8x^2, \quad c=3$

65. $f(x)=\sqrt{x}, \quad c=1$

66. $f(x)=\sqrt{2x}, \quad c=5$

In Problems 67–72, find the limit of the difference quotient for each function f. That is, find $\lim_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$.

67. $f(x)=4x-3$

68. $f(x)=3x+5$

PAGE 98 **69.** $f(x)=3x^2+4x+1$

70. $f(x)=2x^2+x$

71. $f(x)=\dfrac{2}{x}$

72. $f(x)=\dfrac{3}{x^2}$

In Problems 73–80, find $\lim_{x\to c^-} f(x)$ and $\lim_{x\to c^+} f(x)$ for the given number c. Based on the results, determine whether $\lim_{x\to c} f(x)$ exists.

PAGE 92 **73.** $f(x)=\begin{cases}2x-3 & \text{if } x\le 1\\ 3-x & \text{if } x>1\end{cases}$ at $c=1$

74. $f(x)=\begin{cases}5x+2 & \text{if } x<2\\ 1+3x & \text{if } x\ge 2\end{cases}$ at $c=2$

75. $f(x)=\begin{cases}3x-1 & \text{if } x<1\\ 4 & \text{if } x=1\\ 2x & \text{if } x>1\end{cases}$ at $c=1$

76. $f(x)=\begin{cases}3x-1 & \text{if } x<1\\ 2 & \text{if } x=1\\ 2x & \text{if } x>1\end{cases}$ at $c=1$

77. $f(x)=\begin{cases}x-1 & \text{if } x<1\\ \sqrt{x-1} & \text{if } x>1\end{cases}$ at $c=1$

78. $f(x)=\begin{cases}\sqrt{9-x^2} & \text{if } 0<x<3\\ \sqrt{x^2-9} & \text{if } x>3\end{cases}$ at $c=3$

79. $f(x)=\begin{cases}\dfrac{x^2-9}{x-3} & \text{if } x\ne 3\\ 6 & \text{if } x=3\end{cases}$ at $c=3$

80. $f(x)=\begin{cases}\dfrac{x-2}{x^2-4} & \text{if } x\ne 2\\ 1 & \text{if } x=2\end{cases}$ at $c=2$

Applications and Extensions

Heaviside Functions *In Problems 81 and 82, find the limit, if it exists, of the given Heaviside function at c.*

PAGE 93 **81.** $u_1(t)=\begin{cases}0 & \text{if } t<1\\ 1 & \text{if } t\ge 1\end{cases}$ at $c=1$

82. $u_3(t)=\begin{cases}0 & \text{if } t<3\\ 1 & \text{if } t\ge 3\end{cases}$ at $c=3$

In Problems 83–92, find each limit.

83. $\lim_{h\to 0}\dfrac{(x+h)^2-x^2}{h}$

84. $\lim_{h\to 0}\dfrac{\sqrt{x+h}-\sqrt{x}}{h}$

85. $\lim_{h\to 0}\dfrac{\dfrac{1}{x+h}-\dfrac{1}{x}}{h}$

86. $\lim_{h\to 0}\dfrac{\dfrac{1}{(x+h)^3}-\dfrac{1}{x^3}}{h}$

87. $\lim_{x\to 0}\left[\frac{1}{x}\left(\frac{1}{4+x}-\frac{1}{4}\right)\right]$

88. $\lim_{x\to -1}\left[\frac{2}{x+1}\left(\frac{1}{3}-\frac{1}{x+4}\right)\right]$

89. $\lim_{x\to 7}\frac{x-7}{\sqrt{x+2}-3}$

90. $\lim_{x\to 2}\frac{x-2}{\sqrt{x+2}-2}$

91. $\lim_{x\to 1}\frac{x^3-3x^2+3x-1}{x^2-2x+1}$

92. $\lim_{x\to -3}\frac{x^3+7x^2+15x+9}{x^2+6x+9}$

93. **Cost of Water** The Jericho Water District determines quarterly water costs, in dollars, using the following rate schedule:

Water used (in thousands of gallons)	Cost
$0\le x\le 10$	\$9.00
$10<x\le 30$	\$9.00 + 0.95 for each thousand gallons in excess of 10,000 gallons
$30<x\le 100$	\$28.00 + 1.65 for each thousand gallons in excess of 30,000 gallons
$x>100$	\$143.50 + 2.20 for each thousand gallons in excess of 100,000 gallons

Source: Jericho Water District, Syosset, NY.

(a) Find a function C that models the quarterly cost, in dollars, of using x thousand gallons of water.

(b) What is the domain of the function C?

(c) Find each of the following limits. If the limit does not exist, explain why.

$$\lim_{x\to 5}C(x)\qquad \lim_{x\to 10}C(x)\qquad \lim_{x\to 30}C(x)\qquad \lim_{x\to 100}C(x)$$

(d) What is $\lim_{x\to 0^+}C(x)$?

(e) Graph the function C.

94. **Cost of Electricity** In January 2019, Florida Power and Light charged customers living in single-family residences for their electric usage according to the following table.

Monthly customer charge	for electricity:
\$7.98	per household, plus
\$0.08692	for each kWH used less than or equal to 1000 kWH, plus
\$0.10708	for each kWH used in excess of 1000 kWH

Source: Florida Power and Light, Miami, FL.

(a) Find a function C that models the monthly cost, in dollars, of using x kWH of electricity.

(b) What is the domain of the function C?

(c) Find $\lim_{x\to 1000}C(x)$, if it exists. If the limit does not exist, explain why.

(d) What is $\lim_{x\to 0^+}C(x)$?

(e) Graph the function C.

95. **Low-Temperature Physics** In thermodynamics, the average molecular kinetic energy (energy of motion) of a gas having molecules of mass m is directly proportional to its temperature T on the absolute (or Kelvin) scale. This can be expressed as $\frac{1}{2}mv^2=\frac{3}{2}kT$, where $v=v(T)$ is the speed of a typical molecule at time t, and k is a constant, known as the **Boltzmann constant.**

(a) What limit does the molecular speed v approach as the gas temperature T approaches absolute zero (0 K or $-273\,°C$ or $-469\,°F$)?

(b) What does this limit suggest about the behavior of a gas as its temperature approaches absolute zero?

96. For the function $f(x)=\begin{cases}3x+5 & \text{if } x\le 2\\ 13-x & \text{if } x>2\end{cases}$, find

(a) $\lim_{h\to 0^-}\frac{f(2+h)-f(2)}{h}$

(b) $\lim_{h\to 0^+}\frac{f(2+h)-f(2)}{h}$

(c) Does $\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}$ exist?

97. Use the fact that $|x|=\begin{cases}x & \text{if } x\ge 0\\ -x & \text{if } x<0\end{cases}$ to show that $\lim_{x\to 0}|x|=0$.

98. Use the fact that $|x|=\sqrt{x^2}$ to show that $\lim_{x\to 0}|x|=0$.

99. Find functions f and g for which $\lim_{x\to c}[f(x)+g(x)]$ may exist even though $\lim_{x\to c}f(x)$ and $\lim_{x\to c}g(x)$ do not exist.

100. Find functions f and g for which $\lim_{x\to c}[f(x)g(x)]$ may exist even though $\lim_{x\to c}f(x)$ and $\lim_{x\to c}g(x)$ do not exist.

101. Find functions f and g for which $\lim_{x\to c}\left[\frac{f(x)}{g(x)}\right]$ may exist even though $\lim_{x\to c}f(x)$ and $\lim_{x\to c}g(x)$ do not exist.

102. Find a function f for which $\lim_{x\to c}|f(x)|$ may exist even though $\lim_{x\to c}f(x)$ does not exist.

103. Prove that if g is a function for which $\lim_{x\to c}g(x)$ exists and if k is any real number, then $\lim_{x\to c}[kg(x)]$ exists and $\lim_{x\to c}[kg(x)]=k\lim_{x\to c}g(x)$.

104. Prove that if the number c is in the domain of a rational function $R(x)=\frac{p(x)}{q(x)}$, then $\lim_{x\to c}R(x)=R(c)$.

Challenge Problems

105. Find $\lim_{x\to a}\frac{x^n-a^n}{x-a}$, n a positive integer.

106. Find $\lim_{x\to -a}\frac{x^n+a^n}{x+a}$, n a positive integer.

107. Find $\lim_{x\to 1}\frac{x^m-1}{x^n-1}$, m, n positive integers.

108. Find $\lim_{x\to 0}\frac{\sqrt[3]{1+x}-1}{x}$.

109. Find $\lim_{x\to 0}\frac{\sqrt{(1+ax)(1+bx)}-1}{x}$.

110. Find $\lim_{x\to 0}\frac{\sqrt{(1+a_1x)(1+a_2x)\cdots(1+a_nx)}-1}{x}$.

111. Find $\lim_{h\to 0}\frac{f(h)-f(0)}{h}$ if $f(x)=x|x|$.

Preparing for the AP® Exam

AP® Practice Problems

PAGE 92 1. Consider the piecewise-defined function f given by

$$f(x)=\begin{cases} -x-2 & \text{if } x<-1 \\ x^2 & \text{if } -1\le x<2 \\ -4x+12 & \text{if } x\ge 2 \end{cases}$$

Investigate the limits below and decide which limit does NOT exist.

(A) $\lim\limits_{x\to -1^+} f(x)$ (B) $\lim\limits_{x\to 2^-} f(x)$

(C) $\lim\limits_{x\to 2} f(x)$ (D) $\lim\limits_{x\to -1} f(x)$

PAGE 97 2. $\lim\limits_{t\to 5} \dfrac{(5-t)^2}{t-5} =$

(A) -5 (B) 0 (C) 1 (D) 5

PAGE 98 3. Find $\lim\limits_{x\to 2} \dfrac{f(x)-f(2)}{x-2}$ for the function $f(x)=3x^3-4$.

(A) 0 (B) 12 (C) 24 (D) 36

PAGE 97 4. $\lim\limits_{x\to s} \dfrac{x-s}{\sqrt{x}-\sqrt{s}} =$

(A) $2s$ (B) $2\sqrt{s}$ (C) $\sqrt{2s}$ (D) s

PAGE 92 5. For $g(x)=\begin{cases} ax^2-5 \text{ if } x<2 \\ ax+b \text{ if } x>2 \end{cases}$ find values for a and b so that $\lim\limits_{x\to 2} g(x)=7$.

(A) $a=1, b=5$ (B) $a=2, b=3$

(C) $a=3, b=1$ (D) $a=6, b=-5$

PAGE 94 6. $\lim\limits_{x\to 4^+} (5\sqrt{x^2-16}+3x) =$

(A) -12 (B) 0 (C) 12 (D) The limit does not exist.

PAGE 94 7. If $\lim\limits_{x\to 2} \sqrt{\dfrac{[f(x)]^2-8x+3}{x+1}} = 9$ and $f(x)\ge 0$ for all x, find $\lim\limits_{x\to 2} f(x)$.

(A) $\sqrt{22}$ (B) $2\sqrt{10}$ (C) 16 (D) 256

PAGE 94 8. $\lim\limits_{x\to 3} [x^{-1/2}(5x-7)^{1/3}] =$

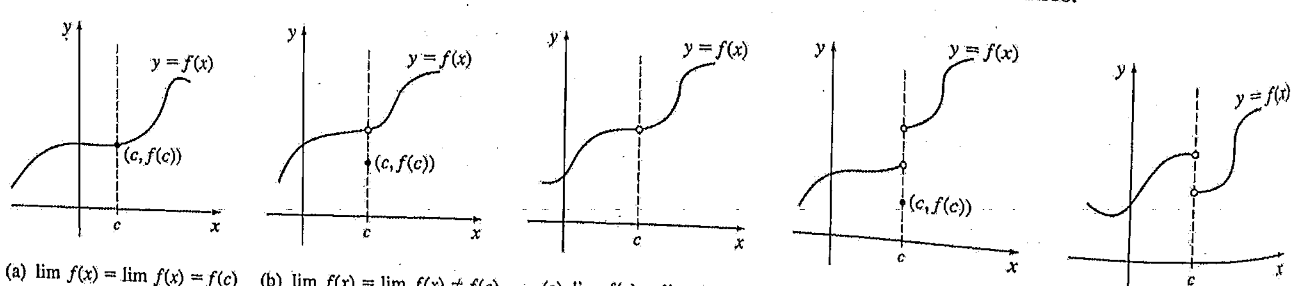
(A) $3^{-1/2}$ (B) $\dfrac{2}{3^{1/2}}$ (C) $\dfrac{8}{3^{1/2}}$ (D) $6^{-1/2}$

1.3 Continuity

OBJECTIVES *When you finish this section, you should be able to:*

1 Determine whether a function is continuous at a number (p. 103)
2 Determine intervals on which a function is continuous (p. 106)
3 Use properties of continuity (p. 108)
4 Use the Intermediate Value Theorem (p. 110)

Sometimes $\lim\limits_{x\to c} f(x)$ equals $f(c)$ and sometimes it does not. In fact, $f(c)$ may not even be defined and yet $\lim\limits_{x\to c} f(x)$ may exist. In this section, we investigate the relationship between $\lim\limits_{x\to c} f(x)$ and $f(c)$. Figure 21 shows some possibilities.



(a) $\lim\limits_{x\to c^-} f(x) = \lim\limits_{x\to c^+} f(x) = f(c)$ (b) $\lim\limits_{x\to c^-} f(x) = \lim\limits_{x\to c^+} f(x) \ne f(c)$ (c) $\lim\limits_{x\to c^-} f(x) = \lim\limits_{x\to c^+} f(x)$; $f(c)$ is not defined. (d) $\lim\limits_{x\to c^-} f(x) \ne \lim\limits_{x\to c^+} f(x)$; $f(c)$ is defined. (e) $\lim\limits_{x\to c^-} f(x) \ne \lim\limits_{x\to c^+} f(x)$; $f(c)$ is not defined.

Figure 21

Of these five graphs, the "nicest" one is Figure 21(a). There, $\lim\limits_{x\to c} f(x)$ exists and is equal to $f(c)$. Functions that have this property are said to be *continuous at the number c*. This agrees with the intuitive notion that a function is continuous if its graph can be drawn without lifting the pencil. The functions in Figures 21(b)–(e) are

X	Y1
1.2	-.032
1.201	-.0263
1.202	-.0205
1.203	-.0148
1.204	-.009
1.205	-.0033
1.206	.00249
1.207	.00827
1.208	.01405
1.209	.01985
1.21	.02566

NORMAL FLOAT AUTO REAL RADIAN MP — PRESS [ENTER] TO EDIT — $Y_1 = X^3+X^2-X-2$

Figure 37

Now subdivide the interval [1.20, 1.21] into 10 subintervals, each of length 0.001. See Figure 37.

We conclude that the zero of the function f is 1.205, correct to three decimal places. ■

Notice that a benefit of the method used in Example 10 is that each additional iteration results in one additional decimal place of accuracy for the approximation.

NOW WORK Problem 65.

1.3 Assess Your Understanding

Concepts and Vocabulary

1. *True or False* A polynomial function is continuous at every real number.
2. *True or False* Piecewise-defined functions are never continuous at numbers where the function changes equations.
3. The three conditions necessary for a function f to be continuous at a number c are ______, ______, and ______.
4. *True or False* If f is continuous at 0, then $g(x) = \frac{1}{4}f(x)$ is continuous at 0.
5. *True or False* If f is a function defined everywhere in an open interval containing c, except possibly at c, then the number c is called a removable discontinuity of f if the function f is not continuous at c.
6. *True or False* If a function f is discontinuous at a number c, then $\lim_{x\to c} f(x)$ does not exist.
7. *True or False* If a function f is continuous on an open interval (a, b), then it is continuous on the closed interval $[a, b]$.
8. *True or False* If a function f is continuous on the closed interval $[a, b]$, then f is continuous on the open interval (a, b).

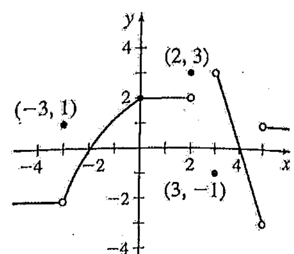
In Problems 9 and 10, explain whether each function is continuous or discontinuous on its domain.

9. The velocity of a ball thrown up into the air as a function of time, if the ball lands 5 seconds after it is thrown and stops.
10. The temperature of an oven used to bake a potato as a function of time.
11. *True or False* If a function f is continuous on a closed interval $[a, b]$, then the Intermediate Value Theorem guarantees that the function takes on every value between $f(a)$ and $f(b)$.
12. *True or False* If a function f is continuous on a closed interval $[a, b]$ and $f(a) \neq f(b)$, but both $f(a) > 0$ and $f(b) > 0$, then according to the Intermediate Value Theorem, f does not have a zero on the open interval (a, b).

Skill Building

In Problems 13–18, use the graph of $y = f(x)$ (top right).
(a) Determine if f is continuous at c.
(b) If f is discontinuous at c, state which condition(s) of the definition of continuity is (are) not satisfied.
(c) If f is discontinuous at c, determine if the discontinuity is removable.
(d) If the discontinuity is removable, define (or redefine) f at c to make f continuous at c.

PAGE 105 13. $c = -3$ 14. $c = 0$
15. $c = 2$ 16. $c = 3$
17. $c = 4$ 18. $c = 5$



In Problems 19–32, determine whether the function f is continuous at c.

PAGE 103 19. $f(x) = x^2 + 1$ at $c = -1$

20. $f(x) = x^3 - 5$ at $c = 5$

PAGE 104 21. $f(x) = \dfrac{x}{x^2+4}$ at $c = -2$

22. $f(x) = \dfrac{x}{x-2}$ at $c = 2$

23. $f(x) = \begin{cases} 2x+5 & \text{if } x \le 2 \\ 4x+1 & \text{if } x > 2 \end{cases}$ at $c = 2$

24. $f(x) = \begin{cases} 2x+1 & \text{if } x \le 0 \\ 2x & \text{if } x > 0 \end{cases}$ at $c = 0$

PAGE 104 25. $f(x) = \begin{cases} 3x-1 & \text{if } x < 1 \\ 4 & \text{if } x = 1 \\ 2x & \text{if } x > 1 \end{cases}$ at $c = 1$

26. $f(x) = \begin{cases} 3x-1 & \text{if } x < 1 \\ 2 & \text{if } x = 1 \\ 2x & \text{if } x > 1 \end{cases}$ at $c = 1$

27. $f(x) = \begin{cases} 3x-1 & \text{if } x < 1 \\ 2x & \text{if } x > 1 \end{cases}$ at $c = 1$

28. $f(x) = \begin{cases} 3x-1 & \text{if } x < 1 \\ 2 & \text{if } x = 1 \\ 3x & \text{if } x > 1 \end{cases}$ at $c = 1$

29. $f(x) = \begin{cases} x^2 & \text{if } x \le 0 \\ 2x & \text{if } x > 0 \end{cases}$ at $c = 0$

30. $f(x) = \begin{cases} x^2 & \text{if } x < -1 \\ 2 & \text{if } x = -1 \\ -3x+2 & \text{if } x > -1 \end{cases}$ at $c = -1$

31. $f(x) = \begin{cases} 4-3x^2 & \text{if } x<0 \\ 4 & \text{if } x=0 \\ \sqrt{\dfrac{16-x^2}{4-x}} & \text{if } 0<x<4 \end{cases}$ at $c=0$

32. $f(x) = \begin{cases} \sqrt{4+x} & \text{if } -4 \le x \le 4 \\ \sqrt{\dfrac{x^2-3x-4}{x-4}} & \text{if } x>4 \end{cases}$ at $c=4$

In Problems 33–36, each function f has a removable discontinuity at c. Define $f(c)$ so that f is continuous at c.

33. $f(x) = \dfrac{x^2-4}{x-2}$, $c=2$

34. $f(x) = \dfrac{x^2+x-12}{x-3}$, $c=3$

PAGE 105 35. $f(x) = \begin{cases} 1+x & \text{if } x<1 \\ 4 & \text{if } x=1 \\ 2x & \text{if } x>1 \end{cases}$ $c=1$

36. $f(x) = \begin{cases} x^2+5x & \text{if } x<-1 \\ 0 & \text{if } x=-1 \\ x-3 & \text{if } x>-1 \end{cases}$ $c=-1$

In Problems 37–40, determine whether each function f is continuous on the given interval. If the answer is no, state the interval(s), if any, on which f is continuous.

PAGE 107 37. $f(x) = \dfrac{x^2-9}{x-3}$ on the interval $[-3, 3)$

38. $f(x) = 1 + \dfrac{1}{x}$ on the interval $[-1, 0)$

39. $f(x) = \dfrac{1}{\sqrt{x^2-9}}$ on the interval $[-3, 3]$

40. $f(x) = \sqrt{9-x^2}$ on the interval $[-3, 3]$

In Problems 41–50, determine where each function f is continuous. First determine the domain of the function. Then support your decision using properties of continuity.

41. $f(x) = 2x^2 + 5x - \dfrac{1}{x}$

42. $f(x) = x + 1 + \dfrac{2x}{x^2+5}$

43. $f(x) = (x-1)(x^2+x+1)$

44. $f(x) = \sqrt{x}(x^3-5)$

PAGE 109 45. $f(x) = \dfrac{x-9}{\sqrt{x}-3}$

46. $f(x) = \dfrac{x-4}{\sqrt{x}-2}$

PAGE 109 47. $f(x) = \sqrt{\dfrac{x^2+1}{2-x}}$

48. $f(x) = \sqrt{\dfrac{4}{x^2-1}}$

49. $f(x) = (2x^2+5x-3)^{2/3}$

50. $f(x) = (x+2)^{1/2}$

In Problems 51–56, use the function

$$f(x) = \begin{cases} \sqrt{15-3x} & \text{if } x<2 \\ \sqrt{5} & \text{if } x=2 \\ 9-x^2 & \text{if } 2<x<3 \\ \lfloor x-2 \rfloor & \text{if } 3 \le x \end{cases}$$

51. Is f continuous at 0? Why or why not?

PAGE 105 52. Is f continuous at 4? Why or why not?

53. Is f continuous at 3? Why or why not?

54. Is f continuous at 2? Why or why not?

55. Is f continuous at 1? Why or why not?

56. Is f continuous at 2.5? Why or why not?

In Problems 57 and 58:

(a) Use technology to graph f using a suitable scale on each axis.

(b) Based on the graph from (a), determine where f is continuous.

(c) Use the definition of continuity to determine where f is continuous.

(d) What advice would you give a fellow student about using technology to determine where a function is continuous?

57. $f(x) = \dfrac{x^3-8}{x-2}$

58. $f(x) = \dfrac{x^2-3x+2}{3x-6}$

In Problems 59–64, use the Intermediate Value Theorem to determine which of the functions must have zeros in the given interval. Indicate those for which the theorem gives no information. Do not attempt to locate the zeros.

PAGE 111 59. $f(x) = x^3 - 3x$ on $[-2, 2]$

60. $f(x) = x^4 - 1$ on $[-2, 2]$

61. $f(x) = \dfrac{x}{(x+1)^2} - 1$ on $[10, 20]$

62. $f(x) = x^3 - 2x^2 - x + 2$ on $[3, 4]$

63. $f(x) = \dfrac{x^3-1}{x-1}$ on $[0, 2]$

64. $f(x) = \dfrac{x^2+3x+2}{x^2-1}$ on $[-3, 0]$

In Problems 65–72, verify that each function has a zero in the indicated interval. Then use the Intermediate Value Theorem to approximate the zero correct to three decimal places by repeatedly subdividing the interval containing the zero into 10 subintervals.

PAGE 112 65. $f(x) = x^3 + 3x - 5$; interval: $[1, 2]$

66. $f(x) = x^3 - 4x + 2$; interval: $[1, 2]$

67. $f(x) = 2x^3 + 3x^2 + 4x - 1$; interval: $[0, 1]$

68. $f(x) = x^3 - x^2 - 2x + 1$; interval: $[0, 1]$

69. $f(x) = x^3 - 6x - 12$; interval: $[3, 4]$

70. $f(x) = 3x^3 + 5x - 40$; interval: $[2, 3]$

71. $f(x) = x^4 - 2x^3 + 21x - 23$; interval: $[1, 2]$

72. $f(x) = x^4 - x^3 + x - 2$; interval: $[1, 2]$

In Problems 73 and 74,

(a) Use the Intermediate Value Theorem to show that f has a zero in the given interval.

(b) Use technology to find the zero rounded to three decimal places.

73. $f(x) = \sqrt{x^2+4x} - 2$ in $[0, 1]$

74. $f(x) = x^3 - x + 2$ in $[-2, 0]$

Applications and Extensions

In Problems 75–78, determine whether each function is (a) continuous from the left (b) continuous from the right at the number c.

75. $f(x) = \begin{cases} x^2 & \text{if } -1<x<1 \\ x-1 & \text{if } |x| \ge 1 \end{cases}$ at $c=-1$

76. $f(x) = \begin{cases} x^2-1 & \text{if } -1<x<1 \\ |x-1| & \text{if } |x| \ge 1 \end{cases}$ at $c=-1$

PAGE 107 **77.** $f(x)=\sqrt{(x+1)(x-5)}$ at $c=-1$

78. $f(x)=\sqrt{(x-1)(x-2)}$ at $c=1$

79. First-Class Mail As of January 2019, the U.S. Postal Service charged \$0.55 postage for first-class letters weighing up to and including 1 ounce, plus a flat fee of \$0.15 for each additional or partial ounce up to 3.5 ounces. First-class letter rates do not apply to letters weighing more than 3.5 ounces.

Source: U.S. Postal Service Notice 123.

(a) Find a function C that models the first-class postage charged for a letter weighing w ounces. Assume $w>0$.
(b) What is the domain of C?
(c) Determine the intervals on which C is continuous.
(d) At numbers where C is not continuous (if any), what type of discontinuity does C have?
(e) What are the practical implications of the answer to (d)?

80. First-Class Mail As of January 2019, the U.S. Postal Service charged \$1.00 postage for first-class large envelopes weighing up to and including 1 ounce, plus a flat fee of \$0.15 for each additional or partial ounce up to 13 ounces. First-class rates do not apply to large envelopes weighing more than 13 ounces.

Source: U.S. Postal Service Notice 123.

(a) Find a function C that models the first-class postage charged for a large envelope weighing w ounces. Assume $w>0$.
(b) What is the domain of C?
(c) Determine the intervals on which C is continuous.
(d) At numbers where C is not continuous (if any), what type of discontinuity does C have?
(e) What are the practical implications of the answer to (d)?

81. Cost of Electricity In January 2019, Florida Power and Light charged customers living in single-family residences for their electric usage according to the following table.

Monthly customer charge for electricity:	
\$7.98	per household, plus
\$0.08692	for each kWH used less than or equal to 1000 kWH, plus
\$0.10708	for each kWH used in excess of 1000 kWH

Source: Florida Power and Light, Miami, FL.

(a) Find a function C that models the monthly cost of using x kWH of electricity.
(b) What is the domain of C?
(c) Determine the intervals on which C is continuous.
(d) At numbers where C is not continuous (if any), what type of discontinuity does C have?
(e) What are the practical implications of the answer to (d)?

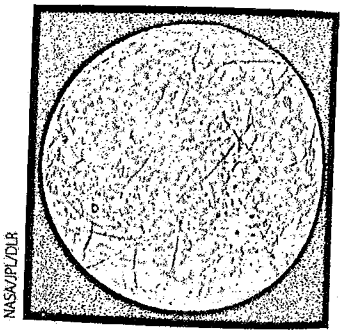
82. Cost of Water The Jericho Water District determines quarterly water costs, in dollars, using the following rate schedule:

Water used (in thousands of gallons)	Cost
$0 \le x \le 10$	\$9.00
$10 < x \le 30$	\$9.00 + 0.95 for each thousand gallons in excess of 10,000 gallons
$30 < x \le 100$	\$28.00 + 1.65 for each thousand gallons in excess of 30,000 gallons
$x > 100$	\$143.50 + 2.20 for each thousand gallons in excess of 100,000 gallons

Source: Jericho Water District, Syosset, NY.

(a) Find a function C that models the quarterly cost of using x thousand gallons of water.
(b) What is the domain of C?
(c) Determine the intervals on which C is continuous.
(d) At numbers where C is not continuous (if any), what type of discontinuity does C have?
(e) What are the practical implications of the answer to (d)?

83. Gravity on Europa


NASA/JPL/DLR

Europa, one of the larger satellites of Jupiter, has an icy surface and appears to have oceans beneath the ice. This makes it a candidate for possible extraterrestrial life. Because Europa is much smaller than most planets, its gravity is weaker. If we think of Europa as a sphere with uniform internal density, then inside the sphere, the gravitational field g is given by $g(r)=\dfrac{Gm}{R^3}r$, $0 \le r < R$, where R is the radius of the sphere, r is the distance from the center of the sphere, and G is the universal gravitation constant. Outside a uniform sphere of mass m, the gravitational field g is given by

$$g(r)=\frac{Gm}{r^2},\quad R<r$$

(a) For the gravitational field of Europa to be continuous at its surface, what must $g(r)$ equal? *Hint:* Investigate $\lim_{r \to R} g(r)$.
(b) Determine the gravitational field at Europa's surface. This will indicate the type of gravity environment organisms will experience. Use the following measured values: Europa's mass is 4.8×10^{22} kilograms, its radius is 1.569×10^{6} meters, and $G=6.67 \times 10^{-11}$.
(c) Compare the result found in (b) to the gravitational field on Earth's surface, which is 9.8 meter/second^2. Is the gravity on Europa less than or greater than that on Earth?

84. Find constants A and B so that the function below is continuous for all x. Graph the resulting function.

$$f(x)=\begin{cases}(x-1)^2 & \text{if } -\infty < x < 0\\ (A-x)^2 & \text{if } 0 \le x < 1\\ x+B & \text{if } 1 \le x < \infty\end{cases}$$

85. Find constants A and B so that the function below is continuous for all x. Graph the resulting function.

$$f(x)=\begin{cases} x+A & \text{if } -\infty<x<4 \\ (x-1)^2 & \text{if } 4\le x\le 9 \\ Bx+1 & \text{if } 9<x<\infty \end{cases}$$

86. For the function f below, find k so that f is continuous at 2.

$$f(x)=\begin{cases} \dfrac{\sqrt{2x+5}-\sqrt{x+7}}{x-2} & \text{if } x\ge -\dfrac{5}{2}, x\ne 2 \\ k & \text{if } x=2 \end{cases}$$

87. Suppose $f(x)=\dfrac{x^2-6x-16}{(x^2-7x-8)\sqrt{x^2-4}}$.

(a) For what numbers x is f defined?
(b) For what numbers x is f discontinuous?
(c) Which discontinuities, if any, found in (b) are removable?

88. **Intermediate Value Theorem**

(a) Use the Intermediate Value Theorem to show that the function $f(x)=\sin x+x-3$ has a zero in the interval $[0,\pi]$.
(b) Approximate the zero rounded to three decimal places.

89. **Intermediate Value Theorem**

(a) Use the Intermediate Value Theorem to show that the function $f(x)=e^x+x-2$ has a zero in the interval [0, 2].
(b) Approximate the zero rounded to three decimal places.

In Problems 90–93, verify that each function intersects the given line in the indicated interval. Then use the Intermediate Value Theorem to approximate the point of intersection correct to three decimal places by repeatedly subdividing the interval into 10 subintervals.

90. $f(x)=x^3-2x^2-1$; line: $y=-1$; interval: (1, 4)
91. $g(x)=-x^4+3x^2+3$; line: $y=3$; interval: (1, 2)
92. $h(x)=\dfrac{x^3-5}{x^2+1}$; line: $y=1$; interval: (1, 3)
93. $r(x)=\dfrac{x-6}{x^2+2}$; line: $y=-1$; interval: (0, 3)
94. Graph a function that is continuous on the closed interval [5, 12], is negative at both endpoints, and has exactly three distinct zeros in this interval. Does this contradict the Intermediate Value Theorem? Explain.
95. Graph a function that is continuous on the closed interval $[-1, 2]$, is positive at both endpoints, and has exactly two zeros in this interval. Does this contradict the Intermediate Value Theorem? Explain.
96. Graph a function that is continuous on the closed interval $[-2, 3]$, is positive at -2 and negative at 3, and has exactly two zeros in this interval. Is this possible? Does this contradict the Intermediate Value Theorem? Explain.
97. Graph a function that is continuous on the closed interval $[-5, 0]$, is negative at -5 and positive at 0, and has exactly three zeros in the interval. Is this possible? Does this contradict the Intermediate Value Theorem? Explain.
98. (a) Explain why the Intermediate Value Theorem gives no information about the zeros of the function $f(x)=x^4-1$ on the interval $[-2, 2]$.
(b) Use technology to determine whether or not f has a zero on the interval $[-2, 2]$.

99. (a) Explain why the Intermediate Value Theorem gives no information about the zeros of the function $f(x)=\ln(x^2+2)$ on the interval $[-2, 2]$.
(b) Use technology to determine whether or not f has a zero on the interval $[-2, 2]$.

100. **Intermediate Value Theorem**

(a) Use the Intermediate Value Theorem to show that the functions $y=x^3$ and $y=1-x^2$ intersect somewhere between $x=0$ and $x=1$.
(b) Use technology to find the coordinates of the point of intersection rounded to three decimal places.
(c) Use technology to graph both functions on the same set of axes. Be sure the graph shows the point of intersection.

101. **Intermediate Value Theorem** An airplane is traveling at a speed of 620 miles per hour and then encounters a slight headwind that slows it to 608 miles per hour. After a few minutes, the headwind eases and the plane's speed increases to 614 miles per hour. Explain why the plane's speed is 610 miles per hour on at least two different occasions during the flight.
Source: Submitted by the students of Millikin University.

102. Suppose a function f is defined and continuous on the closed interval $[a, b]$. Is the function $h(x)=\dfrac{1}{f(x)}$ also continuous on the closed interval $[a, b]$? Discuss the continuity of h on $[a, b]$.

103. Given the two functions f and h:

$$f(x)=x^3-3x^2-4x+12 \qquad h(x)=\begin{cases} \dfrac{f(x)}{x-3} & \text{if } x\ne 3 \\ p & \text{if } x=3 \end{cases}$$

(a) Find all the zeros of the function f.
(b) Find the number p so that the function h is continuous at $x=3$. Justify your answer.
(c) Determine whether h, with the number found in (b), is even, odd, or neither. Justify your answer.

104. The function $f(x)=\dfrac{|x|}{x}$ is not defined at 0. Explain why it is impossible to define $f(0)$ so that f is continuous at 0.

105. Find two functions f and g that are each continuous at c, yet $\dfrac{f}{g}$ is not continuous at c.

106. Discuss the difference between a discontinuity that is removable and one that is nonremovable. Give an example of each.

Bisection Method for Approximating Zeros of a Function *Suppose the Intermediate Value Theorem indicates that a function f has a zero in the interval (a, b). The bisection method approximates the zero by evaluating f at the midpoint m_1 of the interval (a, b). If $f(m_1)=0$, then m_1 is the zero we seek and the process ends. If $f(m_1)\ne 0$, then the sign of $f(m_1)$ is opposite that of either $f(a)$ or $f(b)$ (but not both), and the zero lies in that subinterval. Evaluate f at the midpoint m_2 of this subinterval. Continue bisecting the subinterval containing the zero until the desired degree of accuracy is obtained.*

In Problems 107–114, use the bisection method three times to approximate the zero of each function in the given interval.

107. $f(x)=x^3+3x-5$; interval: [1, 2]

108. $f(x) = x^3 - 4x + 2$; interval: [1, 2]

109. $f(x) = 2x^3 + 3x^2 + 4x - 1$; interval: [0, 1]

110. $f(x) = x^3 - x^2 - 2x + 1$; interval: [0, 1]

111. $f(x) = x^3 - 6x - 12$; interval: [3, 4]

112. $f(x) = 3x^3 + 5x - 40$; interval: [2, 3]

113. $f(x) = x^4 - 2x^3 + 21x - 23$; interval: [1, 2]

114. $f(x) = x^4 - x^3 + x - 2$; interval: [1, 2]

115. Intermediate Value Theorem Use the Intermediate Value Theorem to show that the function $f(x) = \sqrt{x^2 + 4x} - 2$ has a zero in the interval [0, 1]. Then approximate the zero correct to one decimal place.

116. Intermediate Value Theorem Use the Intermediate Value Theorem to show that the function $f(x) = x^3 - x + 2$ has a zero in the interval [−2, 0]. Then approximate the zero correct to two decimal places.

117. Continuity of a Sum If f and g are each continuous at c, prove that $f + g$ is continuous at c.
Hint: Use the Limit of a Sum Property.

118. Intermediate Value Theorem Suppose that the functions f and g are continuous on the interval $[a, b]$. If $f(a) < g(a)$ and $f(b) > g(b)$, prove that the graphs of $y = f(x)$ and $y = g(x)$ intersect somewhere between $x = a$ and $x = b$.
Hint: Define $h(x) = f(x) - g(x)$ and show $h(x) = 0$ for some x between a and b.

Challenge Problems

119. Intermediate Value Theorem Let $f(x) = \dfrac{1}{x-1} + \dfrac{1}{x-2}$. Use the Intermediate Value Theorem to prove that there is a real number c between 1 and 2 for which $f(c) = 0$.

120. Intermediate Value Theorem Prove that there is a real number c between 2.64 and 2.65 for which $c^2 = 7$.

121. Show that the existence of $\lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$ implies f is continuous at $x = a$.

122. Find constants A, B, C, and D so that the function below is continuous for all x. Sketch the graph of the resulting function.

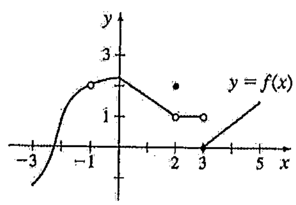
$$f(x) = \begin{cases} \dfrac{x^2 + x - 2}{x - 1} & \text{if } -\infty < x < 1 \\ A & \text{if } x = 1 \\ B(x - C)^2 & \text{if } 1 < x < 4 \\ D & \text{if } x = 4 \\ 2x - 8 & \text{if } 4 < x < \infty \end{cases}$$

123. Let f be a function for which $0 \le f(x) \le 1$ for all x in [0, 1]. If f is continuous on [0, 1], show that there exists at least one number c in [0, 1] such that $f(c) = c$.
Hint: Let $g(x) = x - f(x)$.

Preparing for the AP® Exam

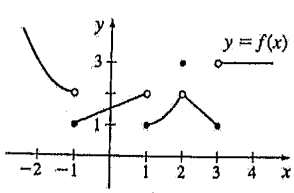
AP® Practice Problems

PAGE 103 **1.** The graph of a function f is shown below. Where on the open interval (−3, 5) is f discontinuous?



(A) 3 only (B) −1 and 3 only
(C) 1 only (D) −1, 2, and 3

PAGE 103 **2.** The graph of a function f is shown below.



If $\lim\limits_{x \to c} f(x)$ exists and f is not continuous at c, then c equals

(A) −1 (B) 1 (C) 2 (D) 3

PAGE 105 **3.** How should the function $f(x) = \dfrac{x^2 - 25}{x + 5}$ be defined at −5 to make it continuous at −5?

(A) −10 (B) −5 (C) 0 (D) 10

PAGE 105 **4.** If $f(x) = \begin{cases} \dfrac{\sqrt{2x+5} - \sqrt{x+15}}{x - 10} & \text{if } x \ne 10 \\ k & \text{if } x = 10 \end{cases}$
and if f is continuous at $x = 10$, then $k =$

(A) 0 (B) $\dfrac{1}{10}$ (C) 1 (D) 10

PAGE 103 **5.** If $\lim\limits_{x \to c} f(x) = L$, where L is a real number, which of the following must be true?

(A) f is defined at $x = c$. (B) f is continuous at $x = c$.
(C) $f(c) = L$. (D) None of the above.

PAGE 111 **6.** If $f(x) = x^3 - 2x + 5$ and if $f(c) = 0$ for only one real number c, then c is between

(A) −4 and −2 (B) −2 and −1 (C) −1 and 1 (D) 1 and 3

PAGE 111 **7.** The function f is continuous at all real numbers, and $f(-8) = 3$ and $f(-1) = -4$. If f has only one real zero (root), then which number x could satisfy $f(x) = 0$?

(A) −10 (B) −5 (C) 0 (D) 2

PAGE 104 8. Let f be the function defined by

$$f(x) = \begin{cases} x^2 & \text{if} \quad x < 0 \\ \sqrt{x} & \text{if} \quad 0 \le x < 1 \\ 2 - x & \text{if} \quad 1 \le x < 2 \\ x - 3 & \text{if} \quad x \ge 2 \end{cases}$$

For what numbers x is f NOT continuous?

(A) 1 only (B) 2 only
(C) 0 and 2 only (D) 1 and 2 only

PAGE 111 9. The function f is continuous on the closed interval $[-2, 6]$. If $f(-2) = 7$ and $f(6) = -1$, then the Intermediate Value Theorem guarantees that

(A) $f(0) = 0$.
(B) $f(c) = 2$ for at least one number c between -2 and 6.
(C) $f(c) = 0$ for at least one number c between -1 and 7.
(D) $-1 \le f(x) \le 7$ for all numbers in the closed interval $[-2, 6]$.

PAGE 111 10. The function f is continuous on the closed interval $[-2, 2]$. Several values of the function f are given in the table below.

x	-2	0	2
$f(x)$	3	c	2

The equation $f(x) = 1$ must have at least two solutions in the interval $[-2, 2]$ if $c =$

(A) $\frac{1}{2}$ (B) 1 (C) 3 (D) 4

PAGE 104 11. The function f is defined by $f(x) = \begin{cases} x^2 - 2x + 3 & \text{if } x \le 1 \\ -2x + 5 & \text{if } x > 1 \end{cases}$

(a) Is f continuous at $x = 1$?
(b) Use the definition of continuity to explain your answer.

1.4 Limits and Continuity of Trigonometric, Exponential, and Logarithmic Functions

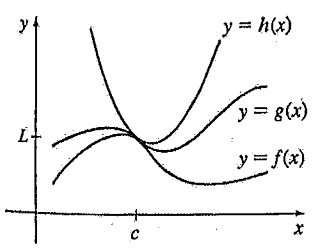
OBJECTIVES *When you finish this section, you should be able to:*

1 Use the Squeeze Theorem to find a limit (p. 117)
2 Find limits involving trigonometric functions (p. 119)
3 Determine where the trigonometric functions are continuous (p. 122)
4 Determine where an exponential or a logarithmic function is continuous (p. 124)

Until now we have found limits using the basic limits

$$\lim_{x \to c} A = A \qquad \lim_{x \to c} x = c$$

and algebraic properties of limits. But there are many limit problems that cannot be found by directly applying these techniques. To find such limits requires different results, such as the *Squeeze Theorem*,* or basic limits involving trigonometric and exponential functions.



$\lim_{x \to c} f(x) = L$, $\lim_{x \to c} h(x) = L$, $\lim_{x \to c} g(x) = L$

Figure 38

1 Use the Squeeze Theorem to Find a Limit

To use the Squeeze Theorem to find $\lim_{x \to c} g(x)$, we need to know, or be able to find, two functions f and h that "sandwich" the function g between them for all x close to c. That is, in some interval containing c, the functions f, g, and h satisfy the inequality

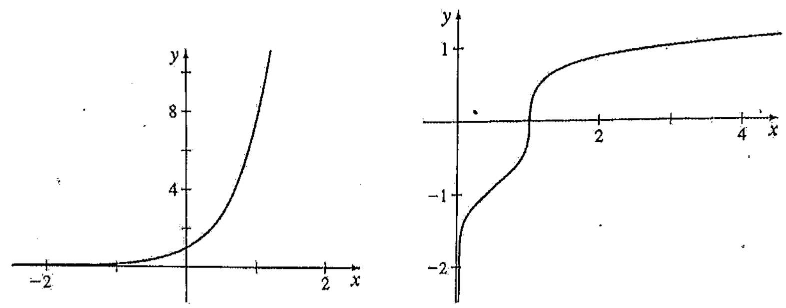
$$f(x) \le g(x) \le h(x)$$

Then if f and h have the same limit L as x approaches c, the function g is "squeezed" to the same limit L as x approaches c. See Figure 38.

We state the Squeeze Theorem here. The proof is given in Appendix B.

*The Squeeze Theorem is also known as the Sandwich Theorem and the Pinching Theorem.

Figures 47(a) and 47(b) illustrate the graphs of f and F.



(a) $f(x) = e^{2x}$ (b) $F(x) = \sqrt[3]{\ln x}$

Figure 47

NOW WORK Problem 45 and AP® Practice Problems 3, 5, and 9.

Summary Basic Limits

- $\lim_{x\to c} A = A$, where A is a constant • $\lim_{x\to c} x = c$ • $\lim_{x\to 0} \sin x = 0$ • $\lim_{x\to 0} \cos x = 1$
- $\lim_{x\to c} \sin x = \sin c$ • $\lim_{x\to c} \cos x = \cos c$ • $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$ • $\lim_{\theta\to 0} \frac{\cos\theta - 1}{\theta} = 0$
- $\lim_{x\to c} a^x = a^c, a > 0, a \neq 1$ • $\lim_{x\to c} \log_a x = \log_a c,\ c > 0, a > 0, a \neq 1$

1.4 Assess Your Understanding

Concepts and Vocabulary

1. $\lim_{x\to 0} \sin x =$ ________
2. *True or False* $\lim_{x\to 0} \frac{\cos x - 1}{x} = 1$
3. The Squeeze Theorem states that if the functions f, g, and h have the property $f(x) \le g(x) \le h(x)$ for all x in an open interval containing c, except possibly at c, and if $\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = L$, then $\lim_{x\to c} g(x) =$ ________.
4. *True or False* $f(x) = \csc x$ is continuous for all real numbers except $x = 0$.

Skill Building

In Problems 5–8, use the Squeeze Theorem to find each limit.

5. (PAGE 118) Suppose $-x^2 + 1 \le g(x) \le x^2 + 1$ for all x in an open interval containing 0. Find $\lim_{x\to 0} g(x)$.
6. Suppose $-(x-2)^2 - 3 \le g(x) \le (x-2)^2 - 3$ for all x in an open interval containing 2. Find $\lim_{x\to 2} g(x)$.
7. Suppose $\cos x \le g(x) \le 1$ for all x in an open interval containing 0. Find $\lim_{x\to 0} g(x)$.
8. Suppose $-x^2 + 1 \le g(x) \le \sec x$ for all x in an open interval containing 0. Find $\lim_{x\to 0} g(x)$.

In Problems 9–22, find each limit.

9. $\lim_{x\to 0}(x^3 + \sin x)$
10. $\lim_{x\to 0}(x^2 - \cos x)$
11. $\lim_{x\to \pi/3}(\cos x + \sin x)$
12. $\lim_{x\to \pi/3}(\sin x - \cos x)$
13. $\lim_{x\to 0} \frac{\cos x}{1 + \sin x}$
14. $\lim_{x\to 0} \frac{\sin x}{1 + \cos x}$
15. $\lim_{x\to 0} \frac{3}{1 + e^x}$
16. $\lim_{x\to 0} \frac{e^x - 1}{1 + e^x}$
17. $\lim_{x\to 0}(e^x \sin x)$
18. $\lim_{x\to 0}(e^{-x} \tan x)$
19. $\lim_{x\to 1} \ln\left(\frac{e^x}{x}\right)$
20. $\lim_{x\to 1} \ln\left(\frac{x}{e^x}\right)$
21. $\lim_{x\to 0} \frac{e^{2x}}{1 + e^x}$
22. $\lim_{x\to 0} \frac{1 - e^x}{1 - e^{2x}}$

In Problems 23–34, find each limit.

23. (PAGE 121) $\lim_{x\to 0} \frac{\sin(7x)}{x}$
24. $\lim_{x\to 0} \frac{\sin \frac{x}{3}}{x}$
25. (PAGE 121) $\lim_{\theta\to 0} \frac{\theta + 3\sin\theta}{2\theta}$
26. $\lim_{x\to 0} \frac{2x - 5\sin(3x)}{x}$
27. $\lim_{\theta\to 0} \frac{\sin\theta}{\theta + \tan\theta}$
28. $\lim_{\theta\to 0} \frac{\tan\theta}{\theta}$
29. $\lim_{\theta\to 0} \frac{5}{\theta \cdot \csc\theta}$
30. $\lim_{\theta\to 0} \frac{\sin(3\theta)}{\sin(2\theta)}$
31. (PAGE 122) $\lim_{\theta\to 0} \frac{1 - \cos^2\theta}{\theta}$
32. $\lim_{\theta\to 0} \frac{\cos(4\theta) - 1}{2\theta}$
33. $\lim_{\theta\to 0}(\theta \cdot \cot\theta)$
34. $\lim_{\theta\to 0}\left[\sin\theta \cdot \frac{\cot\theta - \csc\theta}{\theta}\right]$

In Problems 35–38, determine whether f is continuous at the number c.

35. $f(x)=\begin{cases} 3\cos x & \text{if } x<0 \\ 3 & \text{if } x=0 \\ x+3 & \text{if } x>0 \end{cases}$ at $c=0$

36. $f(x)=\begin{cases} \cos x & \text{if } x<0 \\ 0 & \text{if } x=0 \\ e^x & \text{if } x>0 \end{cases}$ at $c=0$

37. $f(\theta)=\begin{cases} \sin\theta & \text{if } \theta\le\frac{\pi}{4} \\ \cos\theta & \text{if } \theta>\frac{\pi}{4} \end{cases}$ at $c=\frac{\pi}{4}$

38. $f(x)=\begin{cases} \tan^{-1}x & \text{if } x<1 \\ \ln x & \text{if } x\ge 1 \end{cases}$ at $c=1$

In Problems 39–46, determine where f is continuous.

39. $f(x)=\sin\left(\frac{x^2-4x}{x-4}\right)$

40. $f(x)=\cos\left(\frac{x^2-5x+1}{2x}\right)$

41. $f(\theta)=\frac{1}{1+\sin\theta}$

42. $f(\theta)=\frac{1}{1+\cos^2\theta}$

43. $f(x)=\frac{\ln x}{x-3}$

44. $f(x)=\ln(x^2+1)$

45. $f(x)=e^{-x}\sin x$

46. $f(x)=\frac{e^x}{1+\sin^2 x}$

Applications and Extensions

In Problems 47–50, use the Squeeze Theorem to find each limit.

47. $\lim_{x\to 0}\left(x^2\sin\frac{1}{x}\right)$

48. $\lim_{x\to 0}\left[x\left(1-\cos\frac{1}{x}\right)\right]$

49. $\lim_{x\to 0}\left[x^2\left(1-\cos\frac{1}{x}\right)\right]$

50. $\lim_{x\to 0}\left[\sqrt{x^3+3x^2}\sin\frac{1}{x}\right]$

In Problems 51–54, show that each statement is true.

51. $\lim_{x\to 0}\frac{\sin(ax)}{\sin(bx)}=\frac{a}{b};\quad b\ne 0$

52. $\lim_{x\to 0}\frac{\cos(ax)}{\cos(bx)}=1$

53. $\lim_{x\to 0}\frac{\sin(ax)}{bx}=\frac{a}{b};\quad b\ne 0$

54. $\lim_{x\to 0}\frac{1-\cos(ax)}{bx}=0;\quad a\ne 0, b\ne 0$

55. Show that $\lim_{x\to 0}\frac{1-\cos x}{x^2}=\frac{1}{2}$.

56. Squeeze Theorem If $0\le f(x)\le 1$ for every number x, show that $\lim_{x\to 0}[x^2f(x)]=0$.

57. Squeeze Theorem If $0\le f(x)\le M$ for every x, show that $\lim_{x\to 0}[x^2f(x)]=0$.

58. The function $f(x)=\frac{\sin(\pi x)}{x}$ is not defined at 0. Decide how to define $f(0)$ so that f is continuous at 0.

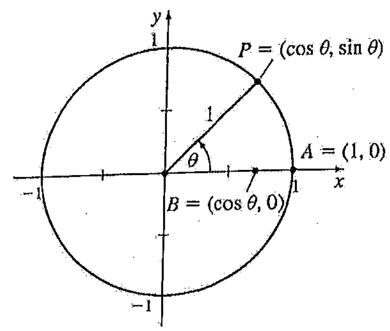
59. Define $f(0)$ and $f(1)$ so that the function $f(x)=\frac{\sin(\pi x)}{x(1-x)}$ is continuous on the interval [0, 1].

60. Is $f(x)=\begin{cases} \frac{\sin x}{x} & \text{if } x\ne 0 \\ 1 & \text{if } x=0 \end{cases}$ continuous at 0?

61. Is $f(x)=\begin{cases} \frac{1-\cos x}{x} & \text{if } x\ne 0 \\ 0 & \text{if } x=0 \end{cases}$ continuous at 0?

62. Squeeze Theorem Show that $\lim_{x\to 0}\left[x^n\sin\left(\frac{1}{x}\right)\right]=0$, where n is a positive integer. *Hint:* Look first at Problem 56.

63. Prove $\lim_{\theta\to 0}\sin\theta=0$. *Hint:* Use a unit circle as shown in the figure, first assuming $0<\theta<\frac{\pi}{2}$. Then use the fact that $\sin\theta$ is less than the length of the arc AP, and the Squeeze Theorem, to show that $\lim_{\theta\to 0^+}\sin\theta=0$. Then use a similar argument with $-\frac{\pi}{2}<\theta<0$ to show $\lim_{\theta\to 0^-}\sin\theta=0$.



64. Prove $\lim_{\theta\to 0}\cos\theta=1$. Use either the proof outlined in Problem 63 or a proof using the result $\lim_{\theta\to 0}\sin\theta=0$ and a Pythagorean identity.

65. Without using limits, explain how you can decide whether $f(x)=\cos(5x^3+2x^2-8x+1)$ is continuous.

66. Explain the Squeeze Theorem. Draw a graph to illustrate your explanation.

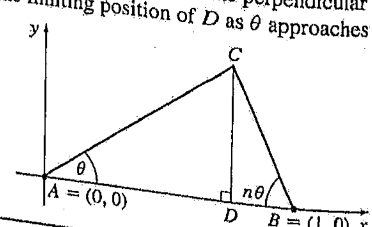
Challenge Problems

67. Use the Sum Formulas $\sin(a+b)=\sin a\cos b+\cos a\sin b$ and $\cos(a+b)=\cos a\cos b-\sin a\sin b$ to show that the sine function and cosine function are continuous on their domains.

68. Find $\lim_{x\to 0}\frac{\sin x^2}{x}$.

69. Squeeze Theorem If $f(x)=\begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$ show that $\lim_{x\to 0}[xf(x)]=0$.

70. Suppose points A and B with coordinates (0, 0) and (1, 0), respectively, are given. Let n be a number greater than 0, and let θ be an angle with the property $0<\theta<\frac{\pi}{1+n}$. Construct a triangle ABC where $\overline{AC}$ and $\overline{AB}$ form the angle θ, and $\overline{BA}$ form the angle $n\theta$ (see the figure below). Let D be the point of intersection of $\overline{AB}$ with the perpendicular from C to $\overline{AB}$. What is the limiting position of D as θ approaches 0?



Preparing for the AP® Exam

AP® Practice Problems

PAGE 121 1. The function $g(x) = \begin{cases} \dfrac{\sin(2x)}{2x} & \text{if } x \neq 0 \\ k & \text{if } x = 0 \end{cases}$ is continuous at $x = 0$. What is the value of k?

(A) 0 (B) $\frac{1}{2}$ (C) 1 (D) 2

PAGE 121 2. $\lim\limits_{x\to 0} \dfrac{\sin(4x)}{2x} =$

(A) 0 (B) $\frac{1}{2}$ (C) 1 (D) 2

PAGE 125 3. The function $f(x) = \begin{cases} x^3 + 2x^2 & \text{if } x \leq -2 \\ e^{2x+4} & \text{if } x > -2 \end{cases}$. Find $\lim\limits_{x\to -2} f(x)$ if it exists.

(A) 0 (B) 1 (C) 16 (D) The limit does not exist.

PAGE 121 4. $\lim\limits_{x\to 0} \dfrac{1 - \cos^2(3x)}{x^2} =$

(A) 0 (B) 1 (C) 3 (D) 9

PAGE 125 5. Which of the following functions are continuous for all real numbers x?

I. $f(x) = x^{1/3}$
II. $g(x) = \sec x$
III. $h(x) = e^{-x}$

(A) I only (B) I and II only
(C) I and III only (D) I, II, and III

PAGE 121 6. Find $\lim\limits_{x\to 0} \dfrac{1}{x \csc x}$ if it exists.

(A) −1 (B) 0 (C) 1 (D) The limit does not exist.

PAGE 121 7. $\lim\limits_{x\to \pi/3} \dfrac{\sin\left(x - \frac{\pi}{3}\right)}{x - \frac{\pi}{3}} =$

(A) $-\frac{\pi}{3}$ (B) 0 (C) 1 (D) $\frac{\pi}{3}$

PAGE 119 8. $\lim\limits_{x\to 0} \dfrac{1 - \cos x}{3 \sin^2 x} =$

(A) $\frac{1}{6}$ (B) $\frac{1}{3}$ (C) $\frac{1}{2}$ (D) 1

PAGE 125 9. If $f(x) = \begin{cases} \ln x & \text{if } 0 < x < 3 \\ (2x - 3)\ln 3 & \text{if } x \geq 3 \end{cases}$ then $\lim\limits_{x\to 3} f(x) =$

(A) ln 3 (B) 3 (C) ln 9 (D) The limit does not exist.

PAGE 121 10. $\lim\limits_{x\to 0} \dfrac{\tan(2x)}{3x} =$

(A) $\frac{1}{3}$ (B) $\frac{1}{2}$ (C) $\frac{2}{3}$ (D) 2

PAGE 118 11. $\lim\limits_{x\to 0} \left(x^3 \sin \dfrac{1}{x}\right) =$

(A) −1 (B) 0 (C) 1 (D) The limit does not exist.

1.5 Infinite Limits; Limits at Infinity; Asymptotes

OBJECTIVES *When you finish this section, you should be able to:*

RECALL The symbols ∞ (infinity) and −∞ (negative infinity) are not numbers. The symbol ∞ expresses unboundedness in the positive direction and −∞ expresses unboundedness in the negative direction.

We have described $\lim\limits_{x\to c} f(x) = L$ by saying if a function f is defined everywhere in an open interval containing c, except possibly at c, then the value $f(x)$ can be made as close as we please to L by choosing numbers x sufficiently close to c. Here c and L are real numbers. In this section, we extend the language of limits to allow c to be ∞ or −∞ (*limits at infinity*) and to allow L to be ∞ or −∞ (*infinite limits*). These limits are useful for locating *asymptotes* that aid in graphing some functions.

We begin with infinite limits.

1.5 Assess Your Understanding

Concepts and Vocabulary

1. *True or False* ∞ is a number.

2. (a) $\lim_{x\to 0^-} \frac{1}{x} =$ ____; (b) $\lim_{x\to 0^+} \frac{1}{x} =$ ____;
   (c) $\lim_{x\to 0^+} \ln x =$ ____

3. *True or False* The graph of a rational function has a vertical asymptote at every number x at which the function is not defined.

4. If $\lim_{x\to 4} f(x) = \infty$, then the line $x = 4$ is a(n) ____ asymptote of the graph of f.

5. (a) $\lim_{x\to\infty} \frac{1}{x} =$ ____; (b) $\lim_{x\to\infty} \frac{1}{x^2} =$ ____; (c) $\lim_{x\to\infty} \ln x =$ ____

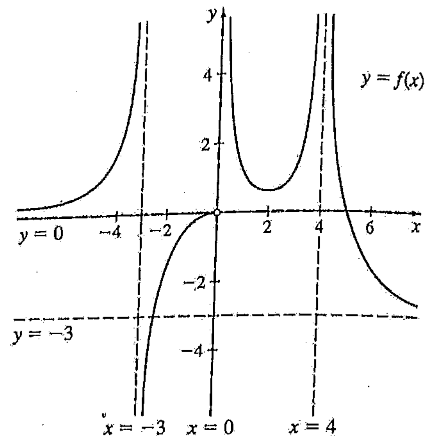
6. *True or False* $\lim_{x\to -\infty} 5 = 0$.

7. (a) $\lim_{x\to -\infty} e^x =$ ____; (b) $\lim_{x\to\infty} e^x =$ ____; (c) $\lim_{x\to\infty} e^{-x} =$ ____

8. *True or False* The graph of a function can have at most two horizontal asymptotes.

25. Identify all vertical asymptotes.

26. Identify all horizontal asymptotes.



Skill Building

In Problems 9–16, use the accompanying graph of $y = f(x)$.

9. Find $\lim_{x\to\infty} f(x)$.

10. Find $\lim_{x\to -\infty} f(x)$.

11. Find $\lim_{x\to -1^-} f(x)$.

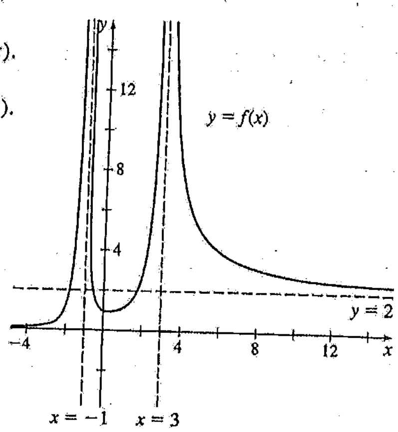
12. Find $\lim_{x\to -1^+} f(x)$.

13. Find $\lim_{x\to 3^-} f(x)$.

14. Find $\lim_{x\to 3^+} f(x)$.

15. Identify all vertical asymptotes.

16. Identify all horizontal asymptotes.



In Problems 17–26, use the graph (top, right) of $y = f(x)$.

17. Find $\lim_{x\to\infty} f(x)$.

18. Find $\lim_{x\to -\infty} f(x)$.

19. Find $\lim_{x\to -3^-} f(x)$.

20. Find $\lim_{x\to -3^+} f(x)$.

21. Find $\lim_{x\to 0^-} f(x)$.

22. Find $\lim_{x\to 0^+} f(x)$.

23. Find $\lim_{x\to 4^-} f(x)$.

24. Find $\lim_{x\to 4^+} f(x)$.

In Problems 27–42, find each limit.

27. $\lim_{x\to 2^-} \frac{3x}{x-2}$

28. $\lim_{x\to -4^+} \frac{2x+1}{x+4}$

29. $\lim_{x\to 2^+} \frac{5}{x^2-4}$

30. $\lim_{x\to 1^-} \frac{2x}{x^3-1}$

31. $\lim_{x\to -1^+} \frac{5x+3}{x(x+1)}$

32. $\lim_{x\to 0^-} \frac{5x+3}{5x(x-1)}$

33. $\lim_{x\to -3^-} \frac{1}{x^2-9}$

34. $\lim_{x\to 2^+} \frac{x}{x^2-4}$

35. $\lim_{x\to 3} \frac{1-x}{(3-x)^2}$

36. $\lim_{x\to -1} \frac{x+2}{(x+1)^2}$

37. $\lim_{x\to \pi^-} \cot x$

38. $\lim_{x\to -\pi/2^-} \tan x$

39. $\lim_{x\to \pi/2^+} \csc(2x)$

40. $\lim_{x\to -\pi/2^-} \sec x$

41. $\lim_{x\to -1^+} \ln(x+1)$

42. $\lim_{x\to 1^+} \ln(x-1)$

In Problems 43–60, find each limit.

43. $\lim_{x\to\infty} \frac{5}{x^2+4}$

44. $\lim_{x\to -\infty} \frac{1}{x^2-9}$

45. $\lim_{x\to\infty} \frac{2x+4}{5x}$

46. $\lim_{x\to\infty} \frac{x+1}{x}$

47. $\lim_{x\to\infty} \frac{x^3+x^2+2x-1}{x^3+x+1}$

48. $\lim_{x\to\infty} \frac{2x^2-5x+2}{5x^2+7x-1}$

49. $\lim_{x\to -\infty} \frac{x^2+1}{x^3-1}$

50. $\lim_{x\to\infty} \frac{x^2-2x+1}{x^3+5x+4}$

51. $\lim_{x\to\infty}\left(\frac{3x}{2x+5}-\frac{x^2+1}{4x^2+8}\right)$

52. $\lim_{x\to\infty}\left(\frac{1}{x^2+x+4}-\frac{x+1}{3x-1}\right)$

53. $\lim_{x\to-\infty}\left(2e^x\cdot\frac{5x+1}{3x}\right)$

54. $\lim_{x\to-\infty}\left(e^x\cdot\frac{x^2+x-3}{2x^3-x^2}\right)$

55. $\lim_{x\to\infty}\frac{\sqrt{x}+2}{3x-4}$

56. $\lim_{x\to\infty}\frac{\sqrt{3x^3+2}}{x^2+6}$

[PAGE 134] 57. $\lim_{x\to\infty}\sqrt{\frac{3x^2-1}{x^2+4}}$

58. $\lim_{x\to\infty}\left(\frac{16x^3+2x+1}{2x^3+3x}\right)^{2/3}$

[PAGE 136] 59. $\lim_{x\to-\infty}\frac{5x^3}{x^2+1}$

60. $\lim_{x\to-\infty}\frac{x^4}{x-2}$

In Problems 61–66, find any horizontal or vertical asymptotes of the graph of f.

61. $f(x)=3+\frac{1}{x}$

62. $f(x)=2-\frac{1}{x^2}$

[PAGE 131] [PAGE 138] 63. $f(x)=\frac{x^2}{x^2-1}$

64. $f(x)=\frac{2x^2-1}{x^2-1}$

65. $f(x)=\frac{\sqrt{2x^2-x+10}}{2x-3}$

66. $f(x)=\frac{\sqrt[3]{x^2+5x}}{x-6}$

In Problems 67–72, for each rational function R:

(a) Find the domain of R.

(b) Find any horizontal asymptotes of R.

(c) Find any vertical asymptotes of R.

(d) Discuss the behavior of the graph at numbers where R is not defined.

67. $R(x)=\frac{-2x^2+1}{2x^3+4x^2}$

68. $R(x)=\frac{x^3}{x^4-1}$

[PAGE 139] 69. $R(x)=\frac{x^2+3x-10}{2x^2-7x+6}$

70. $R(x)=\frac{x(x-1)^2}{(x+3)^3}$

71. $R(x)=\frac{x^3-1}{x-x^2}$

72. $R(x)=\frac{4x^5}{x^3-1}$

Applications and Extensions

In Problems 73 and 74:

(a) Sketch a graph of a function f that has the given properties.

(b) Define a function that describes the graph.

73. $f(3)=0$, $\lim_{x\to\infty}f(x)=1$, $\lim_{x\to-\infty}f(x)=1$, $\lim_{x\to1^-}f(x)=\infty$, $\lim_{x\to1^+}f(x)=-\infty$

74. $f(2)=0$, $\lim_{x\to\infty}f(x)=0$, $\lim_{x\to-\infty}f(x)=0$, $\lim_{x\to0}f(x)=\infty$, $\lim_{x\to5^-}f(x)=-\infty$, $\lim_{x\to5^+}f(x)=\infty$

75. **Newton's Law of Cooling** Suppose an object is heated to a temperature u_0. Then at time $t=0$, the object is put into a medium with a constant lower temperature T causing the object to cool. **Newton's Law of Cooling** states that the temperature u of the object at time t is given by $u=u(t)=(u_0-T)e^{kt}+T$, where $k<0$ is a constant.

(a) Find $\lim_{t\to\infty}u(t)$. Is this the value you expected? Explain why or why not.

(b) Find $\lim_{t\to0^+}u(t)$. Is this the value you expected? Explain why or why not.

Source: Submitted by the students of Millikin University.

76. **Environment** A utility company burns coal to generate electricity. The cost C, in dollars, of removing $p\%$ of the pollutants emitted into the air is

$$C=\frac{70{,}000p}{100-p},\qquad 0\le p<100$$

Find the cost of removing:

(a) 45% of the pollutants.

(b) 90% of the pollutants.

(c) Find $\lim_{p\to100^-}C$.

(d) Interpret the answer found in (c).

77. **Pollution Control** The cost C, in thousands of dollars, to remove a pollutant from a lake is

$$C(x)=\frac{5x}{100-x},\qquad 0\le x<100$$

where x is the percent of pollutant removed. Find $\lim_{x\to100^-}C(x)$. Interpret your answer.

78. **Population Model** A rare species of insect was discovered in the Amazon Rain Forest. To protect the species, entomologists declared the insect endangered and transferred 25 insects to a protected area. The population P of the new colony t days after the transfer is

$$P(t)=\frac{50(1+0.5t)}{2+0.01t}$$

(a) What is the projected size of the colony after 1 year (365 days)?

(b) What is the largest population that the protected area can sustain? That is, find $\lim_{t\to\infty}P(t)$.

(c) Graph the population P as a function of time t.

(d) Use the graph from (c) to describe the regeneration of the insect population. Does the graph support the answer to (b)?

[PAGE 138] 79. **Population of an Endangered Species** Often environmentalists capture several members of an endangered species and transport them to a controlled environment where they can produce offspring and regenerate their population. Suppose six American bald eagles are captured, tagged, transported to Montana, and set free. Based on past experience, the environmentalists expect the population to grow according to the model

$$P(t)=\frac{500}{1+82.3e^{-0.162t}}$$

where t is measured in years.

(continued on next page)

(a) If the model is correct, how many bald eagles can the environment sustain? That is, find $\lim_{t\to\infty} P(t)$.

(b) Graph the population P as a function of time t.

(c) Use the graph from (b) to describe the growth of the bald eagle population. Does the graph support the answer to (a)?

80. Hailstones Hailstones typically originate at an altitude of about 3000 meters (m). If a hailstone falls from 3000 m with no air resistance, its speed when it hits the ground would be about 240 meters/second (m/s), which is 540 miles/hour (mi/h)! That would be deadly! But air resistance slows the hailstone considerably. Using a simple model of air resistance, the speed $v = v(t)$ of a hailstone of mass m as a function of time t is given by $v(t) = \frac{mg}{k}(1 - e^{-kt/m})$ m/s, where $g = 9.8$ m/s^2 and k is a constant that depends on the size of the hailstone, its mass, and the conditions of the air. For a hailstone with a diameter $d = 1$ centimeter (cm) and mass $m = 4.8 \times 10^{-4}$ kg, k has been measured to be 3.4×10^{-4} kg/s.

(a) Determine the limiting speed of the hailstone by finding $\lim_{t\to\infty} v(t)$. Express your answer in meters per second and miles per hour, using the fact that 1 mi/h $\approx$ 0.447 m/s. This speed is called the **terminal speed** of the hailstone.

(b) Graph $v = v(t)$. Does the graph support the answer to (a)?

81. Damped Harmonic Motion The motion of a spring is given by the function

$$x(t) = 1.2e^{-t/2}\cos t + 2.4e^{-t/2}\sin t$$

where x is the distance in meters from the the equilibrium position and t is the time in seconds.

(a) Graph $y = x(t)$. What is $\lim_{t\to\infty} x(t)$, as suggested by the graph?

(b) Find $\lim_{t\to\infty} x(t)$.

(c) Compare the results of (a) and (b). Is the answer to (b) supported by the graph in (a)?

82. Decomposition of Chlorine in a Pool Under certain water conditions, the free chlorine (hypochlorous acid, HOCl) in a swimming pool decomposes according to the law of uninhibited decay, $C = C(t) = C(0)e^{kt}$, where $C = C(t)$ is the amount (in parts per million, ppm) of free chlorine present at time t (in hours) and k is a negative number that represents the rate of decomposition. After shocking his pool, Ben immediately tested the water and found the concentration of free chlorine to be $C_0 = C(0) = 2.5$ ppm. Twenty-four hours later, Ben tested the water again and found the amount of free chlorine to be 2.2 ppm.

(a) What amount of free chlorine will be left after 72 hours?

(b) When the free chlorine reaches 1.0 ppm, the pool should be shocked again. How long can Ben go before he must shock the pool again?

(c) Find $\lim_{t\to\infty} C(t)$.

(d) Interpret the answer found in (c).

83. Decomposition of Sucrose Reacting with water in an acidic solution at 35°C, the amount A of sucrose ($C_{12}H_{22}O_{11}$) decomposes into glucose ($C_6H_{12}O_6$) and fructose ($C_6H_{12}O_6$) according to the law of uninhibited decay $A = A(t) = A(0)e^{kt}$, where $A = A(t)$ is the amount (in moles) of sucrose present at time t (in minutes) and k is a negative number that represents the rate of decomposition. An initial amount $A_0 = A(0) = 0.40$ mole of sucrose decomposes to 0.36 mole in 30 minutes.

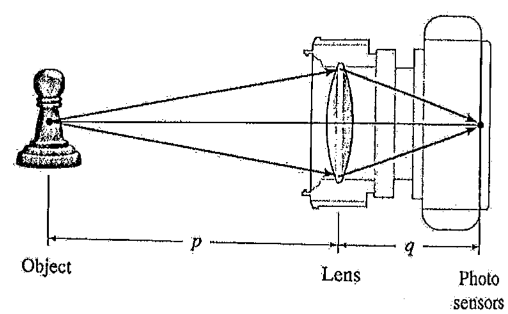
(a) How much sucrose will remain after 2 hours?

(b) How long will it take until 0.10 mole of sucrose remains?

(c) Find $\lim_{t\to\infty} A(t)$.

(d) Interpret the answer found in (c).

84. Macrophotography A camera lens can be approximated by a thin lens. A thin lens of focal length f obeys the thin-lens equation $\frac{1}{f} = \frac{1}{p} + \frac{1}{q}$, where $p > f$ is the distance from the lens to the object being photographed and q is the distance from the lens to the image formed by the lens. See the figure below. To photograph an object, the object's image must be formed on the photo sensors of the camera, which can only occur if q is positive.



(a) Is the distance q of the image from the lens continuous as the distance of the object being photographed approaches the focal length f of the lens? *Hint*: First solve the thin-lens equation for q and then find $\lim_{p\to f^+} q$.

(b) Use the result from (a) to explain why a camera (or any lens) cannot focus on an object placed close to its focal length.

In Problems 85 and 86, find conditions on a, b, c, and d so that the graph of f has no horizontal or vertical asymptotes.

85. $f(x) = \dfrac{ax^3 + b}{cx^4 + d}$ **86.** $f(x) = \dfrac{ax + b}{cx + d}$

87. Explain why the following properties are true. Give an example of each.

(a) If n is an even positive integer, then $\lim_{x\to c} \dfrac{1}{(x-c)^n} = \infty$.

(b) If n is an odd positive integer, then $\lim_{x\to c^-} \dfrac{1}{(x-c)^n} = -\infty$.

(c) If n is an odd positive integer, then $\lim_{x\to c^+} \dfrac{1}{(x-c)^n} = \infty$.

88. Explain why a rational function, whose numerator and denominator have no common zeros, will have vertical asymptotes at each point of discontinuity.

89. Explain why a polynomial function of degree 1 or higher cannot have any asymptotes.

90. If P and Q are polynomials of degree m and n, respectively, discuss $\lim\limits_{x\to\infty}\dfrac{P(x)}{Q(x)}$ when:

(a) $m>n$ (b) $m=n$ (c) $m<n$

91. (a) Use a table to investigate $\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^x$.

CAS (b) Find $\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^x$.

(c) Compare the results from (a) and (b). Explain the possible causes of any discrepancy.

92. Prove that $\lim\limits_{x\to\pm\infty}\dfrac{k}{x^p}=0$, for any real number k, and $p>0$, provided x^p is defined if $x<0$.

Challenge Problems

93. **Kinetic Energy** At low speeds the kinetic energy K, that is, the energy due to the motion of an object of mass m and speed v, is given by the formula $K=K(v)=\frac{1}{2}mv^2$. But this formula is only an approximation to the general formula and works only for speeds much less than the speed of light, c. The general formula, which holds for all speeds, is

$$K_{\text{gen}}(v)=mc^2\left[\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}-1\right]$$

(a) As an object is accelerated closer and closer to the speed of light, what does its kinetic energy K_{gen} approach?

(b) What does the result suggest about the possibility of reaching the speed of light?

94. $\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)=1$, but $\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^x>1$. Discuss why the property $\lim\limits_{x\to\infty}[f(x)]^n=\left[\lim\limits_{x\to\infty}f(x)\right]^n$ cannot be used to find the second limit.

Preparing for the AP® Exam

AP® Practice Problems

PAGE 138 1. For $x>0$, the line $y=1$ is an asymptote of the graph of a function f. Which of the following statements must be true?

(A) $f(x)\neq 1$ for $x>0$ (B) $\lim\limits_{x\to 1}f(x)=\infty$

(C) $\lim\limits_{x\to\infty}f(x)=1$ (D) $\lim\limits_{x\to-\infty}f(x)=1$

PAGE 134 PAGE 136 2. $\lim\limits_{x\to\infty}\dfrac{3x^3+4x^2-x+10}{2x^4-x^3+2x^2-2}=$

(A) -5 (B) 0 (C) $\dfrac{3}{2}$ (D) ∞

PAGE 133 PAGE 136 3. $\lim\limits_{x\to\infty}\dfrac{5x^3-x}{8-x^3}=$

(A) -5 (B) $\dfrac{5}{8}$ (C) 5 (D) ∞

PAGE 138 4. Find all the horizontal asymptotes of the graph of $y=\dfrac{2+3^x}{4-3^x}$.

(A) $y=-1$ only (B) $y=\dfrac{1}{2}$ only

(C) $y=-1$ and $y=0$ (D) $y=-1$ and $y=\dfrac{1}{2}$

PAGE 131 5. Find all the vertical asymptotes of the graph of

$$r(x)=\frac{x^2+5x+6}{x^3-4x}$$

(A) $x=0$ and $x=-2$ (B) $x=0$ and $x=2$

(C) $x=-2$ and $x=2$ (D) $x=0$, $x=-2$ and $x=2$

PAGE 134 6. $\lim\limits_{x\to-\infty}\dfrac{\sqrt{8x^2-4x}}{x+2}=$

(A) $-\infty$ (B) $-2\sqrt{2}$ (C) 4 (D) $2\sqrt{2}$

PAGE 138 7. The graph of which of the following functions has an asymptote of $y=1$?

(A) $y=\cos x$ (B) $y=\dfrac{x-1}{x}$

(C) $y=e^{-x}$ (D) $y=\ln x$

PAGE 139 8. If the graph of $f(x)=\dfrac{ax-b}{x+c}$ has a vertical asymptote $x=-5$ and horizontal asymptote $y=-3$, then $a+c=$

(A) -8 (B) -2 (C) $\dfrac{3}{5}$ (D) 2

PAGE 130 9. $\lim\limits_{x\to 1^-}\dfrac{x}{\ln x}=$

(A) $-\infty$ (B) -1 (C) 1 (D) ∞

PAGE 139 10. The function $f(x)=\dfrac{2x}{|x|-1}$ has

(A) no vertical asymptote and one horizontal asymptote.
(B) one vertical asymptote and one horizontal asymptote.
(C) two vertical asymptotes and one horizontal asymptote.
(D) two vertical asymptotes and two horizontal asymptotes.

PAGE 130 11. $\lim\limits_{x\to 2^-}\dfrac{5x+1}{2x-4}=$

(A) $-\infty$ (B) $-\dfrac{5}{2}$ (C) $\dfrac{5}{2}$ (D) ∞

Preparing for the AP® Exam

AP® REVIEW PROBLEMS: CHAPTER 1

1. Which line is an asymptote to the graph of $f(x)=e^{2x}$?

   (A) $x=0$ (B) $y=0$ (C) $y=2$ (D) $y=x$

2. $\lim_{x\to\infty}\dfrac{3x^3+4x}{5-2x^4}=$

   (A) $-\dfrac{3}{2}$ (B) 0 (C) $\dfrac{3}{5}$ (D) $\dfrac{3}{2}$

3. If $f(x)=5x^3-1$, then $\lim_{x\to 0}\dfrac{f(x)-f(0)}{x^3}=$

   (A) 0 (B) 1 (C) 5 (D) The limit does not exist.

4. $\lim_{\theta\to 0}\dfrac{\theta^2}{1-\cos\theta}=$

   (A) 0 (B) 1 (C) 2 (D) 4

5. The table gives values of three functions:

x	−0.15	−0.1	−0.05	0	0.05	0.1	0.15
$f(x)$	0.075	0.05	0.025	−4	0.025	0.05	0.075
$g(x)$	−8.3	−8.2	−8.1	undefined	−7.9	−7.8	−7.7
$h(x)$	1.997	1.99	1.9975	1	1.005	1.02	1.045

   For which of these functions does the table suggest that the limit as x approaches 0 exists?

   (A) f only (B) h only
   (C) f and g only (D) f and h only

6. If a function f is continuous on the closed interval [1, 4] and if $f(1)=6$ and $f(4)=-1$, then which of the following must be true?

   (A) $f(c)=0$ for some number c in the open interval $(-1, 6)$.
   (B) $f(c)=1$ for some number c in the open interval $(1, 4)$.
   (C) $f(c)=1$ for some number c in the open interval $(-1, 6)$.
   (D) $f(c)\neq -2$ for any number c in the open interval $(1, 4)$.

7. Which are the equations of the asymptotes of the graph of the function $f(x)=\dfrac{x}{x(x^2-9)}$?

   (A) $x=-3, x=0, x=3, y=0$
   (B) $x=-3, x=0, x=3, y=1$
   (C) $x=-3, x=3, y=0$
   (D) $x=-3, x=3, y=1$

8. Find the value of k that makes the function

   $$f(x)=\begin{cases} x^2+2 & \text{if } x\le -1 \\ kx+4 & \text{if } x>-1 \end{cases}$$

   continuous for all real numbers.

   (A) −3 (B) −1 (C) 1 (D) 3

9. An odd function f is continuous for all real numbers. If $\lim_{x\to\infty} f(x)=-2$, then which of the statements must be true?

   I. f has no vertical asymptotes.
   II. $\lim_{x\to 0} f(x)=0$
   III. The horizontal asymptotes of the graph of f are $y=-2$ and $y=2$.

   (A) I only (B) III only
   (C) I and III only (D) I, II, and III

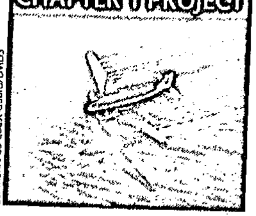
10. $\lim_{x\to 0}\dfrac{\sin(2x)}{\tan(3x)}=$

    (A) 0 (B) $\dfrac{2}{3}$ (C) 1 (D) $\dfrac{3}{2}$

11. Suppose the function f is continuous for all real numbers. If $f(x)=\dfrac{x^3+8}{x+2}$ when $x\neq -2$, then $f(-2)=$

    (A) 0 (B) 4 (C) 8 (D) 12

CHAPTER 1 PROJECT



Petty Officer 3rd Class Stephen Lehmann/US Coast Guard/DVIDS

Pollution in Clear Lake

The Toxic Waste Disposal Company (TWDC) specializes in the disposal of a particularly dangerous pollutant, Agent Yellow (AY). Unfortunately, instead of safely disposing of this pollutant, the company simply dumped AY in (formerly) Clear Lake. Fortunately, they have been caught and are now defending themselves in court.

The facts below are not in dispute. As a result of TWDC's activity, the current concentration of AY in Clear Lake is now 10 ppm (parts per million). Clear Lake is part of a chain of rivers and lakes. Fresh water flows into Clear Lake, and the contaminated water flows downstream from it. The Department of Environmental Protection estimates that the level of contamination in Clear Lake will fall by 20% each year. These facts can be modeled as

$$p(0)=10 \qquad p(t+1)=0.80p(t)$$

where $p=p(t)$, measured in ppm, is the concentration of pollutants in the lake at time t, in years.

1. Explain how the above equations model the facts.
2. Create a table showing the values of t for $t=0, 1, 2, \ldots, 20$.
3. Show that $p(t)=10(0.8)^t$.
4. Use technology to graph $p=p(t)$.
5. What is $\lim\limits_{t\to\infty} p(t)$?

Lawyers for TWDC looked at the results in 1–5 above and argued that their client has not done any real damage. They concluded that Clear Lake would eventually return to its former clear and unpolluted state. They even called in a mathematician, who wrote the following on a blackboard:

$$\lim_{t\to\infty} p(t)=0$$

and explained that this bit of mathematics means, descriptively, that after many years the concentration of AY will, indeed, be close to zero.

Concerned citizens booed the mathematician's testimony. Fortunately, one of them has taken calculus and knows a little bit about limits. She noted that, although "after many years the concentration of AY will approach zero," the townspeople like to swim in Clear Lake and state regulations prohibit swimming unless the concentration of AY is below 2 ppm. She proposed a fine of \$100,000 per year for each full year that the lake is unsafe for swimming. She also questioned the mathematician, saying, "Your testimony was correct as far as it went, but I remember from studying calculus that talking about the eventual concentration of AY after many, many years is only a small part of the story. The more precise meaning of your statement $\lim\limits_{t\to\infty} p(t)=0$ is that given some tolerance T for the concentration of AY, there is some time N (which may be far in the future) so that for all $t>N$, $p(t)<T$."

6. Using a table or a graph for $p=p(t)$, find N so that if $t>N$, then $p(t)<2$.
7. How much is the fine?

Her words were greeted by applause. The town manager sprang to his feet and noted that although a tolerance of 2 ppm was fine for swimming, the town used Clear Lake for its drinking water and until the concentration of AY dropped below 0.5 ppm, the water would be unsafe for drinking. He proposed a fine of \$200,000 per year for each full year the water was unfit for drinking.

8. Using a table or a graph for $p=p(t)$, find N so that if $t>N$, then $p(t)<0.5$.
9. How much is the fine?
10. How would you find if you were on the jury trying TWDC? If the jury found TWDC guilty, what fine would you recommend? Explain your answers.

Chapter Review

THINGS TO KNOW

1.1 Limits of Functions Using Numerical and Graphical Techniques

- Slope of a secant line: $m_{\text{sec}}=\dfrac{f(x)-f(c)}{x-c}$ (p. 78)
- Slope of a tangent line: $m_{\tan}=\lim\limits_{x\to c}\dfrac{f(x)-f(c)}{x-c}$ (p. 78)
- $\lim\limits_{x\to c} f(x)=L$: read, "The limit as x approaches c of $f(x)$ is equal to the number L." (p. 79)
- $\lim\limits_{x\to c} f(x)=L$: interpreted as, "The value $f(x)$ can be made as close as we please to L, for x sufficiently close to c, but not equal to c." (p. 79)
- One-sided limits (p. 80)
- The limit L of a function $y=f(x)$ as x approaches a number c does not depend on the value of f at c. (p. 82)
- The limit L of a function $y=f(x)$ as x approaches a number c is unique. A function cannot have more than one limit as x approaches c. (p. 82)
- The limit L of a function $y=f(x)$ as x approaches a number c exists if and only if both one-sided limits exist at c and both one-sided limits are equal. That is, $\lim\limits_{x\to c} f(x)=L$ if and only if $\lim\limits_{x\to c^-} f(x)=\lim\limits_{x\to c^+} f(x)=L$. (p. 83)

1.2 Analytic Techniques for Finding Limits of Functions

Basic Limits

- $\lim\limits_{x\to c} A=A$, A a constant (p. 90)
- $\lim\limits_{x\to c} x=c$, c a real number (p. 90)

Properties of Limits If f and g are functions for which $\lim_{x \to c} f(x)$ and $\lim_{x \to c} g(x)$ both exist and if k is any real number, then:

- $\lim_{x \to c}[f(x) \pm g(x)] = \lim_{x \to c} f(x) \pm \lim_{x \to c} g(x)$ (pp. 90, 91)
- $\lim_{x \to c}[f(x) \cdot g(x)] = \lim_{x \to c} f(x) \cdot \lim_{x \to c} g(x)$ (p. 91)
- $\lim_{x \to c}[kg(x)] = k \lim_{x \to c} g(x)$ (p. 92)
- $\lim_{x \to c}[f(x)]^n = \left[\lim_{x \to c} f(x)\right]^n$, $n \geq 2$ is an integer (p. 93)
- $\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to c} f(x)}$, provided $f(x) > 0$ if n is even (p. 94)
- $\lim_{x \to c}[f(x)]^{m/n} = \left[\lim_{x \to c} f(x)\right]^{m/n}$, provided $[f(x)]^{m/n}$ is defined for positive integers m and n (p. 94)
- $\lim_{x \to c}\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}$, provided $\lim_{x \to c} g(x) \neq 0$ (p. 95)
- If P is a polynomial function, then $\lim_{x \to c} P(x) = P(c)$. (p. 95)
- If R is a rational function and if c is in the domain of R, then $\lim_{x \to c} R(x) = R(c)$. (p. 96)

1.3 Continuity

Definitions

- Continuity at a number (p. 103)
- Removable discontinuity (p. 105)
- One-sided continuity at a number (p. 105)
- Continuity on an interval (p. 106)
- Continuity on a domain (p. 107)

Properties of Continuity

- A polynomial function is continuous on its domain, all real numbers. (p. 107)
- A rational function is continuous on its domain. (p. 107)
- If the functions f and g are continuous at a number c, and if k is a real number, then the functions $f+g$, $f-g$, $f \cdot g$, and kf are also continuous at c. If $g(c) \neq 0$, the function $\dfrac{f}{g}$ is continuous at c. (p. 108)
- If a function g is continuous at c and a function f is continuous at $g(c)$, then the composite function $(f \circ g)(x) = f(g(x))$ is continuous at c. (p. 109)
- If f is a one-to-one function that is continuous on its domain, then its inverse function f^{-1} is also continuous on its domain. (p. 110)

The Intermediate Value Theorem Let f be a function that is continuous on a closed interval $[a, b]$ with $f(a) \neq f(b)$. If N is any number between $f(a)$ and $f(b)$, then there is at least one number c in the open interval (a, b) for which $f(c) = N$. (p. 110)

1.4 Limits and Continuity of Trigonometric, Exponential, and Logarithmic Functions

Basic Limits

- $\lim_{\theta \to 0} \dfrac{\sin\theta}{\theta} = 1$ (p. 119)
- $\lim_{\theta \to 0} \dfrac{\cos\theta - 1}{\theta} = 0$ (p. 122)
- $\lim_{x \to c} \sin x = \sin c$ (p. 123)
- $\lim_{x \to c} \cos x = \cos c$ (p. 123)
- $\lim_{x \to c} a^x = a^c$; $a > 0$, $a \neq 1$ (p. 124)
- $\lim_{x \to c} \log_a x = \log_a c$; $a > 0$, $a \neq 1$, and $c > 0$ (p. 124)

Squeeze Theorem If the functions f, g, and h have the property that for all x in an open interval containing c, except possibly at c, $f(x) \leq g(x) \leq h(x)$, and if $\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L$, then $\lim_{x \to c} g(x) = L$. (p. 119)

Properties of Continuity

- The six trigonometric functions are continuous on their domains. (p. 123)
- The six inverse trigonometric functions are continuous on their domains. (p. 123)
- An exponential function is continuous on its domain, all real numbers. (p. 124)
- A logarithmic function is continuous on its domain, all positive real numbers. (p. 124)

1.5 Infinite Limits; Limits at Infinity; Asymptotes

Basic Limits

- $\lim_{x \to 0^-} \dfrac{1}{x} = -\infty$ $\lim_{x \to 0^+} \dfrac{1}{x} = \infty$ (p. 128)
- $\lim_{x \to 0} \dfrac{1}{x^2} = \infty$ (p. 128)
- $\lim_{x \to 0^+} \ln x = -\infty$ (p. 129)
- $\lim_{x \to \infty} \dfrac{1}{x} = 0$ $\lim_{x \to -\infty} \dfrac{1}{x} = 0$ (p. 132)
- $\lim_{x \to \infty} \ln x = \infty$ (p. 136)
- $\lim_{x \to -\infty} e^x = 0$ $\lim_{x \to \infty} e^x = \infty$ (p. 136)

Definitions

- Vertical asymptote (p. 131)
- Horizontal asymptote (p. 138)

Properties of Limits at Infinity (p. 132): If k is a real number, $n \geq 2$ is an integer, and the functions f and g approach real numbers as $x \to \infty$, then:

- $\lim_{x \to \infty} A = A$, where A is a constant
- $\lim_{x \to \infty} [kf(x)] = k \lim_{x \to \infty} f(x)$
- $\lim_{x \to \infty} [f(x) \pm g(x)] = \lim_{x \to \infty} f(x) \pm \lim_{x \to \infty} g(x)$
- $\lim_{x \to \infty} [f(x)g(x)] = \left[\lim_{x \to \infty} f(x)\right]\left[\lim_{x \to \infty} g(x)\right]$
- $\lim_{x \to \infty} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to \infty} f(x)}{\lim_{x \to \infty} g(x)}$ provided $\lim_{x \to \infty} g(x) \neq 0$
- $\lim_{x \to \infty} [f(x)]^n = \left[\lim_{x \to \infty} f(x)\right]^n$
- $\lim_{x \to \infty} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to \infty} f(x)}$, where $f(x) > 0$ if n is even

1.6 The ε-δ Definition of a Limit

Definitions

- Limit of a Function (p. 145)
- Limit at Infinity (p. 150)
- Infinite Limit (p. 151)
- Infinite Limit at Infinity (p. 151)

Properties of Limits

- If $\lim_{x \to c} f(x) > 0$, then there is an open interval around c, for which $f(x) > 0$ everywhere in the interval, except possibly at c. (p. 150)
- If $\lim_{x \to c} f(x) < 0$, then there is an open interval around c, for which $f(x) < 0$ everywhere in the interval, except possibly at c. (p. 150)

OBJECTIVES

Section	You should be able to ...	Example	Review Exercises	Preparing for the AP® Exam: AP® Review Problems
1.1	1 Discuss the idea of a limit (p. 79)	1	4	
	2 Investigate a limit using a table (p. 80)	2–4	1	5
	3 Investigate a limit using a graph (p. 81)	5–8	2, 3	
1.2	1 Find the limit of a sum, a difference, and a product (p. 90)	1–6	8, 10, 12, 14, 22, 26, 29, 30, 47, 48	
	2 Find the limit of a power and the limit of a root (p. 93)	7–9	11, 18, 28, 55	
	3 Find the limit of a polynomial (p. 95)	10	10, 22	
	4 Find the limit of a quotient (p. 95)	11–14	13–17, 19–21, 23–25, 27, 56	3
	5 Find the limit of an average rate of change (p. 98)	15	37	
	6 Find the limit of a difference quotient (p. 98)	16	5, 6, 49	
1.3	1 Determine whether a function is continuous at a number (p. 103)	1–4	31–36	11
	2 Determine intervals on which a function is continuous (p. 106)	5, 6	39–42	8
	3 Use properties of continuity (p. 108)	7, 8	39–42	
	4 Use the Intermediate Value Theorem (p. 110)	9, 10	38, 44–46	6
1.4	1 Use the Squeeze Theorem to find a limit (p. 117)	1	7, 69	
	2 Find limits involving trigonometric functions (p. 119)	2, 3	9, 51–55	4, 10
	3 Determine where the trigonometric functions are continuous (p. 122)	4	63–65	
	4 Determine where an exponential or a logarithmic function is continuous (p. 124)	5	43	
1.5	1 Investigate infinite limits (p. 128)	1–3	57, 58	
	2 Find the vertical asymptotes of a graph (p. 131)	4	61, 62	1
	3 Investigate limits at infinity (p. 131)	5–10	59, 60	2
	4 Find the horizontal asymptotes of a graph (p. 138)	11	61, 62	1
	5 Find the asymptotes of the graph of a rational function (p. 139)	12	67, 68	7
1.6	1 Use the ε-δ definition of a limit (p. 144)	1–7	50, 66	

REVIEW EXERCISES

1. Use a table of numbers to investigate $\lim_{x \to 0} \frac{1 - \cos x}{1 + \cos x}$.

In Problems 2 and 3, use a graph to investigate $\lim_{x \to c} f(x)$.

2. $f(x) = \begin{cases} 2x - 5 & \text{if } x < 1 \\ 6 - 9x & \text{if } x \geq 1 \end{cases}$ at $c = 1$

3. $f(x) = \begin{cases} x^2 + 2 & \text{if } x < 2 \\ 2x + 1 & \text{if } x \geq 2 \end{cases}$ at $c = 2$

4. For $f(x) = x^2 - 3$:
   - (a) Find the slope of the secant line joining $(1, -2)$ and $(2, 1)$.
   - (b) Find the slope of the tangent line to the graph of f at $(1, -2)$.

In Problems 5 and 6, for each function find the limit of the difference quotient $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

5. $f(x) = \frac{3}{x}$

6. $f(x) = 3x^2 + 2x$

7. Find $\lim_{x \to 0} f(x)$ if $1 + \sin x \leq f(x) \leq |x| + 1$

In Problems 8–22, find each limit.

8. $\lim_{x \to 2} \left(2x - \frac{1}{x}\right)$

9. $\lim_{x \to \pi} (x \cos x)$

10. $\lim_{x \to -1} (x^3 + 3x^2 - x - 1)$

11. $\lim_{x \to 0} \sqrt[3]{x(x+2)^3}$

12. $\lim_{x\to 0}[(2x+3)(x^5+5x)]$

13. $\lim_{x\to 3}\dfrac{x^3-27}{x-3}$

14. $\lim_{x\to 3}\left(\dfrac{x^2}{x-3}-\dfrac{3x}{x-3}\right)$

15. $\lim_{x\to 2}\dfrac{x^2-4}{x-2}$

16. $\lim_{x\to -1}\dfrac{x^2+3x+2}{x^2+4x+3}$

17. $\lim_{x\to -2}\dfrac{x^3+5x^2+6x}{x^2+x-2}$

18. $\lim_{x\to 1}\left(x^2-3x+\dfrac{1}{x}\right)^{15}$

19. $\lim_{x\to 2}\dfrac{3-\sqrt{x^2+5}}{x^2-4}$

20. $\lim_{x\to 0}\left\{\dfrac{1}{x}\left[\dfrac{1}{(2+x)^2}-\dfrac{1}{4}\right]\right\}$

21. $\lim_{x\to 0}\dfrac{(x+3)^2-9}{x}$

22. $\lim_{x\to 1}[(x^3-3x^2+3x-1)(x+1)^2]$

In Problems 23–28, find each one-sided limit, if it exists.

23. $\lim_{x\to -2^+}\dfrac{x^2+5x+6}{x+2}$

24. $\lim_{x\to 5^+}\dfrac{|x-5|}{x-5}$

25. $\lim_{x\to 1^-}\dfrac{|x-1|}{x-1}$

26. $\lim_{x\to 3/2^+}\lfloor 2x\rfloor$

27. $\lim_{x\to 4^-}\dfrac{x^2-16}{x-4}$

28. $\lim_{x\to 1^+}\sqrt{x-1}$

In Problems 29 and 30, find $\lim_{x\to c^-} f(x)$ and $\lim_{x\to c^+} f(x)$ for the given c. Determine whether $\lim_{x\to c} f(x)$ exists.

29. $f(x)=\begin{cases}2x+3 & \text{if } x<2\\ 9-x & \text{if } x\geq 2\end{cases}$ at $c=2$

30. $f(x)=\begin{cases}3x+1 & \text{if } x<3\\ 10 & \text{if } x=3\\ 4x-2 & \text{if } x>3\end{cases}$ at $c=3$

In Problems 31–36, determine whether f is continuous at c.

31. $f(x)=\begin{cases}5x-2 & \text{if } x<1\\ 5 & \text{if } x=1\\ 2x+1 & \text{if } x>1\end{cases}$ at $c=1$

32. $f(x)=\begin{cases}x^2 & \text{if } x<-1\\ 2 & \text{if } x=-1\\ -3x-2 & \text{if } x>-1\end{cases}$ at $c=-1$

33. $f(x)=\begin{cases}4-3x^2 & \text{if } x<0\\ 4 & \text{if } x=0\\ \sqrt{16-x^2} & \text{if } 0<x\leq 4\end{cases}$ at $c=0$

34. $f(x)=\begin{cases}\sqrt{4+x} & \text{if } -4\leq x\leq 4\\ \sqrt{\dfrac{x^2-16}{x-4}} & \text{if } x>4\end{cases}$ at $c=4$

35. $f(x)=\lfloor 2x\rfloor$ at $c=\dfrac{1}{2}$

36. $f(x)=|x-5|$ at $c=5$

37. (a) Find the average rate of change of $f(x)=2x^2-5x$ from 1 to x.
(b) Find the limit as x approaches 1 of the average rate of change found in (a).

38. A function f is defined on the interval $[-1, 1]$ with the following properties: f is continuous on $[-1, 1]$ except at 0, negative at -1, positive at 1, but with no zeros. Does this contradict the Intermediate Value Theorem?

In Problems 39–43, find all numbers x for which f is continuous.

39. $f(x)=\dfrac{x}{x^3-27}$

40. $f(x)=\dfrac{x^2-3}{x^2+5x+6}$

41. $f(x)=\dfrac{2x+1}{x^3+4x^2+4x}$

42. $f(x)=\sqrt{x-1}$

43. $f(x)=2^{-x}$

44. Use the Intermediate Value Theorem to determine whether $2x^3+3x^2-23x-42=0$ has a zero in the interval $[3, 4]$.

In Problems 45 and 46, use the Intermediate Value Theorem to approximate the zero correct to three decimal places.

45. $f(x)=8x^4-2x^2+5x-1$ on the interval $[0, 1]$.

46. $f(x)=3x^3-10x+9$; zero between -3 and -2.

47. Find $\lim_{x\to 0^+}\dfrac{|x|}{x}(1-x)$ and $\lim_{x\to 0^-}\dfrac{|x|}{x}(1-x)$.
What can you say about $\lim_{x\to 0}\dfrac{|x|}{x}(1-x)$?

48. Find $\lim_{x\to 2}\left(\dfrac{x^2}{x-2}-\dfrac{2x}{x-2}\right)$. Then comment on the statement that this limit is given by $\lim_{x\to 2}\dfrac{x^2}{x-2}-\lim_{x\to 2}\dfrac{2x}{x-2}$.

49. Find $\lim_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$ for $f(x)=\sqrt{x}$.

50. For $\lim_{x\to 3}(2x+1)=7$, find the largest possible δ that "works" for $\varepsilon=0.01$.

In Problems 51–60, find each limit.

51. $\lim_{x\to 0}\cos(\tan x)$

52. $\lim_{x\to 0}\dfrac{\sin\dfrac{x}{4}}{x}$

53. $\lim_{x\to 0}\dfrac{\tan(3x)}{\tan(4x)}$

54. $\lim_{x\to 0}\dfrac{\cos\dfrac{x}{3}-1}{x}$

55. $\lim_{x\to 0}\left(\dfrac{\cos x-1}{x}\right)^{10}$

56. $\lim_{x\to 0}\dfrac{e^{4x}-1}{e^x-1}$

57. $\lim_{x\to \pi/2^+}\tan x$

58. $\lim_{x\to -3}\dfrac{2+x}{(x+3)^2}$

59. $\lim_{x\to\infty}\dfrac{3x^3-2x+1}{x^3-8}$

60. $\lim_{x\to\infty}\dfrac{3x^4+x}{2x^2}$

In Problems 61 and 62, find any vertical and horizontal asymptotes of f.

61. $f(x)=\dfrac{4x-2}{x+3}$

62. $f(x)=\dfrac{2x}{x^2-4}$

63. Let $f(x)=\begin{cases}\dfrac{\tan x}{2x} & \text{if } x\neq 0\\ \dfrac{1}{2} & \text{if } x=0\end{cases}$. Is f continuous at 0?

64. Let $f(x)=\begin{cases}\dfrac{\sin(3x)}{x} & \text{if } x\neq 0\\ 1 & \text{if } x=0\end{cases}$. Is f continuous at 0?

65. The function $f(x) = \dfrac{\cos\left(\pi x + \frac{\pi}{2}\right)}{x}$ is not defined at 0.
Decide how to define $f(0)$ so that f is continuous at 0.

66. Use the ε-δ definition of a limit to prove $\lim\limits_{x \to -3} (x^2 - 9) \neq 18$.

67. (a) Sketch a graph of a function f that has the following properties:

$$f(-1) = 0 \quad \lim_{x \to \infty} f(x) = 2 \quad \lim_{x \to -\infty} f(x) = 2$$

$$\lim_{x \to 4^-} f(x) = -\infty \quad \lim_{x \to 4^+} f(x) = \infty$$

(b) Define a function that describes your graph.

68. (a) Find the domain and the intercepts (if any) of

$$R(x) = \frac{2x^2 - 5x + 2}{5x^2 - x - 2}.$$

(b) Discuss the behavior of the graph of R at numbers where R is not defined.

(c) Find any vertical or horizontal asymptotes of the function R.

69. If $1 - x^2 \leq f(x) \leq \cos x$ for all x in the interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$, show that $\lim\limits_{x \to 0} f(x) = 1$.