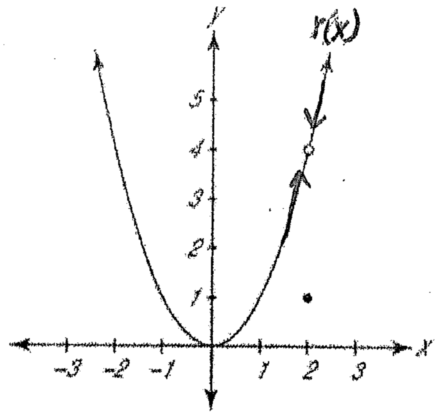
Key ①

Calculus AB   Ch. 1.2   Notes on Limits

Definition: The Limit is the y-value that a function or graph approaches as the x-value moves closer to a given constant

Function Value is finding the location of the y-value of the graph at a specific x-value.

Example 1:



* The y-value of the graph when x =2 is 1"

Notation: $r(2)=1$

* "The Limit of r(x) as x approaches 2 is 4"

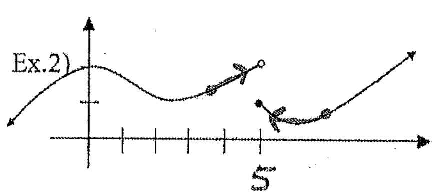
Notation: $\lim_{x\to 2} r(x) = 4$ $\cancel{\lim_{x\to 2} = 4}$

*watch notation

*In order for a limit to exist, the graph MUST approach the same **Real Number** y-value from both sides of the target x-value constant.

Examples where the Limit does not exist:

Ex.2)



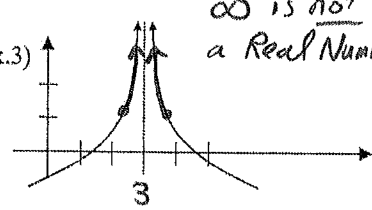
*Jump discontinuity

Example 2:

$$\lim_{x\to 5} f(x) = d.n.e.$$

(does not exist)

Ex.3)

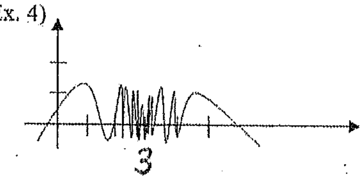


∞ is not a Real Number

*Vertical Asymptote

Example 3: $\lim_{x\to 3} f(x) = d.n.e.$ $(+\infty)$

Ex. 4)



*Graphs with oscillating behavior

Example 4: $\lim_{x\to 3} \sin\left(\frac{1}{x}\right) = d.n.e.$

*If there is some sort of break in the graph, the limit and function value will always be different from each other.

Example 5: Find the limit using a table of values given that $f(x) = \frac{x^3-1}{x-1}$

x	0.9	.99	.999	1	1.0001	1.001	1.01	1.1
f(x)	2.71	2.97	2.997	Undefined	3.0003	3.003	3.03	3.31

Approaches 3   Approaches 3

$$\lim_{x\to 1} \frac{x^3-1}{x-1} = \boxed{3}$$

However, $f(1)$ = undefined

2

Key

Calculus Ch. 1.2 Classwork Problems **Evaluating Limits Graphically**

*Does Not Exist (dne)

1) $\lim_{x\to -5} f(x) =$ d.n.e.

2) $\lim_{x\to -4} f(x) =$ 1

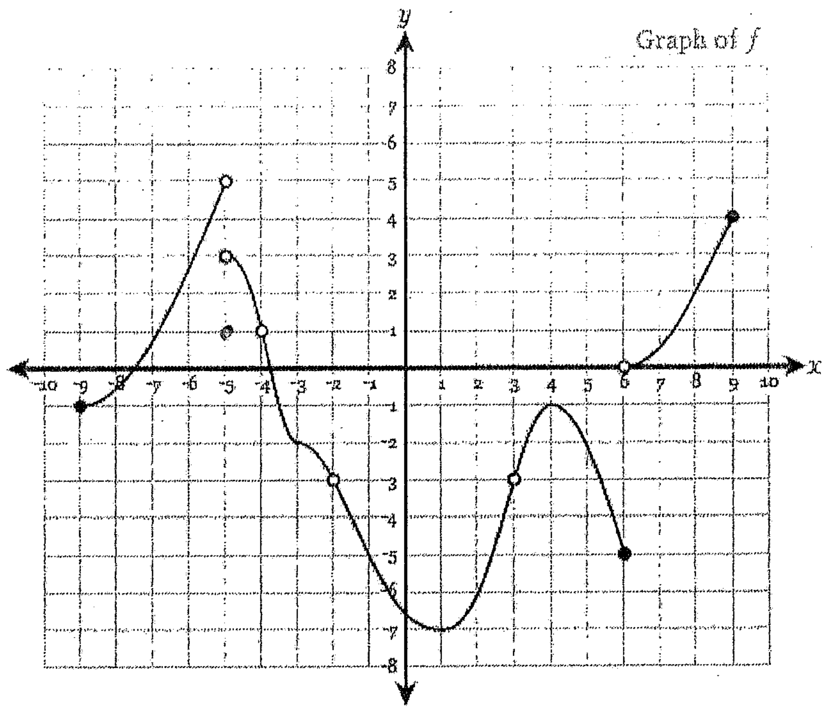
3) $f(-3) =$ −2

4) $\lim_{x\to -3} f(x) =$ −2

5) $f(3) =$ undefined

6) $\lim_{x\to 3} f(x) =$ −3

7) $\lim_{x\to 6} f(x) =$ dne.



8) $\lim_{x\to -8} f(x) =$ 1

9) $\lim_{x\to -7} f(x) =$ dne.

10) $f(-3) =$ −3

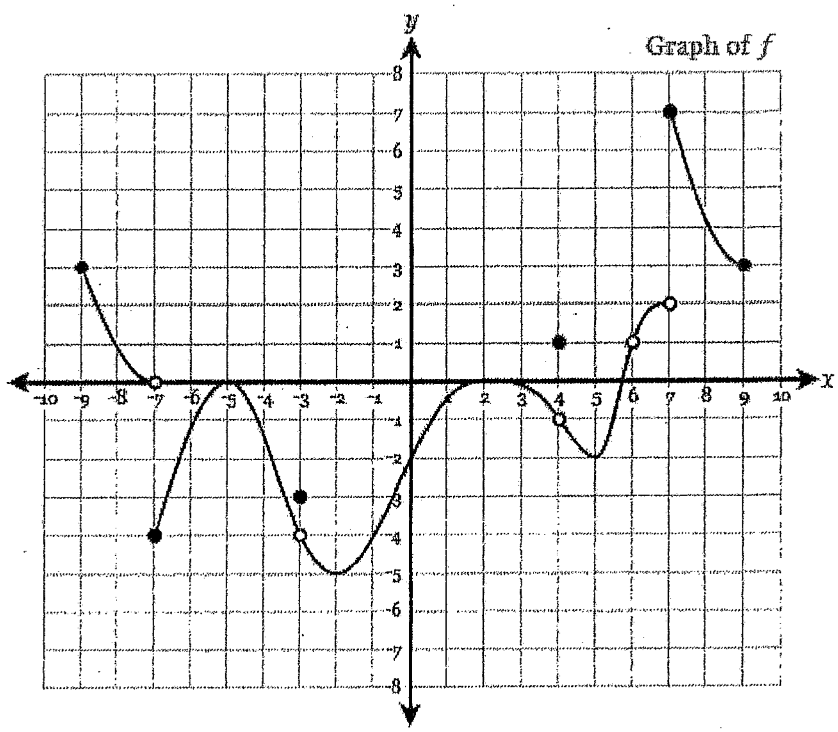
11) $\lim_{x\to 4} f(x) =$ −1

12) $f(4) =$ 1

13) $f(6) =$ d.n.e (undefined)

14) $\lim_{x\to 6} f(x) =$ 1

15) $\lim_{x\to 7} f(x) =$ d.n.e.



16) $\lim_{x\to -9} f(x) =$ d.n.e.

17) $\lim_{x\to -6} f(x) = -5$

18) $\lim_{x\to -4} f(x) =$ d.n.e. $(+\infty)$

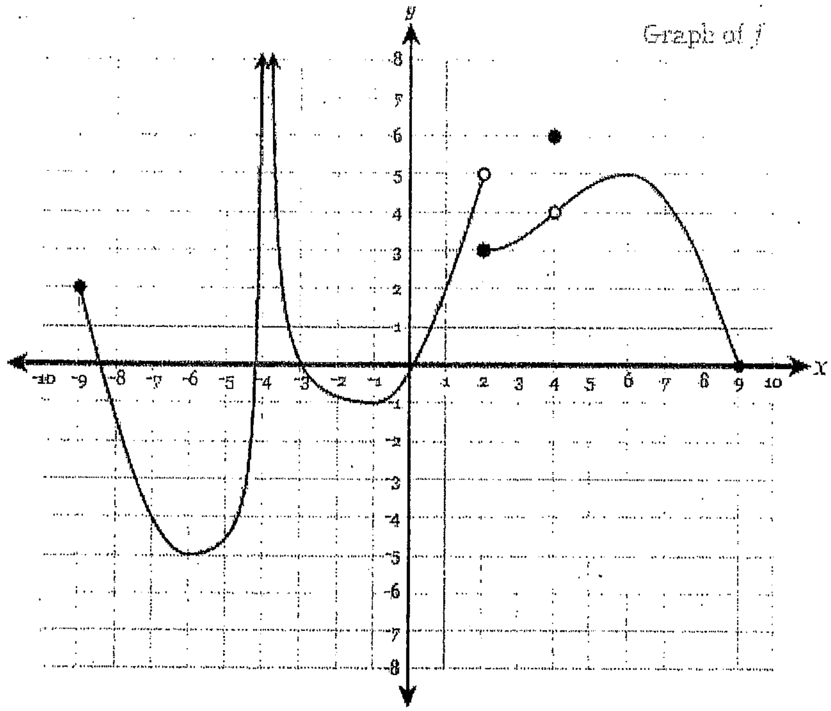
19) $f(-4) =$ undefined

20) $\lim_{x\to 2} f(x) =$ d.n.e.

21) $f(2) = 3$

22) $\lim_{x\to 4} f(x) = 4$

23) $f(4) = 6$



24) $\lim_{x\to -6} f(x) = 0$

25) $\lim_{x\to -4} f(x) = 3$

26) $f(-4) = 2$

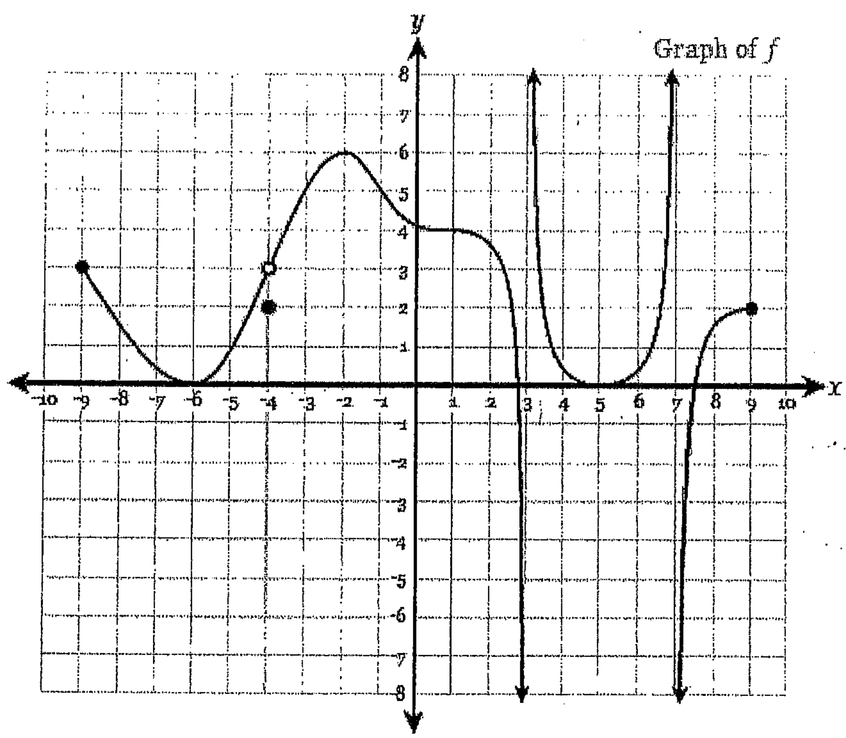
27) $f(3) =$ d.n.e

28) $\lim_{x\to 3} f(x) =$ d.n.e.

29) $\lim_{x\to 5} f(x) = 0$

30) $\lim_{x\to 7} f(x) =$ d.n.e.

31) $\lim_{x\to 9} f(x) =$ d.n.e.



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[redacted]

Key

Calculus Ch. 1.2 Classwork Problems Worksheet #2

Sketch graph of a function satisfying the given descriptions:

1) $\lim_{x \to -5} f(x) = 3$

2) $f(-5) = -2$

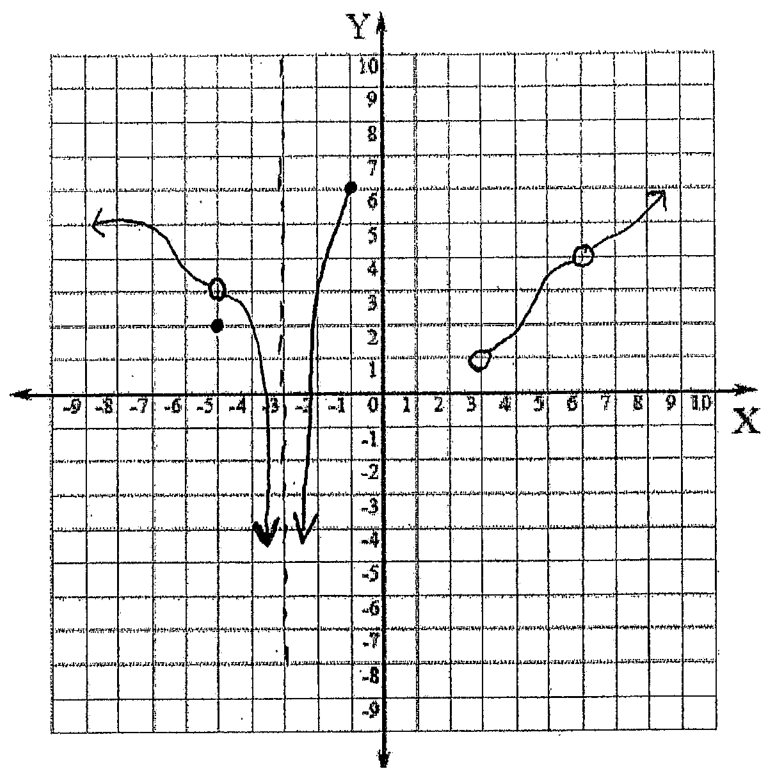
3) $f(-1) = 6$

4) $\lim_{x \to -3} f(x) = -\infty$

5) $f(3) = undefined$

6) $\lim_{x \to 3} f(x)$ *does not exist*

7) $\lim_{x \to 6} f(x) = 4$



8) $\lim_{x \to -8} f(x) = DNE$

9) $\lim_{x \to -7} f(x) = 5$

10) $f(-3) = 5$

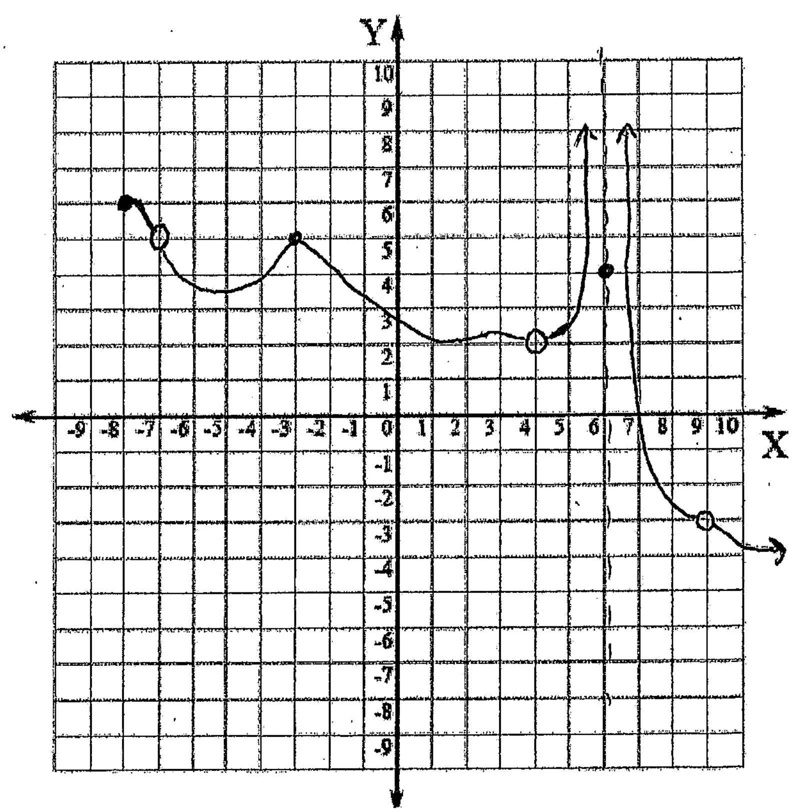
11) $\lim_{x \to 4} f(x) = 2$

12) $f(4) = undefined$

13) $f(6) = 4$

14) $\lim_{x \to 6} f(x) = \infty$

15) $\lim_{x \to 9} f(x) = -3$



Key (5)

Ch. 1.3a **Evaluating Limits Algebraically**

Rules:

1) a) $\lim_{x \to c} b = b$ 2) Suppose $\lim_{x \to c} f(x) = L$ then $\lim_{x \to c} bf(x) = bL$

I. **Direct Substitution Method:** To find limits for a function, first try to evaluate the argument in the expression (plug in the value). If the resulting value is a Real Number, then the value is the limit (answer).

Example 1:

Extension Question: Why do the limits for these problems all produce same value as the function value?

a) $\lim_{x \to 2} x^2 + 3x = 2^2 + 3(2) = \boxed{10}$

b) $\lim_{x \to 2} 5 = \boxed{5}$

c) $\lim_{x \to -1} 3x^5 - 2x^2 + 7x + 4 = 3(-1)^5 - 2(-1)^2 + 7(-1) + 4$
$= -3 - 2 - 7 + 4 = \boxed{-8}$

d) $\lim_{x \to \pi} x\cos x = \pi\cos(\pi) = \pi(-1)$
$= \boxed{-\pi}$

Answer: All are continuous functions: limit and function value are equal

II. **Simplify/Reduction Method** (Factor/Simplify/Substitute) Steps:

1) Evaluate argument First! (plug in value into expression)
2) If direct substitution produces $\frac{0}{0}$ (indeterminate form), we need to evaluate further

*Note: $\frac{0}{0}$ does not mean the Limit is Undefined. $\frac{0}{0}$ just means our problem is incomplete and unfinished. (It's true that the function value is undefined because there's a hole in the graph, but the limit most times does still exist)

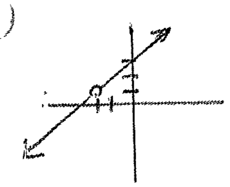
3) Factor/Reduce/Simplify: Try finding common factors in order to reduce expression
4) Using the Reduced expression, re-evaluate the limit
5) Confirm resulting value is now a Real Number, therefore the limit (answer)

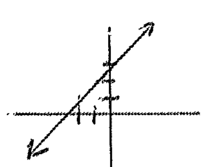
Example 2: Evaluate first!

function value is undefined, but the limit exists! (Graph is a hole)

a) $\lim_{x \to -2} \frac{x^2 + 5x + 6}{x + 2} = \frac{(-2)^2 + 5(-2) + 6}{-2 + 2} \to \frac{0}{0}$

$\lim_{x \to -2} \frac{(x+3)\cancel{(x+2)}}{\cancel{x+2}} \to -2 + 3 = \boxed{1}$





$y = x + 3$ (clone version of original function)

b) $\lim_{x \to 1} \frac{x^2 + 5x + 6}{x - 1} = \frac{1^2 + 5(1) + 6}{1 - 1} \to \frac{12}{0}$

$\boxed{\text{does not exist (d.n.e.)}}$

vertical Asymptote at $x = 1$, so therefore limit does not exist.

(6)

Simplify/Reduction Method (Factor/Simplify/Substitute) Steps:

1) Evaluate argument First! (plug in value into expression)
2) If direct substitution produces $\frac{0}{0}$ (indeterminate form), we need to evaluate further
3) Factor/Reduce/Simplify: Try finding common factors in order to reduce expression
4) Using the Reduced expression, re-evaluate the limit
5) Confirm resulting value is now a Real Number, therefore the limit (answer)

Distinguishing variations of zeros:

i) $\frac{0}{0} \to$ indeterminate form $\to$ hole in graph $\to$ keep going! to find limit — limit still hidden from view

ii) $\frac{12}{0} \to$ vertical asymptote $\to$ limit does not exist

iii) $\frac{0}{4} \to$ Real Number $\to \boxed{0}$

Example 2 (continued):

c) $\lim_{x\to 2} \frac{x^2+5x+6}{x+2} = \frac{2^2+5(2)+6}{2+2}$

$\frac{4+10+6}{4} = \frac{20}{4} = \boxed{5}$

d) $\lim_{x\to 2} \frac{x^2-4}{x+2} = \frac{2^2-4}{2+2} \to \frac{0}{4} = \boxed{0}$

Practice Problems:

1) $\lim_{x\to 1} \frac{2x^2-x-3}{x-1}$ $\quad \frac{2-1-3}{1-1} \to \frac{-2}{0}$

$\boxed{\text{does not exist}}$

2) $\lim_{x\to 3} \frac{4x^2-7x-2}{x-2} \to \frac{4(3)^2-7(3)-2}{3-2} = \frac{13}{1}$

$= \boxed{13}$

3) $\lim_{x\to -1} \frac{x^2-1}{x+1} \to \frac{0}{0}$

$\lim_{x\to -1} \frac{(x-1)\cancel{(x+1)}}{\cancel{(x+1)}} = \boxed{-2}$

4) $\lim_{x\to 1} \frac{x^2-4}{x^2-3x+2}$ $\quad \frac{1-4}{1-3+2} = \frac{-3}{0}$ V.A. at $x=1$

limit $\boxed{\text{dne}}$ (does not exist)

5) $\lim_{x\to -3} \frac{x^2+x-6}{x^2-9}$ $\quad \frac{(-3)^2-3-6}{(-3)^2-9}$

$\frac{9-3-6}{9-9} \to \frac{0}{0}$ $\quad \lim_{x\to -3} \frac{\cancel{(x+3)}(x-2)}{\cancel{(x+3)}(x-3)}$

$\frac{-3-2}{-3-3} = \frac{-5}{-6} = \boxed{\frac{5}{6}}$

6) $\lim_{x\to 5} \frac{5-x}{x^2-25}$ $\quad \frac{0}{25-25} \to \frac{0}{0}$

$\lim_{x\to 5} \frac{(5-x)}{(x-5)(x+5)} \to \lim_{x\to 5} \frac{-1\cancel{(x-5)}}{\cancel{(x-5)}(x+5)}$

$\lim_{x\to 5} \frac{-1}{x+5} \to \frac{-1}{5+5} = \boxed{\frac{-1}{10}}$

Key ⑦

Ch. 1.3b (More) Evaluating Limits Algebraically

Recap Steps: **Simplify/Reduction Method** (Factor/Simplify/Substitute) Steps:

1) Evaluate argument First! (plug in value into expression)
2) If direct substitution produces $\frac{0}{0}$ (indeterminate form), we need to evaluate further

*Note: $\frac{0}{0}$ does not mean the Limit is Undefined. $\frac{0}{0}$ just means our problem is incomplete and unfinished. (It's true that the function value is undefined because there's a hole in the graph, but the limit most times does still exist)

3) **Factor/Reduce/Simplify: Try finding common factors in order to reduce expression**
4) Using the Reduced expression, re-evaluate the limit
5) Confirm resulting value is now a Real Number, therefore the limit (answer)

I. Simplify using conjugate method

- If there is a sum or difference of 2 terms in the numerator, then multiply the numerator and denominator by the **conjugate** term.

*leave the denominator in factored form, unexpanded.

Example 1: $\lim\limits_{x\to 4}\frac{6-\sqrt{x+32}}{x-4} \to \frac{6-\sqrt{4+32}}{4-4} \to \frac{0}{0}$

$$\lim_{x\to 4}\frac{(6-\sqrt{x+32})}{(x-4)}\cdot\frac{(6+\sqrt{x+32})}{(6+\sqrt{x+32})} \to \lim_{x\to 4}\frac{36-(x+32)}{(x-4)(6+\sqrt{x+32})}$$

$$\lim_{x\to 4}\frac{36-x-32}{(x-4)(6+\sqrt{x+32})} \to 4-x$$

$$\lim_{x\to 4}\frac{-1\cancel{(x-4)}}{\cancel{(x-4)}(6+\sqrt{x+32})}$$

$$\frac{-1}{6+\sqrt{4+32}} = \frac{-1}{6+\sqrt{36}} = \boxed{\frac{-1}{12}}$$

II. Simplify by finding Common Denominator

Example 2: $\lim\limits_{x\to 0}\frac{\frac{1}{x+4}-\frac{1}{4}}{x}$

$$\lim_{x\to 0}\frac{\frac{1}{0+4}-\frac{1}{4}}{0} \to \frac{0}{0}$$

$$\lim_{x\to 0}\frac{\left(\frac{1}{x+4}-\frac{1}{4}\right)}{x}\cdot\frac{4(x+4)}{4(x+4)} \to \frac{1}{\cancel{x+4}}\cdot\frac{4\cancel{(x+4)}}{1}-\frac{1}{\cancel{4}}\cdot\frac{\cancel{4}(x+4)}{1}$$

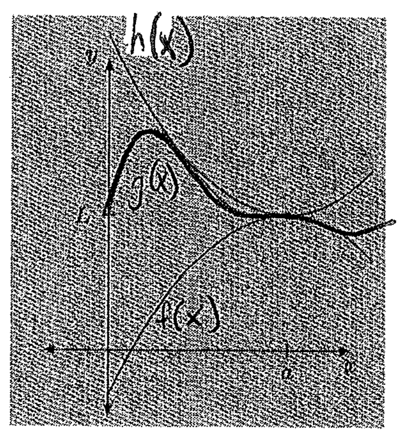
$$\lim_{x\to 0}\frac{4-(x+4)}{4(x+4)\cdot x} \to \frac{4-x-4}{4x(x+4)}$$

$$\lim_{x\to 0}\frac{-\cancel{x}}{4\cancel{x}(x+4)} = \frac{-1}{4(0+4)} = \frac{-1}{4(4)} = \boxed{\frac{-1}{16}}$$

III. Squeeze Theorem

In the graph below, the lower and upper functions have the same limit value at x=a. The middle function has the same limit value because it is trapped between the two outer functions.

The middle function is "squeezed" to Limit **L** as x approaches **a**



Definition: *Suppose $f(x) \le g(x) \le h(x)$ for all x in an open interval*

Suppose that $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} h(x) = L$ *Then,* $\lim\limits_{x\to a} g(x) = L$

Example 3: Let $h(x) = 1$, $f(x) = x^2 + 1$. *If* $f(x) \le g(x) \le h(x)$ *find* $\lim\limits_{x\to 0} g(x)$

$$\lim_{x\to 0} x^2+1 \le \lim_{x\to 0} g(x) \le \lim_{x\to 0} 1$$

$$1 \le \lim_{x\to 0} g(x) \le 1$$

By squeeze theorem

$$\boxed{\lim_{x\to 0} g(x) = 1}$$

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1.3b Practice Problems:

Simplify/Reduction Method (Factor/Simplify/Substitute) Steps:

a) Evaluate argument First! (plug in value into expression)
b) If direct substitution produces $\frac{0}{0}$ (indeterminate form), we need to evaluate further
c) Factor/Reduce/Simplify: Try finding common factors in order to reduce expression
d) Using the Reduced expression, re-evaluate the limit
e) Confirm resulting value is now a Real Number, therefore the limit (answer)

1) $\lim_{x\to4}\frac{\sqrt{x}-2}{x-4}$ $\quad \frac{2-2}{4-4}\to\frac{0}{0}$

$\lim_{x\to4}\frac{(\sqrt{x}-2)}{(x-4)}\cdot\frac{(\sqrt{x}+2)}{(\sqrt{x}+2)}$

$\lim_{x\to4}\frac{\cancel{x-4}}{(\cancel{x-4})(\sqrt{x}+2)}$

$\lim_{x\to4}\frac{1}{\sqrt{x}+2}=\frac{1}{\sqrt{4}+2}=\boxed{\frac{1}{4}}$

2) $\lim_{x\to0}\frac{\frac{1}{3+x}-\frac{1}{3}}{x}\to\frac{0}{0}\to\frac{\frac{3}{3(3+x)}-\frac{3+x}{3(3+x)}}{x}$

$\lim_{x\to0}\frac{\frac{3-3-x}{3(3+x)}}{x}\cdot\frac{\frac{1}{x}}{\frac{1}{x}}$ $\quad \lim_{x\to0}\frac{-\cancel{x}}{3(3+x)}\cdot\frac{1}{\cancel{x}}\to\lim_{x\to0}\frac{-1}{3(3+x)}$

$\frac{-1}{3(3+0)}=\boxed{-\frac{1}{9}}$

3) $\lim_{x\to0}\frac{x^4-5x^2}{x^2}\to\frac{0}{0}$

$\lim_{x\to0}\frac{\cancel{x^2}(x^2-5)}{\cancel{x^2}}\to0^2-5=\boxed{-5}$

4) $\lim_{x\to16}\frac{4-\sqrt{x}}{x-16}\to\frac{0}{0}$

$\lim_{x\to16}\frac{4-\sqrt{x}}{x-16}\cdot\frac{(4+\sqrt{x})}{(4+\sqrt{x})}$

$\lim_{x\to16}\frac{16-x}{(x-16)(4+\sqrt{x})}$

$\lim_{x\to16}\frac{-1(\cancel{x-16})}{(\cancel{x-16})(4+\sqrt{x})}$

$\frac{-1}{4+\sqrt{16}}=\frac{-1}{4+4}$

$=\boxed{\frac{-1}{8}}$

5) $\lim_{x\to4}\frac{x^2+7x-44}{x^2-6x+8}\to\frac{16+28-44}{16-24+8}\to\frac{0}{0}$

$\lim_{x\to4}\frac{(x+11)(\cancel{x-4})}{(\cancel{x-4})(x-2)}\to\frac{4+11}{4-2}=\boxed{\frac{15}{2}}$

6) $\lim_{x\to0}\frac{\sqrt{1+2x}-1}{3x}\cdot\frac{(\sqrt{1+2x}+1)}{(\sqrt{1+2x}+1)}$ $\to\frac{0}{0}$

$\lim_{x\to0}\frac{1+2x-1}{3x(\sqrt{1+2x}+1)}$

$\lim_{x\to0}\frac{2\cancel{x}}{3\cancel{x}(\sqrt{1+2x}+1)}$

$\frac{2}{3(\sqrt{1+0}+1)}=\frac{2}{3(2)}$

$=\boxed{\frac{1}{3}}$

7) $\lim_{x\to0}\frac{\frac{1}{4}+\frac{1}{x-4}}{x}\to\frac{0}{0}$ $\quad\to\frac{\frac{x-4}{4(x-4)}+\frac{4}{4(x-4)}}{x}$

$\lim_{x\to0}\frac{\frac{x-4+4}{4(x-4)}}{x}\cdot\frac{\frac{1}{x}}{\frac{1}{x}}\to\lim_{x\to0}\frac{\cancel{x}}{4(x-4)}\cdot\frac{1}{\cancel{x}}$

$\lim_{x\to0}\frac{1}{4(x-4)}=\frac{1}{4(0-4)}=\boxed{\frac{-1}{16}}$

8) $\lim_{x\to5}\frac{\sqrt{x-1}-2}{x^2-25}\to\frac{2-2}{25-25}=\frac{0}{0}$

$\lim_{x\to5}\frac{\sqrt{x-1}-2}{x^2-25}\cdot\frac{(\sqrt{x-1}+2)}{(\sqrt{x-1}+2)}$

$\lim_{x\to5}\frac{x-1-4}{(x^2-25)(\sqrt{x-1}+2)}$

$\lim_{x\to5}\frac{\cancel{x-5}}{(\cancel{x-5})(x+5)(\sqrt{x-1}+2)}$

$\lim_{x\to5}\frac{1}{(x+5)(\sqrt{x-1}+2)}$

$\frac{1}{(5+5)(2+2)}$

$\frac{1}{10(4)}=\boxed{\frac{1}{40}}$

Key

Ch. 1.2-1.3 Limits Quiz Review Worksheet #2

1) Find the values

a. $\lim_{x \to -8} g(x) = (-\infty)$ DNE

b. $g(-8) =$ undefined

c. $\lim_{x \to -5} g(x) =$ DNE

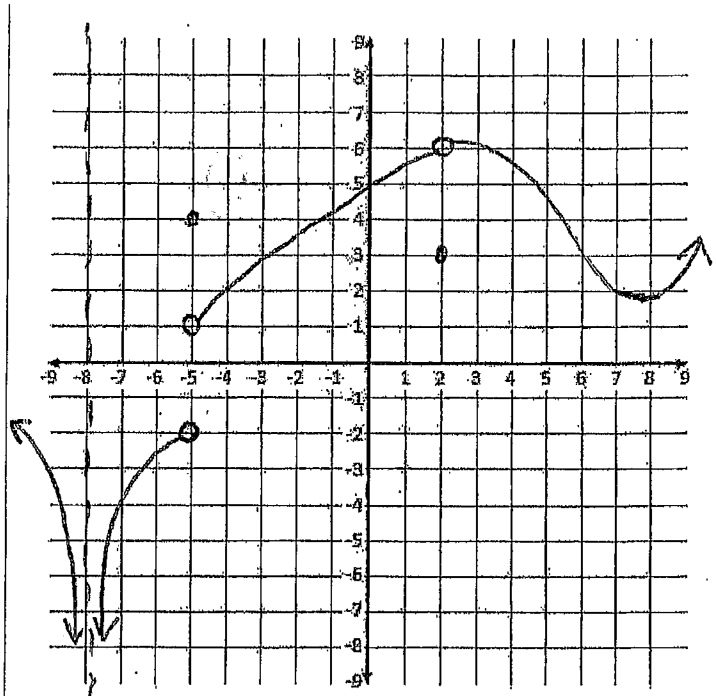
d. $g(-5) =$ 4

e. $\lim_{x \to 2} g(x) =$ 6

f. $g(2) =$ 3

g. $g(7) =$ 2

h. $\lim_{x \to 7} g(x) =$ 2



2) Sketch a graph with the following characteristics:

a) $\lim_{x \to -5} f(x) = -4$

b) $g(-5) = undefined$

c) $g(-2) = -8$

d) $\lim_{x \to -2} f(x) = \infty$

e) $g(2) = 7$

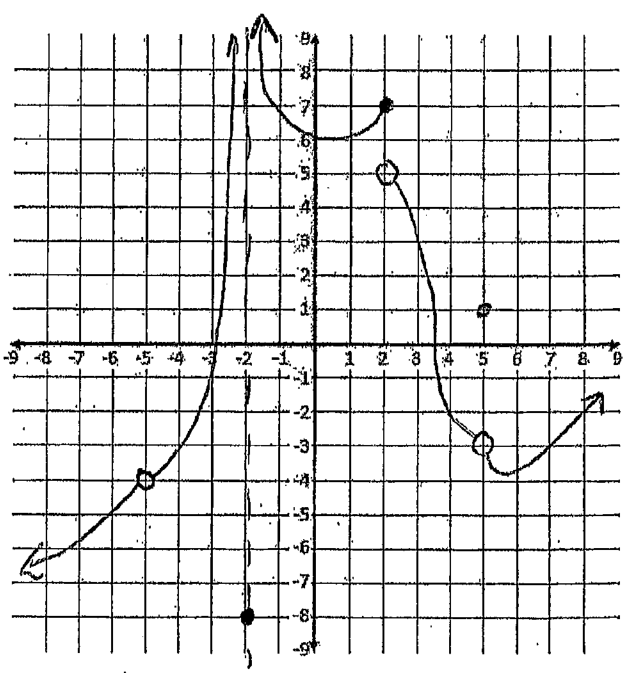
f) $\lim_{x \to 2} f(x) =$ *does not exist*

g) $g(5) = 1$

h) $\lim_{x \to 5} f(x) = -3$

i) $g(7) = -3$

j) $\lim_{x \to 7} f(x) = -3$



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Evaluate the Limit:

WS #2 Key

3) $\lim_{x\to 0}\dfrac{\boxed{\dfrac{1}{x+6}-\dfrac{1}{6}}}{x} \to \dfrac{0}{0}$

$(6)\dfrac{1}{x+6}-\dfrac{1}{6}(x+6)$

$\dfrac{6}{6(x+6)}-\dfrac{(x+6)}{6(x+6)}$

$\lim_{x\to 0}\dfrac{\dfrac{\cancel{6}-x-\cancel{6}}{6(x+6)}\cdot\dfrac{1}{x}}{x\cdot\dfrac{1}{x}}$

$\lim_{x\to 0}\dfrac{-\cancel{x}}{6(x+6)}\cdot\dfrac{1}{\cancel{x}}$

$\lim_{x\to 0}\dfrac{-1}{6(x+6)}$

$\dfrac{-1}{6(6)}$

$\boxed{\dfrac{-1}{36}}$

4) $\lim_{x\to 1}\dfrac{2x^2+2x-3}{x-1} \to \dfrac{2+2-3}{1-1} \to \dfrac{+1}{0}$

vertical asymptote

$\boxed{d.n.e.}$

5) $\lim_{x\to 5}\dfrac{4-\sqrt{11+x}}{x-5} \to \dfrac{0}{0}$

$\lim_{x\to 5}\dfrac{(4-\sqrt{11+x})}{(x-5)}\cdot\dfrac{(4+\sqrt{11+x})}{(4+\sqrt{11+x})}$

$\lim_{x\to 5}\dfrac{16-(11+x)}{(x-5)(4+\sqrt{11+x})}$

$\lim_{x\to 5}\dfrac{16-11-x}{(x-5)(4+\sqrt{11+x})}$

$\lim_{x\to 5}\dfrac{5-x}{(x-5)(4+\sqrt{11+x})}$

$\lim_{x\to 5}\dfrac{-1\cancel{(x-5)}}{\cancel{(x-5)}(4+\sqrt{11+x})}$

$\dfrac{-1}{4+\sqrt{16}} = \boxed{-1/8}$

6) $\lim_{x\to 1}\dfrac{4x^2-x-2}{x-3} \to \dfrac{4-1-2}{1-3} = \dfrac{1}{-2} = \boxed{-\dfrac{1}{2}}$

7) $\lim_{x\to 3}\dfrac{6x^2-15x-9}{x-3} \to \dfrac{0}{0}$

$-6 \quad 1 \quad -6 \quad 2 \quad -5 \quad 2$

$\lim_{x\to 3}\dfrac{3(2x^2-5x-3)}{x-3}$

$\lim_{x\to 3}\dfrac{3(x-3)(x+1/2)}{x-3}$

$\lim_{x\to 3}\dfrac{3\cancel{(x-3)}(2x+1)}{\cancel{(x-3)}}$

$3(2(3)+1) = \boxed{21}$

8) $\lim_{x\to 0}\dfrac{\sqrt{5+x}-\sqrt{5}}{x} \to \dfrac{0}{0}$

$\lim_{x\to 0}\dfrac{\sqrt{5+x}-\sqrt{5}}{x}\cdot\dfrac{(\sqrt{5+x}+\sqrt{5})}{(\sqrt{5+x}+\sqrt{5})}$

$\lim_{x\to 0}\dfrac{\cancel{5}+x-\cancel{5}}{x(\sqrt{5+x}+\sqrt{5})}$

$\lim_{x\to 0}\dfrac{\cancel{x}}{\cancel{x}(\sqrt{5+x}+\sqrt{5})}$

$\lim_{x\to 0}\dfrac{1}{\sqrt{5+x}+\sqrt{5}}$

$\dfrac{1}{\sqrt{5+0}+\sqrt{5}}$

$\boxed{\dfrac{1}{2\sqrt{5}}}$

9) $\lim_{x\to 0}\dfrac{\boxed{\dfrac{1}{2-x}-\dfrac{1}{2}}}{x} \to \dfrac{0}{0}$

$(2)\dfrac{1}{2-x}-\dfrac{1}{2}(2-x)$

$\dfrac{2}{2(2-x)}-\dfrac{(2-x)}{2(2-x)}$

$\dfrac{\cancel{2}-\cancel{2}+x}{2(2-x)}$

$\lim_{x\to 0}\dfrac{\dfrac{x}{2(2-x)}\cdot\dfrac{1}{x}}{x\cdot\dfrac{1}{x}}$

$\lim_{x\to 0}\dfrac{\cancel{x}}{2(2-x)}\cdot\dfrac{1}{\cancel{x}}$

$\lim_{x\to 0}\dfrac{1}{2(2-x)} \to \dfrac{1}{2(2-0)}$

$\to \boxed{\dfrac{1}{4}}$

10) $\lim_{x\to 2}\dfrac{\boxed{\dfrac{2}{x}-1}}{x-2} \to \dfrac{0}{0}$

$\dfrac{2}{x}-\dfrac{1}{1}(x)$

$\dfrac{2}{x}-\dfrac{x}{x}$

$\dfrac{2-x}{x}$

$\lim_{x\to 2}\dfrac{\dfrac{2-x}{x}\cdot\dfrac{1}{x-2}}{x-2\cdot\dfrac{1}{x-2}}$

$\lim_{x\to 2}\dfrac{2-x}{x}\cdot\dfrac{1}{x-2}$

$\lim_{x\to 2}\dfrac{-1\cancel{(x-2)}}{x}\cdot\dfrac{1}{\cancel{x-2}}$

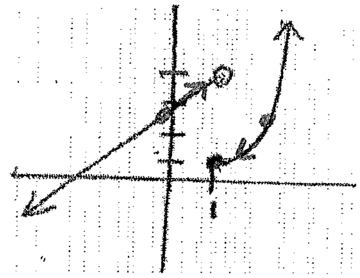
$\lim_{x\to 2}\dfrac{-1}{x} = \boxed{\dfrac{-1}{2}}$

Calculus **Ch. 1.4a Notes** **Continuity and One-Sided Limits**

Definition: **One-Sided Limits** – describes the function's behavior from the left or the right side of an x-value

Example 1:

$$f(x) = \begin{cases} x^2 , & x \geq 1 \\ x+3 , & x < 1 \end{cases}$$



a) $\lim\limits_{x\to 1} f(x) =$ does not exist

b) left handed limit: $\lim\limits_{x\to 1^-} f(x) = 4$

In other words: "The Limit (y-value that the graph approaches) **from the left side** of x = 1 is 4
(start left of point, move right)

c) right handed limit: $\lim\limits_{x\to 1^+} f(x) = 1$

In other words: "The Limit (y-value that the graph approaches) **from the right side** of x = 1 is 1
(start right of point, move left)

- Recall that the limit of $f(x)$ as $x \to c$ exists only if $\lim\limits_{x\to c^-} f(x) = \lim\limits_{x\to c^+} f(x)$.

Continuity

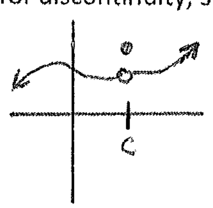
can I walk along the graph without any interruptions? Can I draw the graph without ever lifting my pen/pencil? If so, the path or graph is continuous at that point.

Continuity Conditions: (*IMPORTANT: KNOW THIS*)

For a function, f, to be continuous at c, the following 3 conditions must be met.

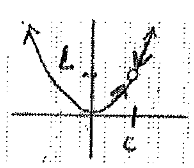
1. $f(c)$ is defined *point exists
2. $\lim\limits_{x\to c} f(x)$ exists $\left(\lim\limits_{x\to c^-} f(x) = \lim\limits_{x\to c^+} f(x)\right)$ *the limit exists
3. $\lim\limits_{x\to c} f(x) = f(c)$ * the limit exists at same location as point

- When checking for discontinuity, step through each of the conditions above in order.



Types of Continuity:

1) **Removable Discontinuity** (hole in graph) – a graph with removable discontinuity can be made continuous by filling in a single point.



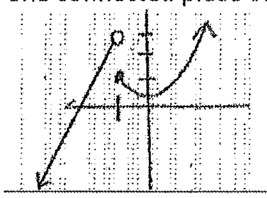
*Removable discontinuity passes the 2nd condition but fails the 3rd condition.

✓ → $\lim\limits_{x\to c} f(x)$ exists

X → $\lim\limits_{x\to c} f(x) \neq f(c)$

(12)

2) **Nonremovable Discontinuity** (step, jump discontinuity) – this is a discontinuity where the graph jumps from one connected piece of graph to another.



*Non-removable discontinuity fails the 2nd continuity condition:

$\lim_{x \to -1^-} f(x) = 3$ $\lim_{x \to -1^+} f(x) = 1$ then $\lim_{x \to -1} f(x) =$ does not exist

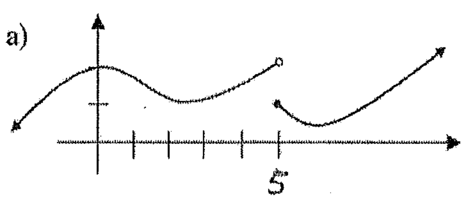
Continuity Conditions revisited

$i.\ f(c) \text{ is defined}$	*If first condition fails, function not continuous at the point, but continue to test next condition to categorize removable/nonremovable
$ii.\ \lim_{x \to c} f(x)$ exists ($\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x)$)	*If 2nd condition fails, then the limit does not exist, and this function must have **non-removable discontinuity** at that point * Test 3rd condition only if 2nd condition passes.
$iii.\ \lim_{x \to c} f(x) = f(c)$	*If 2nd condition passes, but 3rd condition fails, then this function must have **removable discontinuity** at that point *If all 3 condition passes, then the function is continuous at that point.

(* since 2nd condition fails, we stop and don't test 3rd condition)

Class Example 2: Using continuity conditions, determine the reason why the following graphs are discontinuous. Then categorize as removable or nonremovable discontinuity

a)

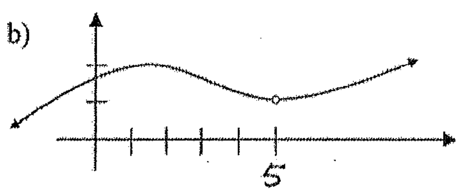


✓ $i)\ f(5) = 1$

× $ii)\ \lim_{x \to 5} f(x)$ does not exist $\left[\lim_{x \to 5^-} f(x) \neq \lim_{x \to 5^+} f(x)\right]$

* therefore, nonremovable discontinuity at $x = 5$.

b)



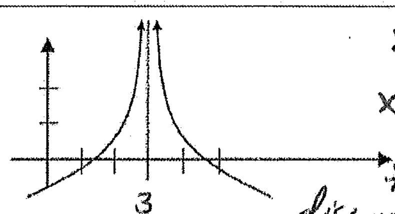
× $i)\ f(5)$ undefined

✓ $ii)\ \lim_{x \to 5} f(x) = 1$

× $iii)\ \lim_{x \to 5} f(x) \neq f(5)$

* therefore, removable discontinuity at $x = 5$.

c)



× $i)\ f(3)$ undefined.

× $ii)\ \lim_{x \to 3} f(x)$ does not exist

* therefore, nonremovable discontinuity at $x = 3$

* Extension questions:
1) can you sketch example graph condition 1 and 2 passes, 3rd fails?
2) Is there example where #1, 2 fails but #3 passes?

d) Find the point (x-value) of discontinuity for the function $f(x) = \dfrac{x^2 - 9}{x - 3}$. Is it removable? If so, what would we need to set f(x) equal to at that value for the function to be continuous? (Step through continuity conditions to support your answer)

continuity conditions.

* Discontinuity at $x = 3$

$i)\ f(3) = \dfrac{3^2 - 9}{3 - 3} \to \dfrac{0}{0} \to f(3)$ undefined

$ii)\ \lim_{x \to 3} \dfrac{x^2 - 9}{x - 3} \to \dfrac{0}{0} \to \lim_{x \to 3} \dfrac{\cancel{(x-3)}(x+3)}{\cancel{(x-3)}} \to \lim_{x \to 3} x + 3 = \boxed{6}$

$iii)\ \lim_{x \to 3} f(x) \neq f(3)$

* therefore, removable discontinuity at $x = 3$.

* To make f(x) continuous, let $f(3) = 6$

* Recall that a graph with removable discontinuity is only 1 point away from becoming continuous.

Calculus **Ch. 1.4b Notes** **Intermediate Value Theorem (IVT)**

Review continuity conditions:

i) $f(c)$ is defined **ii)** $\lim_{x \to c} f(x)$ exists ($\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x)$) **iii)** $\lim_{x \to c} f(x) = f(c)$

Warm-up problem: Prove that the following is discontinuous at x = 2. Is it removable? If so, redefine f(2) to make the function continuous. (step through continuity conditions)

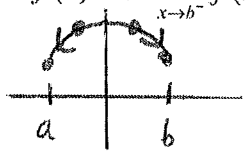
$$f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2}, & x \neq 2 \\ x + 5, & x = 2 \end{cases}$$

i) $f(2) = 2 + 5 = 7$

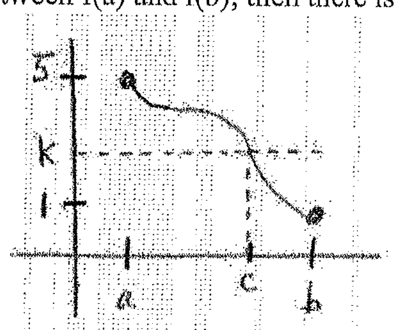
ii) $\lim_{x \to 2} \frac{x^2 - 4}{x - 2} \to \frac{0}{0} \to \lim_{x \to 2} \frac{\cancel{(x-2)}(x+2)}{\cancel{(x-2)}} \to 2 + 2 = 4$

iii) $\lim_{x \to 2} f(x) \neq f(2)$ therefore removable discontinuity at x=2.

Continuity on a closed interval: If a function is continuous on an open interval (a, b) and $\lim_{x \to a^+} f(x) = f(a)$ and $\lim_{x \to b^-} f(x) = f(b)$, then the function is continuous on the closed interval [a, b].



Intermediate Value Theorem: If f is continuous on a closed interval [a, b] and k is any number between f(a) and f(b), then there is at least one number c in [a, b] such that f(c) = k.



*In other words, if a function is continuous, then the graph has to touch all the y-values between the 2 endpoints (at least once)

Example 1: Use the IVT to show that there is a zero in the interval [0, 1] for the function $f(x) = x^3 + 2x - 1$.

f(c) = 0 ; y-value of 0 ; x-values endpoints of 0 and 1

*test endpoints

* f(x) is continuous [0,1] ← Establishing and stating continuity is important!

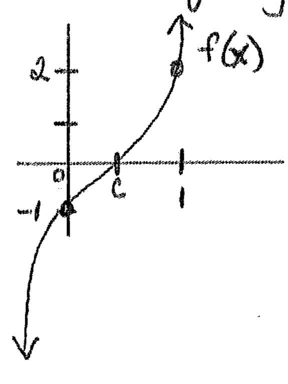
$f(0) = 0^3 + 2(0) - 1 = -1$ → compare these y-values with our target: y = 0

$f(1) = 1^3 + 2(1) - 1 = 2$

By IVT, since $f(0) = -1 < 0 \leq 2 = f(1)$

then there must be a value of c

where $f(c) = 0$.



(14)

Example 2: Verify the IVT applies to $f(x)=\frac{x^2+x}{x-1}$ on the interval $\left[\frac{5}{2},4\right]$ for f(c) = 6 and find c.

x-values

y-value

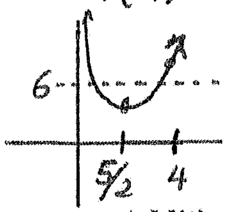
specific x-value of the ordered pair.

* V.A. at $x=1$

* $f(x)$ continuous $[5/2, 4]$

$f(5/2)=\frac{2.5^2+2.5}{2.5-1}=\frac{35}{6}\approx 5.8$

$f(4)=\frac{4^2+4}{4-1}=\frac{20}{3}=6.7$



a) By IVT, since $5.8 < f(c)=6 < 6.7$, $f(c)=6$ on interval $[5/2, 4]$

b) To find c, set $f(x)=6$, solve for x.

$\frac{x^2+x}{x-1}=6 \rightarrow \frac{x^2+x}{x-1}=\frac{6}{1} \rightarrow x^2+x=6(x-1)$

$\rightarrow x^2+x=6x-6 \rightarrow x^2+x-6x+6=0$

$x^2-5x+6=0$

$(x-3)(x-2)=0$

$x=3, \cancel{x=2}$

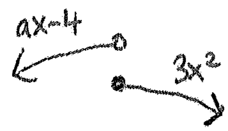
$\boxed{c=3}$

Additional Continuity Practice Problems:

Making a Function Continuous In Exercises 61–66, find the constant a, or the constants a and b, such that the function is continuous on the entire real number line.

Continuity conditions:

1. $f(c)$ is defined
2. $\lim_{x\to c} f(x)$ exists $\quad (\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x))$
3. $\lim_{x\to c} f(x) = f(c)$



61. $f(x)=\begin{cases} 3x^2, & x \geq 1 \\ ax-4, & x<1 \end{cases}$

* In this piecewise function, 2 equations define 2 separate graphs (that may or may not be connected)

* step through continuity conditions:

$\boxed{c=1}$ ← find c-value by looking at the restriction defined for the graphs

i) $f(1)=3(1)^2=3$

ii) $\lim_{x\to 1^-} ax-4 = a(1)-4 = \boxed{a-4}$ $\quad$ $\lim_{x\to 1^+} 3x^2 = 3(1)^2 = \boxed{3}$

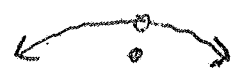
$a-4=3$

$\boxed{a=7}$

If $a=7$, then $\lim_{x\to 1} f(x)=3$

iii) $f(1)=\lim_{x\to 1} f(x) = 3$ ✓

f(x) continuous at $x=1$ if $a=7$



graph

66. $g(x)=\begin{cases} \frac{x^2-a^2}{x-a}, & x \neq a \\ 8, & x=a \end{cases}$

point

* In this piecewise function, one equation defines a graph with a hole that may or may not be filled in by the 2nd equation (defined at a point)

$\boxed{c=a}$

i) $g(a)=8$

ii) $\lim_{x\to a} \frac{x^2-a^2}{x-a} \rightarrow \frac{a^2-a^2}{a-a} \rightarrow \frac{0}{0} \rightarrow \lim_{x\to a} \frac{\cancel{(x-a)}(x+a)}{\cancel{(x-a)}}$

$\lim_{x\to a} x+a = a+a = \boxed{2a}$

iii) $\lim_{x\to a} f(x) = g(a) \rightarrow 2a=8 \rightarrow \boxed{a=4}$

f(x) continuous at $x=a$ if $a=4$

Key 15

Calculus **Ch. 1.5 Notes: Limits Approaching Infinity (Vertical Asymptotes)**

Infinite Limits: a limit where the function increases or decreases without bound (towards infinity) as x approaches c

*If the limit as x approaches c from either right or left is $\pm\infty$, then x = c is a vertical asymptote

* Rational Functions: $y = \frac{f(x)}{g(x)}$ If g(x) has no factors that cancel, then there is a vertical asymptote.

Example 1: Find all the vertical asymptotes of $f(x) = \frac{x^2-3x+2}{x^2-4}$ $= \frac{(x-1)\cancel{(x-2)}}{\underbrace{\cancel{(x-2)}}_{\text{hole}}\underbrace{(x+2)}_{\text{vertical asymptote}}}$

vertical asymptote exists at $\boxed{x=-2}$

Finding One-Sided Limits approaching Vertical Asymptotes:

Steps:

1) Evaluate Limit using the argument (plug in the value)
2) If Limit is undefined ($\frac{nonzero}{zero}$) then there is a vertical asymptote
3) To further evaluate the one-sided limit (determining the direction of arrows as $+\infty$ or $-\infty$)
   a. Test decimals 0.1 to the left of the argument x-value
   b. Test decimal 0.1 to the right of the argument x-value
4) Determine if the resulting fraction is a positive or negative value
   a. A positive decimal value indicates the one-sided limit is $+\infty$
   b. A negative decimal value indicates the one-sided limit is $-\infty$

Example 2: Determine $\lim_{x\to 2} f(x)$ for $f(x) = \frac{x+1}{x-2}$

$\lim_{x\to 2} \frac{x+1}{x-2} \to \frac{2+1}{2-2} \to \frac{3}{0}$ This indicates that vertical asymptote exists at x=2.

$\lim_{x\to 2} f(x) = d.n.e$ (but we can use one-sided limits to further subcategorize) (either $+\infty$ or $-\infty$)

$\lim_{x\to 2^-} \frac{x+1}{x-2} \to \frac{1.9+1}{1.9-2} \to \frac{+}{-} \to \boxed{-\infty}$

test using x=1.9

$\lim_{x\to 2^+} \frac{x+1}{x-2} \to \frac{3}{0} \to \lim_{x\to 2^+} \frac{x+1}{x-2} \to \frac{2.1+1}{2.1-2} \to \frac{+}{+} = \boxed{+\infty}$

test x=2.1

$\boxed{\lim_{x\to 2} f(x) = \text{does not exist}}$

no subcategory since arrows are not same direction

Find the following:

3) $\lim_{x\to -3^-} \frac{9-x^2}{x-4} = \frac{9-(-3)^2}{-3-4} = \frac{0}{-7}$

$\boxed{0}$

4) $\lim_{x\to 0^-} \frac{5x-x^2}{x^2-x} = \frac{0}{0}$

$\lim_{x\to 0^-} \frac{\cancel{x}(5-x)}{\cancel{x}(x-1)} \to \lim_{x\to 0^-} \frac{5-x}{x-1} = \frac{5-0}{0-1}$ $\frac{5}{-1}$

$= \boxed{-5}$

5) $\lim_{x\to -2^-} \frac{x^2+1}{x+2} = \frac{(-2)^2+1}{-2+2} = \frac{5}{0}$

VA., Limit DNE $\nearrow +\infty$ $\searrow -\infty$

test $x=-2.1$

$\frac{(-2.1)^2+1}{-2.1+2} = \frac{+}{-} = \boxed{-\infty}$

6) $\lim_{x\to 5} \frac{3x^2-1}{25-x^2} = \frac{3(5)^2-1}{25-5^2} = \frac{74}{0}$ $\nearrow +\infty$ $\searrow -\infty$

limit does not exist

7) $\lim_{x\to -3^+} \frac{2x^2+3x-9}{x+3}$ $\quad 2(-3)^2-9-9 = \frac{0}{0}$

$\lim_{x\to -3^+} \frac{(2x-3)\cancel{(x+3)}}{\cancel{(x+3)}} = 2(-3)-3 = \boxed{-9}$

8) $\lim_{x\to -4^+} \frac{2x^2-1}{x^2-16} = \frac{2(-4)^2-1}{(-4)^2-16} = \frac{31}{0}$ $\nearrow +\infty$ $\searrow -\infty$

$\frac{2(-3.9)^2-1}{(-3.9)^2-16} = \frac{+}{-} = \boxed{-\infty}$

9) $\lim_{x\to 1^+} \frac{x^2-2}{x^2+2x+1} = \frac{1-2}{1^2+2+1} = \boxed{\frac{-1}{4}}$

10) $\lim_{x\to 3^+} \frac{4x^2-14x+6}{x-3} = \frac{4(9)-14(3)+6}{3-3}$ $\quad \frac{0}{0}$

$\lim_{x\to 3^+} \frac{2(2x^2-7x+3)}{(x-3)}$

$\lim_{x\to 3^+} \frac{2(2x-1)\cancel{(x-3)}}{\cancel{(x-3)}} = 2(2(3)-1) = 2(5)$

$= \boxed{10}$

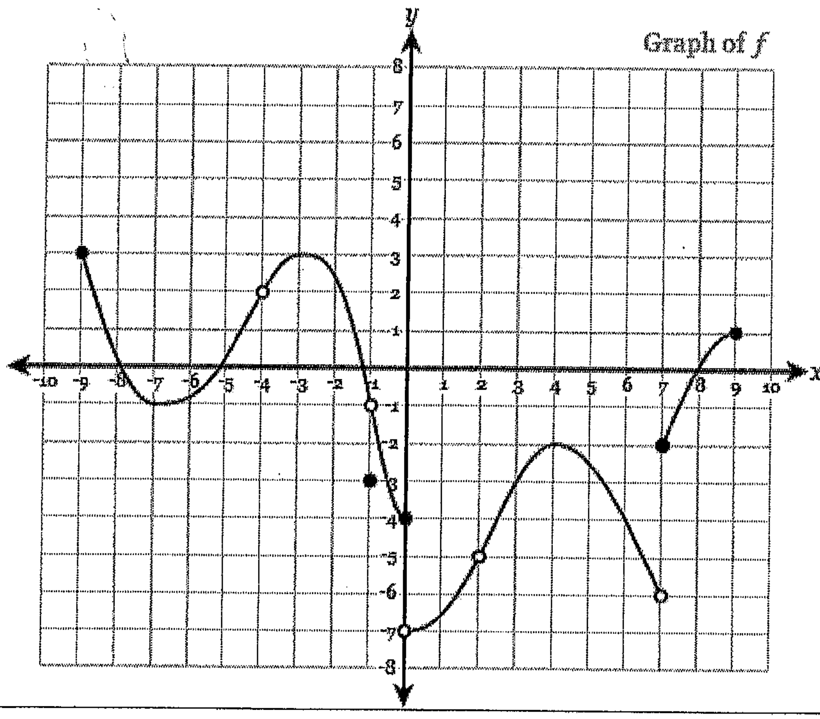
Key 17

Non-AP Calculus **1.4-1.5 Continuity/IVT/Limits Classwork Problems**

Non-Removable discontinuity: point where graph is not continuous and Limit does not exist

Removable Discontinuity: point where graph is not continuous but the limit exists

1) Identify values of x and determine the types of discontinuity for the below graph:



Non-Removable Discontinuity:

$x = 0$

$x = 7$

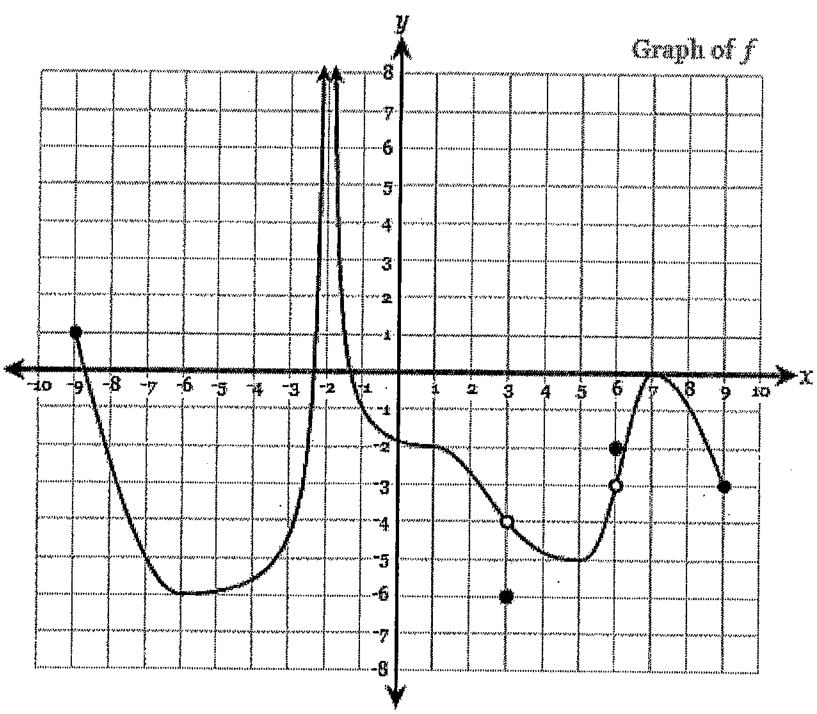
Removable Discontinuity:

$x = -4$

$x = -1$

$x = 2$

2) Identify values of x and determine the types of discontinuity for the below graph:



Non-Removable Discontinuity:

$x = -2$

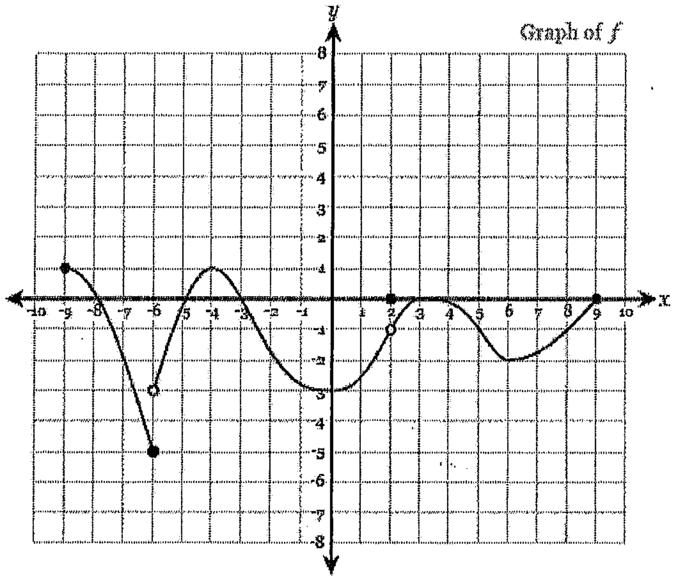
Removable Discontinuity:

$x = 3$

$x = 6$

Continuity Conditions

1. f(c) is defined (point exists on the graph)
2. The $\lim_{x\to c} f(x)$ exists $[\lim_{x\to c^+} f(x) = \lim_{x\to c^-} f(x)]$
3. f(c) = $\lim_{x\to c} f(x)$



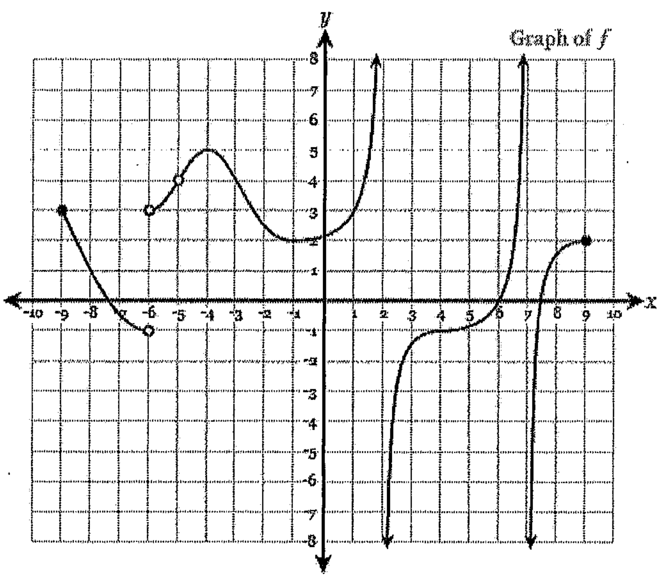
3) Use the definition of continuity to determine whether the function f(x) graphed below is continuous at x= 2.

i) $f(2)=0$

ii) $\lim_{x\to 2} f(x) = -1$

iii) $f(2) \neq \lim_{x\to 2} f(x)$

Removable discontinuity at $x=2$



4) Use the definition of continuity to determine whether the function f(x) graphed below is continuous at x= -6.

i) $f(-6) =$ undefined

ii) $\lim_{x\to -6^-} f(x) = -1 \quad \lim_{x\to -6^+} f(x) = 3$

$\lim_{x\to -6} f(x)$ does not exist

Nonremovable discontinuity at $x=-6$

Continuity Conditions

i) f(c) is defined (point exists on the graph)
ii) The $\lim_{x\to c} f(x)$ exists $[\lim_{x\to c^+} f(x) = \lim_{x\to c^-} f(x)]$
iii) $f(c) = \lim_{x\to c} f(x)$

Use Continuity Conditions to show that f(x) is discontinuous at a point and state reason for discontinuity. Then determine if the discontinuity is removable or non-removable and state why.

5)

$$f(x) = \begin{cases} x^2 - 4x + 6, & x < 3 \\ -x^2 + 4x - 2, & x \geq 3 \end{cases}$$

i) $f(3) = -3^2 + 4(3) - 2 = 1$

ii) $\lim_{x\to 3^-} x^2 - 4x + 6 = 3 \quad \lim_{x\to 3^+} -x^2 + 4x - 2 = 1$

$\lim_{x\to 3} f(x)$ does not exist

Nonremovable discontinuity at $x = 3$

6)

$$f(x) = \begin{cases} x, & x \leq 1 \\ 1 - x, & x > 1 \end{cases}$$

i) $f(1) = 1$

ii) $\lim_{x\to 1^-} x = 1 \quad \lim_{x\to 1^+} 1 - x = 0$, $\lim_{x\to 1} f(x)$ does not exist

Nonremovable discontinuity at $x = 1$

7)

$$f(x) = \begin{cases} \dfrac{x+2}{2}, & x \leq 3 \\ \dfrac{12 - 2x}{3}, & x > 3 \end{cases}$$

i) $f(3) = \frac{3+2}{2} = \frac{5}{2}$

ii) $\lim_{x\to 3^-} \frac{x+2}{2} = \frac{5}{2} \quad \lim_{x\to 3^+} \frac{12 - 2(3)}{3} = \frac{6}{3} = 2$

$\lim_{x\to 3} f(x)$ does not exist

Nonremovable discontinuity at $x = 3$

8)

$$f(x) = \begin{cases} x^2 - 4x + 6, & x < 3 \\ -x^2 + 4x - 2, & x \geq 3 \end{cases}$$

i) $f(3) = 1$

ii) $\lim_{x\to 3} f(x)$ dne

Nonremovable discontinuity at $x = 3$

Using the Intermediate Value Theorem In Exercises 95–98, verify that the Intermediate Value Theorem applies to the indicated interval and find the value of c guaranteed by the theorem.

95. $f(x) = x^2 + x - 1$, $[0, 5]$, $f(c) = 11$

$f(x)$ continuous $[0,5]$

$f(0) = -1$

$f(5) = 29$

By IVT, since $f(0) < 11 < f(5)$
$f(c) = 11$ in interval $[0,5]$

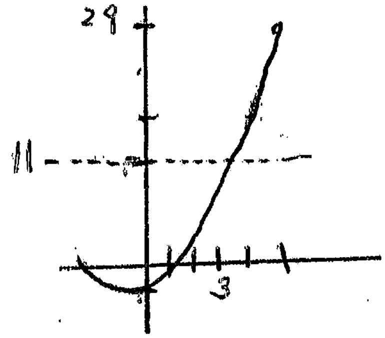
$x^2 + x - 1 = 11$

$x^2 + x - 12 = 0$

$(x+4)(x-3) = 0$

$\cancel{x = -4}$, $x = 3$

$\boxed{c = 3}$



96. $f(x) = x^2 - 6x + 8$, $[0, 3]$, $f(c) = 0$

$f(x)$ continuous $[0,3]$

$f(0) = 8$

$f(3) = -1$

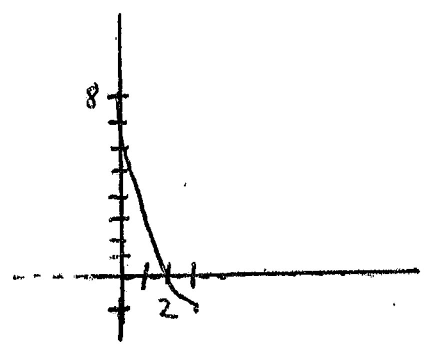
By IVT, since $f(3) < 0 < f(0)$,
$f(c) = 0$ in interval $[0,3]$

$x^2 - 6x + 8 = 0$

$(x-4)(x-2) = 0$

$\cancel{x = 4}$, $x = 2$

$\boxed{c = 2}$



97. $f(x) = x^3 - x^2 + x - 2$, $[0, 3]$, $f(c) = 4$

$f(x)$ continuous $[0,3]$

$f(0) = -2$

$f(3) = 19$

By IVT, since $f(0) < 4 < f(3)$
$f(c) = 4$ in interval $[0,3]$

$x^3 - x^2 + x - 2 = 4$

$x^3 - x^2 + x - 6 = 0$

$(x-2)(x^2+x+3) = 0$

$x - 2 = 0$

$x = 2$

$\boxed{c = 2}$

$x^2 + x + 3 = 0$

no solution

Find the following:

1) $\lim_{x\to -3^-} \frac{9-x^2}{x-4} = \frac{9-(-3)^2}{-3-4} = \frac{0}{-7}$

$\boxed{0}$

2) $\lim_{x\to 0^-} \frac{5x-x^2}{x^2-x} = \frac{0}{0}$

$\lim_{x\to 0^-} \frac{\cancel{x}(5-x)}{\cancel{x}(x-1)} \to \lim_{x\to 0^-} \frac{5-x}{x-1} = \frac{5-0}{0-1}\ \frac{5}{-1}$

$= \boxed{-5}$

3) $\lim_{x\to -2^-} \frac{x^2+1}{x+2} = \frac{(-2)^2+1}{-2+2} = \frac{5}{0}$

VA., Limit DNE $\nearrow +\infty$ $\searrow -\infty$

test $x=-2.1$

$\frac{(-2.1)^2+1}{-2.1+2} = \frac{+}{-} = \boxed{-\infty}$

4) $\lim_{x\to 5} \frac{3x^2-1}{25-x^2} = \frac{3(5)^2-1}{25-5^2} = \frac{74}{0}$ $\nearrow +\infty$ $\searrow -\infty$

limit does not exist

5) $\lim_{x\to -3^+} \frac{2x^2+3x-9}{x+3}$ $\quad 2(-3)^2-9-9 = \frac{0}{0}$

$\lim_{x\to -3^+} \frac{(2x-3)\cancel{(x+3)}}{\cancel{(x+3)}} = 2(-3)-3 = \boxed{-9}$

6) $\lim_{x\to -4^+} \frac{2x^2-1}{x^2-16} = \frac{2(-4)^2-1}{(-4)^2-16} = \frac{31}{0}$ $\nearrow +\infty$ $\searrow -\infty$

$\frac{2(-3.9)^2-1}{(-3.9)^2-16} = \frac{+}{-} = \boxed{-\infty}$

7) $\lim_{x\to 1^+} \frac{x^2-2}{x^2+2x+1} = \frac{1-2}{1^3+2+1} = \boxed{\frac{-1}{4}}$

8) $\lim_{x\to 3^+} \frac{4x^2-14x+6}{x-3} = \frac{4(9)-14(3)+6}{3-3}\ \frac{0}{0}$

$\lim_{x\to 3^+} \frac{2(2x^2-7x+3)}{(x-3)}$

$\lim_{x\to 3^+} \frac{2(2x-1)\cancel{(x-3)}}{\cancel{(x-3)}} = 2(2(3)-1) = 2(5)$

$= \boxed{10}$

22

Finding a One-Sided Limit In Exercises 33–48, find the one-sided limit (if it exists).

33. $\lim_{x \to -1^+} \frac{1}{x+1}$ $\qquad \frac{1}{0}$ V.A. $+\infty$, $-\infty$

-0.9

$\frac{1}{-0.9+1} = \frac{+}{+} = \boxed{+\infty}$

34. $\lim_{x \to 1^-} \frac{-1}{(x-1)^2} \to \frac{-1}{0^2}$ V.A. $+\infty$, $-\infty$

$x=0.9$

$\frac{-1}{(0.9-1)^2} = \frac{-}{+} = \boxed{-\infty}$

35. $\lim_{x \to 2^+} \frac{x}{x-2} \to \frac{2}{0}$ VA $+\infty$, $-\infty$

test $x=2.1$

$\frac{2.1}{2.1-2} = \frac{+}{+} = \boxed{+\infty}$

36. $\lim_{x \to 2^-} \frac{x^2}{x^2+4} = \frac{2^2}{2^2+4} = \frac{4}{8} = \boxed{\frac{1}{2}}$

37. $\lim_{x \to -3^-} \frac{x+3}{x^2+x-6}$ $\qquad \frac{-3+3}{3^2+-3-6} = \frac{0}{0}$

$\lim_{x \to -3^-} \frac{\cancel{(x+3)}}{\cancel{(x+3)}(x-2)}$

$\lim_{x \to -3^-} \frac{1}{x-2} = \frac{1}{-3-2} = \boxed{\frac{-1}{5}}$

50. $f(x) = \frac{x^3-1}{x^2+x+1}$

$\lim_{x \to 1^-} f(x)$

$\lim_{x \to 1^-} \frac{x^3-1}{x^2+x+1} = \frac{1-1}{1+1+1} = \frac{0}{3} = \boxed{0}$

51. $f(x) = \frac{1}{x^2-25}$

$\lim_{x \to 5^-} f(x)$

$\lim_{x \to 5^-} \frac{1}{x^2-25} = \frac{1}{0}$ VA $+\infty$, $-\infty$

test $x=4.9$

$\frac{1}{4.9^2-25} = \frac{+}{-} = \boxed{-\infty}$

Key

Calculus **Ch. 3.5 Notes** **Limits at Infinity (End behavior)**

A. Checking for Horizontal Asymptotes (H.A.) $\left(\lim_{x\to\infty} f(x) \quad or \quad \lim_{x\to-\infty} f(x)\right)$

If $f(x) = \frac{p(x)}{q(x)}$, then **compare the degrees between numerator and denominator**

i) If Numerator degree < Denominator degree, then the H.A. is y = 0

Example 1: $\lim_{x\to\infty} \frac{3x^2-7}{2x^3+1} = \boxed{0}$

ii) If Denominator degree = Numerator degree, then H.A. is $y = \frac{numerator\ coefficient}{denominator\ coefficient}$

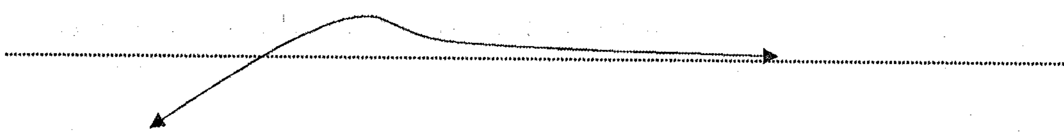
Example 2: $\lim_{x\to\infty} \frac{5x^2+3}{2x^2+4x-9} = \boxed{\frac{5}{2}}$ $\qquad \lim_{x\to-\infty} \frac{5x^2+3}{2x^2+4x-9} = \boxed{\frac{5}{2}}$

iii) If Numerator degree > Denominator degree, then H.A. does not exist (limit is therefore $+\infty$ or $-\infty$)

test $x=100$

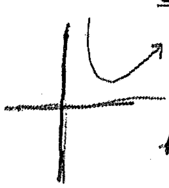
Example 3: $\lim_{x\to\infty} \frac{2x^3+1}{7x^2+5x+10} =$ $+\infty$ / $-\infty$ $\qquad \frac{2(100)^3+1}{7(100)^2+5(100)+10} \to \frac{+}{+}$ $\boxed{+\infty}$

Note: a H.A. is a description of end behavior, not a boundary that the graph can't cross. A function can NEVER cross a vertical asymptote, but it might cross a horizontal asymptote.



Use Horizontal Asymptote Rules for the following:

same degree, take ratio of coefficients



4) $\lim_{x\to\infty} \frac{3x^2+1}{2x-5}$ $+\infty$ / $-\infty$

test $x=100$

$\frac{3(100)^2+1}{2(100)-5} \to \frac{+}{+} \to \boxed{+\infty}$

5) $\lim_{x\to-\infty} \frac{3x^2+1}{2x-5}$ $+\infty$ / $-\infty$

test $x=-100$

$\frac{3(-100)^2+1}{2(-100)-5} \to \frac{+}{-} \to \boxed{-\infty}$

6) $\lim_{x\to-\infty} \frac{3x+1}{5-2x} \to \frac{3}{-2} = \boxed{\frac{-3}{2}}$

7) $\lim_{x\to\infty} \frac{3x+1}{5-2x} \to \boxed{\frac{3}{-2}}$

8) $\lim_{x\to\infty} \frac{3x+1}{2x^2-5} = \boxed{0}$

9) $\lim_{x\to-\infty} \frac{3x^3+1}{2x^2-5}$ $+\infty$ / $-\infty$

test $x=-100$

$\frac{3(-100)^3+1}{2(-100)^2-5} \to \frac{-}{+} \to \boxed{-\infty}$

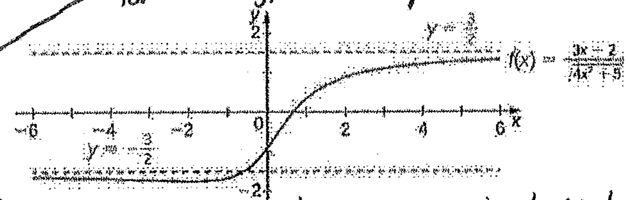
B. Finding Horizontal Asymptotes with **Radicals in denominator**

Think of this as a special case. Split horizontal asymptotes only apply for this type of setup.

Ex. 10: Find the Horizontal asymptotes for:

$$y = \frac{3x-2}{\sqrt{4x^2+5}}$$

*compare degrees



* Evaluate $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$

$$\lim_{x\to\infty}\frac{3x-2}{\sqrt{4x^2+5}} \to \frac{3}{\sqrt{4}} \to \frac{3}{2}$$

$$\lim_{x\to-\infty}\frac{3x-2}{\sqrt{4x^2+5}} \to -\frac{3}{\sqrt{4}} = -\frac{3}{2}$$

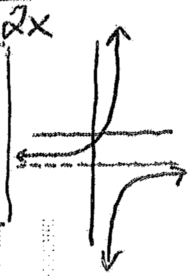
$\boxed{\text{Horizontal Asymptotes at } y=\frac{3}{2} \text{ and } y=-3/2}$

* Need to change sign of ratio when $\lim\limits_{x\to-\infty} f(x)$

* Important Note!
We do not change signs for $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$ for rational functions

Ex: $f(x)=\frac{3x-1}{1-2x}$

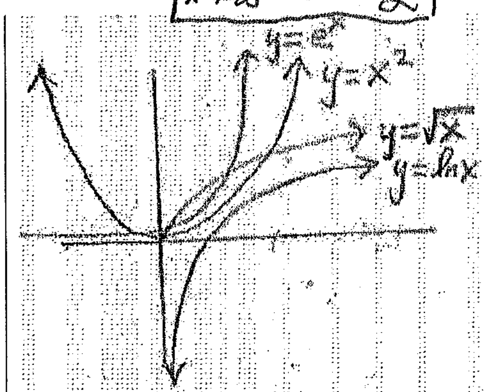
$\boxed{\lim\limits_{x\to\infty} f(x) = \frac{3}{-2} \quad \lim\limits_{x\to-\infty} f(x) = -\frac{3}{2}}$



C. Comparative Growth Rates

*Families of Functions grow at predictable rates in relations to each other as x **approaches** $+\infty$

***Logarithms < Radicals < Polynomial (Algebraic) < Exponential**
(slowest) (fastest)



* $\lim\limits_{x\to\infty} \frac{\text{slower}}{\text{faster}} = \boxed{0}$

* $\lim\limits_{x\to\infty} \frac{\text{faster}}{\text{slower}} \to \boxed{+\infty \text{ or } -\infty}$

test $x=100$ to determine between $+\infty$ and $-\infty$

*Note: Comparative Growth Rates relationship **only apply** when limit approaches infinity. (NOT $-\infty$)

Ex. 11 $\lim\limits_{x\to\infty}\frac{\sqrt{5000x+1000}}{x^2}$ $\quad \frac{\text{Radical}}{\text{Polynomial}}$

$= \boxed{0}$

Ex. 13 $\lim\limits_{x\to\infty}\frac{\ln(40000000x)}{2x} \to \frac{\text{logarithm}}{\text{algebraic}}$

$= \boxed{0}$

Ex. 12 $\lim\limits_{x\to\infty}\frac{-e^{2x}}{1000x^4+x^5}$ $\quad \frac{\text{exponential}}{\text{polynomial}}$

test $x=100$

$$\frac{-e^{2(100)}}{1000(100)^4+100^5} \to \frac{-}{+} \to \boxed{-\infty}$$

Ex. 14 $\lim\limits_{x\to\infty}\frac{-\sqrt{3000x-4}}{\ln(5x+1)} \to \frac{\text{Radical}}{\text{logarithm}}$ $\nearrow +\infty$ $\searrow -\infty$

test $x=100$

$$\frac{-\sqrt{3000(100)-4}}{\ln(5(100)+1)} \to \frac{-}{+} \to \boxed{-\infty}$$