$f'(x) = \begin{cases} 2x - a & \text{if } x \le 1 \\ a & \text{if } x > 1 \end{cases}$

Key

2.2 AP Practice Problems (p.182) – Derivative as a function & differentiability

1. The function $f(x) = \begin{cases} x^2 - ax & \text{if} \quad x \le 1 \\ ax + b & \text{if} \quad x > 1 \end{cases}$, where a and b are constants. If f is differentiable at $x = 1$, then $a + b =$

(A) -3 (B) -2 $\boxed{\text{(C)}}$ 0 (D) 2

* $f(x)$ is differentiable at $x=1$ (set derivatives equal)

$2x - a = a + 0$

$2(1) - a = a + 0$

$2 = 2a$

$\boxed{1 = a}$

* $f(x)$ is continuous at $x = 1$ (set equations equal)

$x^2 - ax = ax + b$

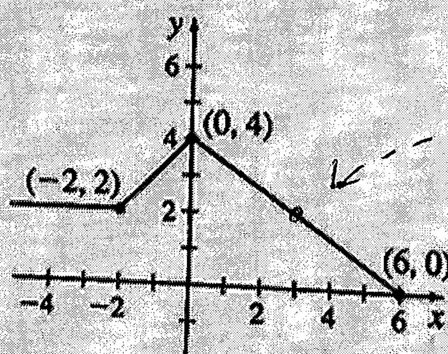
$(1)^2 - a(1) = a(1) + b$

$1 - a = a + b$

$1 = 2a + b \rightarrow 1 = 2(1) + b \rightarrow \boxed{b = -1}$

$a + b \rightarrow 1 + (-1) = \boxed{0}$

2. The graph of the function f, given below, consists of three line segments. Find $\lim_{h \to 0} \frac{f(3+h) - f(3)}{h}$.



(A) -1 $\boxed{\text{(B)}}$ $-\frac{2}{3}$ (C) $-\frac{3}{2}$ (D) does not exist

$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$f'(3) = \lim_{h \to 0} \frac{f(3+h) - f(3)}{h}$

slope of graph at $x = 3$

$f'(3) = \frac{0 - 4}{6 - 0} = \frac{-4}{6}$

$\boxed{f'(3) = -\frac{2}{3}}$

3. If $f(x) = \begin{cases} \dfrac{x^2 - 25}{x - 5} & \text{if} \quad x \ne 5 \\ 5 & \text{if} \quad x = 5 \end{cases}$

which of the following statements about f are true?

✓ I. $\lim_{x \to 5} f$ exists.

× II. f is continuous at $x = 5$.

× III. f is differentiable at $x = 5$.

$\boxed{\text{(A)}}$ I only (B) I and II only

(C) I and III only (D) I, II, and III

* step thru continuity conditions:

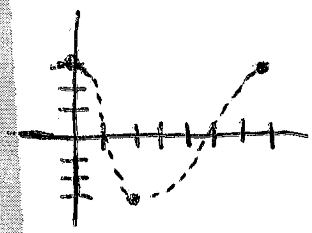
i) $f(5) = 5$

ii) $\lim_{x \to 5} \frac{x^2 - 25}{x - 5} \rightarrow \frac{0}{0} \rightarrow \lim_{x \to 5} \frac{(x+5)\cancel{(x-5)}}{\cancel{(x-5)}} = 10$

iii) $f(5) \ne \lim_{x \to 5} f(x)$

Removable Discontinuity at $x = 5$ (hole at $x = 5$)

*sketch graph first



4. Suppose f is a function that is differentiable on the open interval $(-2, 8)$. If $f(0) = 3$, $f(2) = -3$, and $f(7) = 3$, which of the following must be true?

✓ I. f has at least 2 zeros.

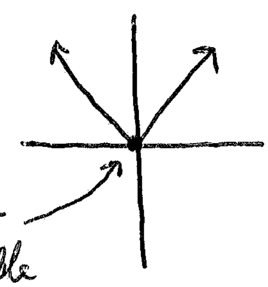
✓ II. f is continuous on the closed interval $[-1, 7]$.

✓ III. For some c, $0 < c < 7$, $f(c) = -2$.

(A) I only (B) I and II only

(C) II and III only $\boxed{\text{(D)}}$ I, II, and III

$f(x) = |x|$



$f(x)$ not differentiable at $x=0$

5. If $f(x) = |x|$, which of the following statements about f are true?

✓ I. f is continuous at 0.

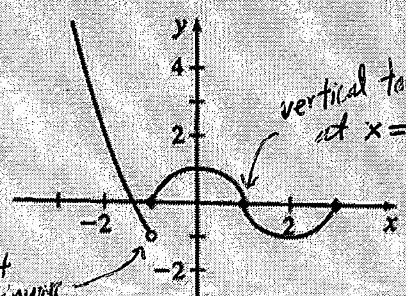
✗ II. f is differentiable at 0.

✓ III. $f(0) = 0$.

(A) I only (B) III only

$\boxed{\text{(C)}}$ I and III only (D) I, II, and III

6. The graph of the function f shown in the figure has horizontal tangent lines at the points $(0, 1)$ and $(2, -1)$ and a vertical tangent line at the point $(1, 0)$. For what numbers x in the open interval $(-2, 3)$ is f not differentiable?



vertical tangent at $x=1$ (not differentiable)

not continuous at $x=-1$

(A) −1 only $\boxed{\text{(B)}}$ −1 and 1 only

(C) −1, 0, and 2 only (D) −1, 0, 1, and 2

7. Let f be a function for which $\lim\limits_{h \to 0} \dfrac{f(1+h) - f(1)}{h} = -3$.
Which of the following must be true?
✓ I. f is continuous at 1.
II. f is differentiable at 1.
III. f' is continuous at 1.

(A) I only (B) II only
$\boxed{\text{(C)}}$ I and II only (D) I, II, and III

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = -3$$

$$\boxed{f'(1) = -3}$$

slope of the graph at $x = 1$ is -3

8. At what point on the graph of $f(x) = x^2 - 4$ is the tangent line parallel to the line $6x - 3y = 2$?

$\boxed{\text{(A)}}$ $(1, -3)$ (B) $(1, 2)$ (C) $(2, 0)$ (D) $(2, 4)$

line: $6x - 3y = 2$
$-3y = -6x + 2$
$y = \frac{-6}{-3}x + \frac{2}{-3}$
$y = 2x - 2/3 \rightarrow$ line has slope of $m = 2$

$f(x) = x^2 - 4$
$f'(x) = 2x$

$2 = 2x$
$x = 1$

$f(1) = (1)^2 - 4$
$\boxed{f(1) = -3}$

9. At $x = 2$, the function $f(x) = \begin{cases} 4x + 1 & \text{if } x \le 2 \\ 3x^2 - 3 & \text{if } x > 2 \end{cases}$ is

(A) Both continuous and differentiable.
$\boxed{\text{(B)}}$ Continuous but not differentiable.
(C) Differentiable but not continuous.
(D) Neither continuous nor differentiable.

Continuity conditions:
i) $f(2) = 4(2) + 1 = 9$
ii) $\lim\limits_{x \to 2^-} 4x + 1 = 9$ $\lim\limits_{x \to 2^+} 3x^2 - 3 = 9$
iii) $f(2) = \lim\limits_{x \to 2} f(x) = 9$
$f(x)$ continuous at $x = 2$

$f'(x) = \begin{cases} 4 & \text{if } x \le 2 \\ 6x & \text{if } x < 2 \end{cases}$

$\lim\limits_{x \to 2^-} 4 = 4$ $\lim\limits_{x \to 2^+} 6x = 12$

Since $\lim\limits_{x \to 2^-} f'(x) \neq \lim\limits_{x \to 2^+} f'(x)$, $f(x)$ not differentiable at $x = 2$ (no consistent slope at $x = 2$)

10. Oil is leaking from a tank. The amount of oil, in gallons, in the tank is given by $G(t) = 4000 - 3t^2$, where t, $0 \le t \le 24$ is the number of hours past midnight.

(a) Find $G'(5)$ using the definition of the derivative.

(b) Using appropriate units, interpret the meaning of $G'(5)$ in the context of the problem.

a) -30

b)

a) $G'(t) = 0 - 6t$

$G'(5) = -6(5) = -30$

$G'(5) = -30$ gallons per hour

b) $G'(5)$ means oil is leaking at rate of -30 gallons per hour when $t = 5$ hrs after midnight.

11. A rod of length 12 cm is heated at one end. The table below gives the temperature $T(x)$ in degrees Celsius at selected numbers x cm from the heated end.

x	0	2	5	7	9	12
$T(x)$	80	71	66	60	54	50

(a) Use the table to approximate $T'(8)$.

(b) Using appropriate units, interpret $T'(8)$ in the context of the problem.

a) -3

a) $T'(8) \approx \frac{T(9) - T(7)}{9 - 7} = \frac{54 - 60}{9 - 7} = \frac{-6}{2} \rightarrow -3 \ °C/cm$

b) $T'(8)$ is the rate of change of temperature per cm from one end when $x = 8$ cm