Solution Key

Calculus Notes Ch. 2.6 - Related Rates

Related Rates: Problems involving finding the rate of change for a variable with respect to time

This is also an application of implicit differentiation: Finding derivatives of variables with respect to time t.

Related Rates Steps:

1. Write what you are given *use units of measurement to help match appropriate variable to values
2. Write what you are trying to find
3. Write an algebraic or geometric equation relating the variables (needs to be in terms of the rates that you are either given or are trying to find)
4. Differentiate equation with respect to time *t*
5. Substitute and solve

*Important Note: Remember that when the item is getting bigger, the rate is positive
If the item is getting smaller, the rate is negative – regardless of direction

Example 1: The sides of a square are increasing at a rate of 5 cm/min. How fast is the area increasing when the sides measure 15 cm in length?

$A = x^2$

represents change in Area w/ respect to time

represents change in side length with respect to time

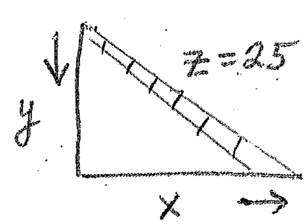
$\frac{dA}{dt} = 2x\left(\frac{dx}{dt}\right)$

Given: $\frac{dx}{dt} = 5\,cm/min$ Find $\frac{dA}{dt} =$ ___

$x = 15\,cm$

$\frac{dA}{dt} = 2(15)(5) = \boxed{150\,cm^2/min}$

Example 2: A 25 foot ladder is leaning against a vertical wall. The bottom of the ladder is being pulled horizontally away from the wall at the constant rate of 3 ft / sec. a) How fast is the top of the ladder moving and in what direction when the bottom of the ladder is 15 ft from the wall? b) at what rate is the area changing when the bottom of the ladder is 15 ft from the wall?



$x = 15$

$y = 20$

$z = 25$

$15^2 + y^2 = 25^2$

$y = 20$

$\frac{dx}{dt} = 3$

$\frac{dy}{dt} =$ ___

$\frac{dz}{dt} = 0$

$x^2 + y^2 = z^2$

$2x\left(\frac{dx}{dt}\right) + 2y\left(\frac{dy}{dt}\right) = 2z\left(\frac{dz}{dt}\right)$

$2(15)(3) + 2(20)\frac{dy}{dt} = 2(25)(0)$

$40\left(\frac{dy}{dt}\right) = -90$ $\boxed{\frac{dy}{dt} = -9/4\,ft/s}$

$A = \frac{1}{2}xy$ $f'g + fg'$

$\frac{dA}{dt} = \frac{1}{2}\left(\frac{dx}{dt}\right)(y) + \frac{1}{2}(x)\left(\frac{dy}{dt}\right)$

$\frac{dA}{dt} = \frac{1}{2}(3)(20) + \frac{1}{2}(15)(-9/4)$

$= 30 - \frac{135}{8} = \boxed{\frac{105}{8}\,ft^2/s}$

$\approx 13.125\,ft^2/s$

②

Example 3: A spherical balloon is being inflated at a rate of 10 cubic centimeters per second. Find the rate of change of the surface area of the balloon at the moment when the surface area is 64π.

change in volume (cm^3/s)

$V = \frac{4}{3}\pi r^3$ and $S = 4\pi r^2$

$\frac{dV}{dt} = 10\, cm^3/s$

$\frac{dS}{dt} = \_\_\_\_$

$S = 64\pi$

$\frac{dV}{dt} = \frac{4}{\cancel{3}}\pi \cdot \cancel{3} r^2 \left(\frac{dr}{dt}\right)$

$\frac{dV}{dt} = 4\pi r^2 \left(\frac{dr}{dt}\right)$

$\frac{dS}{dt} = 8\pi r \left(\frac{dr}{dt}\right)$

$64\pi = 4\pi r^2$

$16 = r^2$

$r = 4$

$10 = 4\pi(4)^2\left(\frac{dr}{dt}\right)$

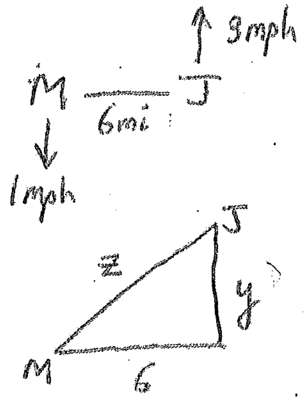
$\frac{10}{64\pi} = \frac{dr}{dt}$

$\frac{dr}{dt} = \frac{5}{32\pi}$

$\frac{dS}{dt} = 8\pi(4)\left(\frac{5}{32\pi}\right)$

$\boxed{\frac{dS}{dt} = 5\, cm^2/sec}$

Example 4: Joe is standing 6 miles straight east of Moe. If Joe walks straight north at 3 mph while Moe walks straight south at 1 mph, at what rate is the distance between them changing after 2 hours?



$\frac{dy}{dt} = 4\, mph$

$y = (4)(2) = 8\, mi$

$z = \underline{10}\, mi$

$6^2 + 8^2 = z^2$

$z = 10$

Find $\frac{dz}{dt} = \_\_\_\_$

$x^2 + y^2 = z^2$

$2x\left(\frac{dx}{dt}\right) + 2y\left(\frac{dy}{dt}\right) = 2z\left(\frac{dz}{dt}\right)$

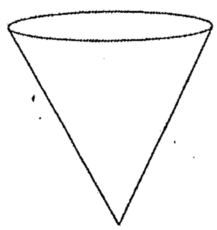
$\cancel{2(6)(0)} + 2(8)(4) = 2(10)\left(\frac{dz}{dt}\right)$

$64 = 20\frac{dz}{dt}$

$\frac{dz}{dt} = \frac{64}{20} = \frac{16}{5} = \boxed{3.2\, mph}$

Example 5: A conical tank (vertex down) is 10 feet across the top and 12 feet deep. If water is flowing into the tank at a rate of 10 cubic feet per minute, find the rate of change of the depth of the water when the water is 8 feet deep.

*Use similar triangles to rewrite equation using less variables. (find relationship between height and radius, use substitution)

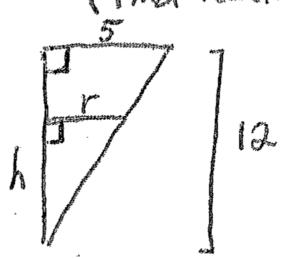


$V = \frac{\pi}{3} r^2 h$

$\frac{dV}{dt} = 10\, ft^3/min$

$\frac{dh}{dt} = \_\_\_\_$

$h = 8\, ft.$



$\frac{r}{5} = \frac{h}{12}$

$5h = 12r$

$\frac{5}{12}h = r$

*Rewrite volume in terms of "h"

$V = \frac{\pi}{3}\left(\frac{5h}{12}\right)^2 h = \frac{\pi}{3} \cdot \frac{25h^2}{144} \cdot h = \frac{25\pi}{432}h^3$

$V = \frac{25\pi}{432}h^3$

$\frac{dV}{dt} = \frac{25\pi}{432} \cdot 3h^2\left(\frac{dh}{dt}\right)$

$10 = \frac{25\pi}{432} \cdot 3(8)^2 \frac{dh}{dt}$

$10 = \frac{4800\pi}{432} \cdot \frac{dh}{dt}$

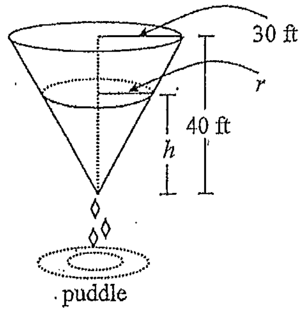
$10 \cdot \frac{432}{4800\pi} = \frac{dh}{dt}$

$\boxed{\frac{dh}{dt} = \frac{9}{10\pi}\, ft/min}$

Solution Key 2014 (3)

A.P. Calculus AB **Worksheet 2-6 – Related Rates WS #1**

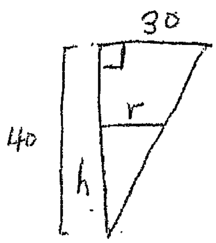
1. A conical water tank with a height of 40 ft and a radius of 30 ft is leaking at the rate of 2 ft^3 / min. When the height (h) of the water in the tank is 30 ft, at what rate is the height of the water changing? (Volume of a cone $= \frac{1}{3}\pi r^2 h$)



$\frac{dV}{dt} = -2\ ft^3/min$

$h = 30$

$\frac{dh}{dt} = \_\_\_$



$\frac{r}{30} = \frac{h}{40}$

$40r = 30h$

$r = \frac{3}{4}h$

$V = \frac{\pi}{3} r^2 h$

$V = \frac{\pi}{3}\left(\frac{3h}{4}\right)^2 h$

$V = \frac{\pi}{3} \cdot \frac{9h^2}{16} h$

$V = \frac{3\pi}{16} h^3$

$\frac{dV}{dt} = \frac{3\pi}{16} \cdot 3h^2 \cdot \frac{dh}{dt}$

$\frac{dV}{dt} = \frac{9\pi}{16} h^2 \frac{dh}{dt}$

$-2 = \frac{9\pi}{16}(30)^2 \frac{dh}{dt}$

$\frac{-2 \cdot 16}{9\pi \cdot 30^2} = \frac{dh}{dt}$

$\boxed{\frac{dh}{dt} = \frac{-8}{2025\pi}\ ft/min.}$

2. The volume of a cube is decreasing at a rate of 10 m^3 / hour. How fast is the total surface area decreasing when the surface area is 54 m^2?

$V = x^3$

$\frac{dV}{dt} = 3x^2\left(\frac{dx}{dt}\right)$

$\frac{dV}{dt} = -10\ m^3/hr.$

$S = 6x^2$

$\frac{dS}{dt} = 12x \frac{dx}{dt}$

$S = 54 m^2$

$54 = 6x^2$ $\quad x = 3$

$\frac{dV}{dt} = 3x^2 \frac{dx}{dt}$

$-10 = 3(3)^2\left(\frac{dx}{dt}\right)$

$\frac{-10}{27} = \frac{dx}{dt}$

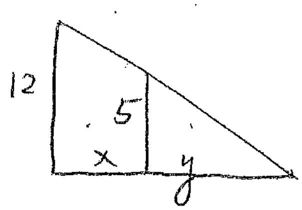
$\frac{dS}{dt} = 12(3)\left(\frac{-10}{27}\right)$

$= \boxed{-\frac{40}{3}\ m^2/hr.}$

3. A light is on the top of a 12 ft tall pole and a 5ft tall person is walking away from the pole at a rate of 2 ft/sec.

(a) At what rate is the tip of the shadow moving away from the pole when the person is 25 ft from the pole?

(b) At what rate is the ~~tip of the shadow~~ moving away from the person when the person is 25 ft from the pole? ROC for shadow length



$\frac{5}{12} = \frac{y}{x+y}$

$5x + 5y = 12y$

$5x = 7y$

$5\frac{dx}{dt} = 7\frac{dy}{dt}$

$\frac{dx}{dt} = 2\ ft/s$

$x = 25$

$5(2) = 7\frac{dy}{dt}$

$10 = 7\frac{dy}{dt}$

$\frac{10}{7} = \frac{dy}{dt}$

a) $\frac{dx}{dt} + \frac{dy}{dt} = 10/7 + 2$

$= \frac{24}{7}\ ft/s$

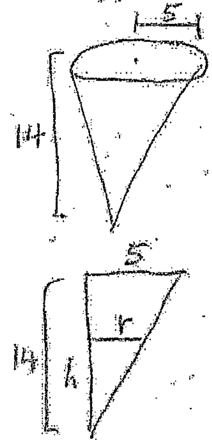
b) $\frac{dy}{dt} = \frac{10}{7}\ ft/s$

4

4. A tank of water in the shape of a cone is leaking water at a constant rate of 2 ft^3/hr. The base radius of the tank is 5 ft and the height of the tank is 14 ft.

(a) At what rate is the depth of the water in the tank changing when the depth of the water is 6 ft?

(b) At what rate is the radius of the top of the water in the tank changing when the depth of the water is 6 ft?



$\frac{r}{5} = \frac{h}{14}$

$\frac{dV}{dt} = -2\ ft^3/hr$

$\frac{dh}{dt} = \_\_\_$

$h = 6\ ft.$

$r = \frac{5h}{14}$

$V = \frac{\pi}{3}\pi r^2 h$

$V = \frac{\pi}{3}\left(\frac{5h}{14}\right)^2 h$

$V = \frac{\pi}{3} \cdot \frac{25}{196} h^3$

$V = \frac{25\pi}{588} h^3$

$\frac{dV}{dt} = \frac{25\pi}{588} 3h^2 \frac{dh}{dt}$

$-2 = \frac{75\pi}{588} (6)^2 \frac{dh}{dt}$

$\frac{-2 \cdot 588}{75\pi \cdot 36} = \frac{dh}{dt}$

$\boxed{\frac{dh}{dt} = \frac{-98}{225\pi}\ ft/hr.}$

Find $\frac{dr}{dt}$

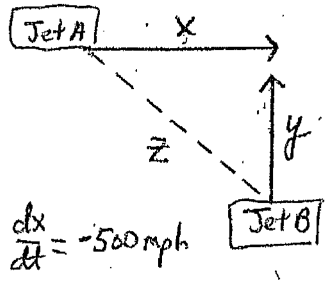
$\frac{r}{5} = \frac{h}{14}$ $\quad 14r = 5h$

$14\left(\frac{dr}{dt}\right) = 5\left(\frac{dh}{dt}\right)$

$\frac{dr}{dt} = \frac{5 \cdot -98}{225\pi} \cdot \frac{1}{14}$

$\boxed{\frac{dr}{dt} = \frac{-7}{45\pi}\ ft/hr.}$

5. Jet A travels due east from San Francisco toward St. Louis at 500 mph. Jet B travels due north from New Orleans toward St. Louis at 600mph. Find the rate of change of the distance between the two jets when they are 300 miles apart, and jet A is 100 miles from St. Louis (round answer to 3 decimal places) *Be sure to draw diagram, and watch your signs!)



$\frac{dx}{dt} = -500\ mph$

$\frac{dy}{dt} = -600\ mph$

$x = 100$

$z = 300$

$y = \underline{282.843}$ miles

$x^2 + y^2 = z^2$

$2x\left(\frac{dx}{dt}\right) + 2y\left(\frac{dy}{dt}\right) = 2z\left(\frac{dz}{dt}\right)$

$2(100)(-500) + 2(282.843)(-600) = 2(300)\left(\frac{dz}{dt}\right)$

$\boxed{\frac{dz}{dt} = -732.353\ mph}$

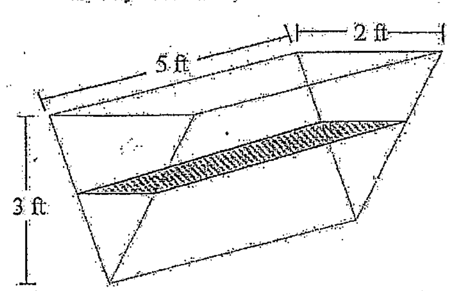
6. The trough shown in the figure below is 5 feet long, and its vertical cross sections are inverted isosceles triangles with base 2 feet and height 3 feet. Water is being siphoned out of the trough at a rate of 2 cubic feet per minute. At any time, t, let h be the depth and V be the volume of water in the trough.

a) Find the volume of water in the trough when it is full.

b) What is the rate of change in h at the instant when the trough is ¼ full by volume?

c) What is the rate of change in the area of the surface of the water (shaded in the figure) at the instant when the trough is ¼ full by volume?

(next page)



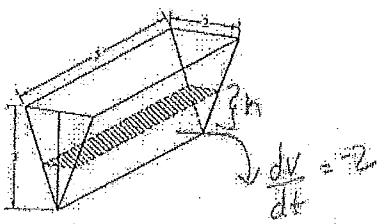
Name

4/23/12

AP Review: Related Rates

#6

1987 AB5



The trough shown in the figure above is 5 feet long, and its vertical cross sections are inverted isosceles triangles with base 2 feet and height 3 feet. Water is being siphoned out of the trough at the rate of 2 cubic feet per minute. At any time t, let h be the depth and V be the volume of water in the trough.

(a) Find the volume of water in the trough when it is full.

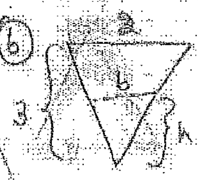
means volume of trough

(b) What is the rate of change in h at the instant when the trough is $\frac{1}{4}$ full by volume?

$\frac{dh}{dt}$

(c) What is the rate of change in the area of the surface of the water (shaded in the figure) at the instant when the trough is $\frac{1}{4}$ full by volume?

(a) $V = (\text{Area of } \triangle)(\text{height}) = \frac{1}{2} \cdot 2 \cdot 3 \cdot 5 = 15 \text{ ft}^3$

(b) 

similar triangles

$\frac{3}{2} = \frac{h}{b}$

$3b = 2h$

$b = \frac{2h}{3}$

$V = \frac{1}{2} bh \cdot 5 = \frac{5}{2} bh$

$V = \frac{5}{2}\left(\frac{2h}{3}\right)h = \frac{5}{3}h^2$

$\frac{dV}{dt} = \frac{5}{3} \cdot 2h \cdot \frac{dh}{dt}$

When trough is $\frac{1}{4}$ full:

$V = \frac{15}{4}$

$V = \frac{15}{4} = \frac{5}{3}h^2$

$h = 3/2$

$\frac{dV}{dt} = \frac{10h}{3} \cdot \frac{dh}{dt}$

$-2 = \frac{10}{3}\left(\frac{3}{2}\right) \cdot \frac{dh}{dt}$

$-\frac{2}{5} = \frac{dh}{dt}$

surface rectangle

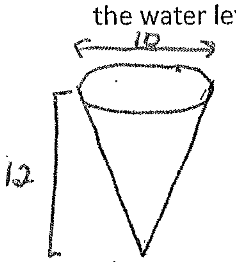
(c) $A = 5b = 5\left(\frac{2}{3}h\right) = \frac{10}{3}h$

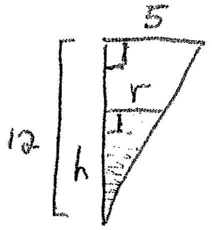
$\frac{dA}{dt} = \frac{10}{3} \cdot \frac{dh}{dt} = \frac{10}{3} \cdot \frac{-2}{5} = \frac{-4}{3}$

(5)

AP Calculus AB **Related Rates Quiz Review** Solutions Key

1. A conical tank (vertex down) full of water is 10 feet across the top and 12 feet deep. If water is flowing out of the tank at a rate of 12 cubic feet per minute, find the rate at which the water level is dropping when radius of the water level is 4 ft.





$\frac{dV}{dt} = -12 \text{ ft}^3/\text{min}$

$r = 4$ ft

* since $\frac{r}{5} = \frac{h}{12}$,

$\frac{4}{5} = \frac{h}{12}$, $5h = 48$

so $h = \frac{48}{5}$ ft.

$\frac{r}{5} = \frac{h}{12}$

$12r = 5h$

$r = \frac{5h}{12}$

Find $\frac{dh}{dt} = $ ____

$V = \frac{\pi}{3} r^2 h$

*Rewrite volume equation in terms of variable h.

$V = \frac{\pi}{3}\left(\frac{5h}{12}\right)^2 h = \frac{\pi}{3}\left(\frac{25h^2}{144}\right)h = \frac{25\pi}{432}h^3$

$V = \frac{25\pi}{432}h^3$

$\frac{dV}{dt} = \frac{25\pi}{432} \cdot 3h^2 \frac{dh}{dt}$

$-12 = \frac{25\pi}{432} \cdot 3\left(\frac{48}{5}\right)^2\left(\frac{dh}{dt}\right)$

$-12 = \frac{\cancel{25}\pi \cdot 3 \cdot 48^2}{432 \cdot \cancel{25}} \frac{dh}{dt}$

$-12 = 16\pi \frac{dh}{dt}$

$\frac{-12}{16\pi} = \frac{dh}{dt}$

$\boxed{\frac{dh}{dt} = \frac{-3}{4\pi} \text{ ft/min}}$

2. A conical tank (vertex down) full of water is 10 feet across the top and 12 feet deep. If water is flowing out of the tank at a rate of 12 cubic feet per minute, find **the rate of change of the radius of water level** when radius of the water level is 4 ft.

$V = \frac{\pi}{3} r^2 h$

$12r = 5h$

$\frac{12r}{5} = h$

Find $\frac{dr}{dt} = $ ____

$r = 4$

$V = \frac{\pi}{3} r^2 \left(\frac{12r}{5}\right)$

$V = \frac{12\pi}{15} r^3$

$V = \frac{4\pi}{5} r^3$

$\frac{dV}{dt} = \frac{4\pi}{5} \cdot 3r^2 \left(\frac{dr}{dt}\right)$

$-12 = \frac{4\pi}{5} \cdot 3(4)^2 \frac{dr}{dt}$

$-12 = \frac{4\pi \cdot 3 \cdot 16}{5} \cdot \frac{dr}{dt}$

$\frac{-12 \cdot 5}{4\pi \cdot 48} = \frac{dr}{dt}$

$\frac{-5}{16\pi} = \frac{dr}{dt}$

$\boxed{\frac{dr}{dt} = \frac{-5}{16\pi} \text{ ft/min}}$

OR

Since we know from part a, $\frac{dh}{dt} = \frac{-3}{4\pi}$.

and $\frac{r}{5} = \frac{h}{12}$

$12r = 5h$

$\downarrow$

$12\left(\frac{dr}{dt}\right) = 5\left(\frac{dh}{dt}\right)$

$\frac{dr}{dt} = \frac{5}{12}\left(\frac{dh}{dt}\right)$

$\frac{dr}{dt} = \frac{5}{12}\left(\frac{-3}{4\pi}\right) = \frac{-15}{48\pi} = \boxed{\frac{-5}{16\pi} \text{ ft/min}}$

6

3. Helium is pumped into a spherical balloon at the constant rate of 25 ft^3/min. At what rate is the surface area of the balloon increasing the moment when the diameter is 16 ft? $V = \frac{4}{3}\pi r^3$ $S = 4\pi r^2$

$\frac{dV}{dt} = 25\ \text{ft}^3/\text{min}$

Find $\frac{dS}{dt} =$ ____

diameter $= 16$, so $r = 8$

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dt} = \frac{4}{3}\pi \cdot 3r^2\left(\frac{dr}{dt}\right)$

$\frac{dV}{dt} = 4\pi r^2\left(\frac{dr}{dt}\right)$

$25 = 4\pi(8)^2\left(\frac{dr}{dt}\right)$

$\frac{25}{4\pi \cdot 64} = \frac{dr}{dt}$

$\frac{dr}{dt} = \frac{25}{256\pi}$ ft/min.

$S = 4\pi r^2$

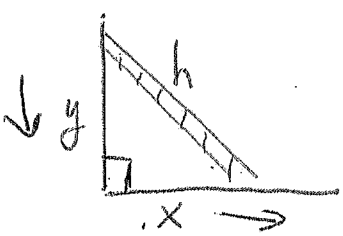
$\frac{dS}{dt} = 8\pi r\left(\frac{dr}{dt}\right)$

$\frac{dS}{dt} = 8\pi(8)\left(\frac{25}{256\pi}\right)$

$\boxed{\frac{dS}{dt} = \frac{25}{4}\ \text{ft}^2/\text{min.}}$

4. A 17 ft ladder is leaning against a wall. If the bottom of the ladder is pulled along the ground away from the wall at a constant rate of 5 ft/s,

a) how fast will the top of the ladder be moving down the wall when it is 8 ft above the ground?
b) At what rate is the area of the triangle changing when the ladder is 8 ft above ground?



$x^2 + y^2 = h^2$

$2x\left(\frac{dx}{dt}\right) + 2y\left(\frac{dy}{dt}\right) = 2z\left(\frac{dz}{dt}\right)$

$x = 15$ $\frac{dx}{dt} = 5$ ft/s

$y = 8$ $\frac{dy}{dt} =$ ____

$h = 17$ $\frac{dh}{dt} = 0$

a) $2(15)(5) + 2(8)\left(\frac{dy}{dt}\right) = \cancel{2(17)(0)}$

$150 + 16\left(\frac{dy}{dt}\right) = 0$ $\frac{dy}{dt} = \frac{-150}{16} = \boxed{\frac{-75}{8}\ \text{ft/s}}$

b) $A = \frac{1}{2}xy$

$\frac{dA}{dt} = \frac{1}{2}\left(\frac{dx}{dt}\right)y + \frac{1}{2}x\left(\frac{dy}{dt}\right)$

$= \frac{1}{2}(5)(8) + \frac{1}{2}(15)\left(\frac{-75}{8}\right)$

$= 20 - \frac{1125}{16}$

$\boxed{\frac{dA}{dt} = \frac{-805}{16}\ \text{ft}^2/\text{s}}$

Calculus AB Related Rates WS #2

1) 1990 problem #4

The radius r of a sphere is increasing at a constant rate of 0.04 centimeters per second.
(Note: the volume of a sphere with radius r is $V = \frac{4}{3}\pi r^3$.)

a) At the time when the radius of the sphere is 10 centimeters, what is the rate of increase of its volume?
b) At the time when the volume of the sphere is 36π cubic centimeters, what is the rate of increase of the area of a cross section through the center of the sphere?
c) At the time when the volume and the radius of the sphere are increasing at the same numerical rate, what is the radius?

a) $\frac{dr}{dt} = 0.04$ cm/s

$r = 10$ cm

$\frac{dV}{dt} =$ _____

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dt} = 4\pi r^2\left(\frac{dr}{dt}\right)$

$\frac{dV}{dt} = 4\pi(10)^2(0.04)$

$= 16\pi\ cm^3/s$

b) $V = 36\pi$

$V = \frac{4}{3}\pi r^3$

$\frac{4}{3}\pi r^3 = 36\pi$

$r^3 = 27$

$\underline{r=3}$

$A = \pi r^2$

$\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$

$\frac{dA}{dt} = 2\pi(3)(0.04)$

$\boxed{0.24\pi\ cm^2/s}$

c) set $\frac{dV}{dt} = \frac{dr}{dt}$

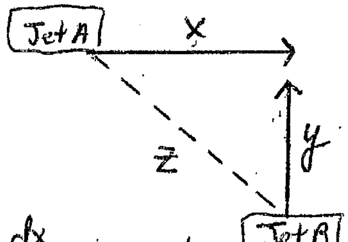
$\frac{dV}{dt} = 4\pi r^2\left(\frac{dr}{dt}\right)$

$\cancel{\frac{dr}{dt}} = 4\pi r^2\left(\cancel{\frac{dr}{dt}}\right)$

$1 = 4\pi r^2$

$\sqrt{\frac{1}{4\pi}} = r$, $\boxed{r = \frac{1}{2\sqrt{\pi}}\ cm}$

2. Jet A travels due east from San Francisco toward St. Louis at 500 mph. Jet B travels due north from New Orleans toward St. Louis at 600mph. Find the rate of change of the distance between the two jets when they are 300 miles apart, and jet A is 100 miles from St. Louis (round answer to 3 decimal places) *Be sure to draw diagram, and watch your signs!)



$\frac{dx}{dt} = -500$ mph

$\frac{dy}{dt} = -600$ mph

$x = 100$

$z = 300$

$y = \underline{282.843}$ miles

$x^2 + y^2 = z^2$

$2x\left(\frac{dx}{dt}\right) + 2y\left(\frac{dy}{dt}\right) = 2z\left(\frac{dz}{dt}\right)$

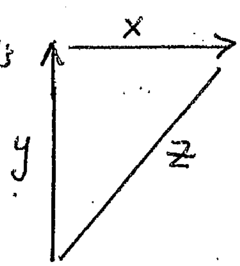
$2(100)(-500) + 2(282.843)(-600) = 2(300)\left(\frac{dz}{dt}\right)$

$\boxed{\frac{dz}{dt} = -732.353\ mph}$

3. A man is driving north at a rate of 17 m/s. He sees a railroad track 20m ahead of him that is perpendicular to the road. There is a train going east on the track crossing the road and the man determines with a radar gun that the engine is 35 m from him and the distance between his car and the engine is increasing at the rate of 5 m/s. What is the speed of the train?

$\frac{dy}{dt} = -17$ m/s

$\frac{dz}{dt} = 5$ m/s



$x = \underline{28.723}$

$y = 20$

$z = 35$

$x^2 + 20^2 = 35^2$

$x = 28.723$

$x^2 + y^2 = z^2$

$2x\left(\frac{dx}{dt}\right) + 2y\left(\frac{dy}{dt}\right) = 2z\left(\frac{dz}{dt}\right)$

$2(28.723)\left(\frac{dx}{dt}\right) + 2(20)(-17) = 2(35)(5)$

$\boxed{\frac{dx}{dt} = 17.929\ m/s}$

b. What is the rate at which the area of the triangle is changing?

$A = \frac{1}{2}xy$

$\frac{dA}{dt} = \frac{1}{2}\left(\frac{dx}{dt}\right)y + \frac{1}{2}x\left(\frac{dy}{dt}\right)$

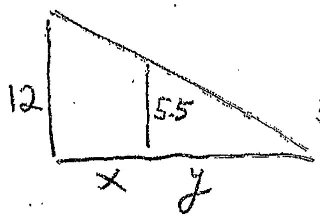
$\frac{dA}{dt} = \frac{1}{2}(17.929)(20) + \frac{1}{2}(28.723)(-17)$

$\boxed{\frac{dA}{dt} = \cancel{279.82}1\ m/s^2}$

$-64.84\ m/s^2$

8

4. A man 5.5 ft tall walks away at a rate of 2 ft/sec away from a lamppost that is 12 feet above ground. When the man is 8 ft away from the base of the light,
   a. At what rate is the length of his shadow changing? (ans: 1.69 ft/s)
   b. At what rate is the tip of his shadow moving? (ans: 3.69 ft/s)



$\frac{dx}{dt} = 2$ ft/s

$\frac{5.5}{12} = \frac{y}{x+y}$

$5.5x + 5.5y = 12y$

$5.5x = 6.5y$

$5.5\frac{dx}{dt} = 6.5\frac{dy}{dt}$

$5.5(2) = 6.5\frac{dy}{dt}$

a) $\frac{dy}{dt} = 1.692$ ft/s

b) $\frac{dx}{dt} + \frac{dy}{dt} = 3.692$ ft/s

5. A spherical balloon is inflated so that its volume is increasing at the rate of 40 cubic feet per minute. How fast is the surface area of the balloon increasing when the radius is 5 feet? (Note: $S = 4\pi r^2$ and $V = \frac{4}{3}\pi r^3$)

$\frac{dV}{dt} = 40$ ft^3/min.

$\frac{dS}{dt} =$ ____

$r = 5$ ft.

$\frac{dS}{dt} = 8\pi r\left(\frac{dr}{dt}\right)$

$\frac{dV}{dt} = 4\pi r^2\left(\frac{dr}{dt}\right)$

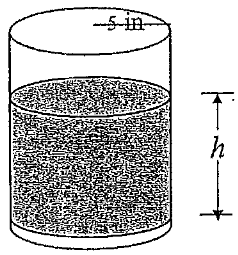
$40 = 4\pi(5)^2\left(\frac{dr}{dt}\right)$

$\frac{2}{5\pi} = \frac{dr}{dt}$

$\frac{dS}{dt} = 8\pi(5)\left(\frac{2}{5\pi}\right)$

$\boxed{\frac{dS}{dt} = 16 \text{ ft}^2/\text{min}}$

6. 2003 AB problem #5



A coffeepot has the shape of a cylinder with radius 5 inches, as shown in the figure above. Let h be the depth of the coffee in the pot, measured in inches, where h is a function of time, t, measured in seconds. The volume, V, of coffee in the pot is changing at the rate of $-5\pi\sqrt{h}$ cubic inches per second. (The volume of a cylinder with radius r and height h is $V = \pi r^2 h$.) Find $\frac{dh}{dt}$ as a function of h.

(This means your answer will contain the variable h)

$\frac{dV}{dt} = -5\pi\sqrt{h}$ in^3/sec

$r = 5$ in.

$V = \pi r^2 h$

$V = \pi(5)^2 h$

$V = 25\pi h$

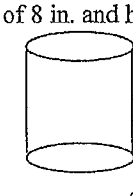
$\frac{dV}{dt} = 25\pi\left(\frac{dh}{dt}\right)$

$-5\pi\sqrt{h} = 25\pi\left(\frac{dh}{dt}\right)$

$\frac{-5\pi\sqrt{h}}{25\pi} = \frac{dh}{dt}$

$\boxed{\frac{dh}{dt} = \frac{-\sqrt{h}}{5} \text{ in/sec.}}$

7. Water is leaking out of a full cylindrical container at a rate of 2 in^3/hr. The container has a diameter of 8 in. and height of 12 in. At what rate is the height of the container changing when the container is half full?



$V = \pi r^2 h$

$\frac{dV}{dt} = -2$ in^3/hr.

$r = 4$ in.

$h = 12$ in.

$\frac{dh}{dt} =$ ____

$V = \pi(4)^2 h$

$V = 16\pi h$

$\frac{dV}{dt} = 16\pi\left(\frac{dh}{dt}\right)$

$-2 = 16\pi\left(\frac{dh}{dt}\right)$

$\frac{dh}{dt} = \frac{-2}{16\pi} = \boxed{\frac{-1}{8\pi} \text{ in/hr.}}$

11

Key 9

Related Rates Notes 2 - Similar Triangles and Shadow Problems

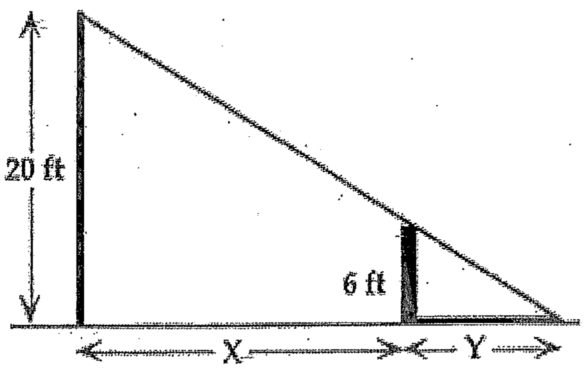
Example 1:

A man who is 6 feet tall is walking away from a lamp post at a rate of 5 feet per ~~minute~~ second.

The lamp post is 20 feet tall. The person casts a shadow on the ground in front of them.

a) How fast is the shadow growing when the person is 30 feet from the lamp post? → $\frac{dy}{dt}$

b) How fast is the tip of the shadow moving when the person is 30 ft from the lamp post? → $\frac{dx}{dt}+\frac{dy}{dt}$



Notes:

1) $\frac{dx}{dt} = rate\ of\ person\ walking$

2) $\frac{dy}{dt} = rate\ of\ change\ of\ shadow\ length$

3) $\frac{dx}{dt}+\frac{dy}{dt} = rate\ of\ change\ of\ tip\ of\ shadow$

$\frac{dx}{dt} = 5\ ft/s$

$x = 30\ ft$

$\frac{6}{20} = \frac{y}{x+y}$

$6(x+y) = 20y$

$6x+6y = 20y$

$6x = 14y$

$6\left(\frac{dx}{dt}\right) = 14\left(\frac{dy}{dt}\right)$

$6(5) = 14\left(\frac{dy}{dt}\right)$

$\frac{30}{14} = \frac{dy}{dt}$

$\frac{15}{7} = \frac{dy}{dt}$

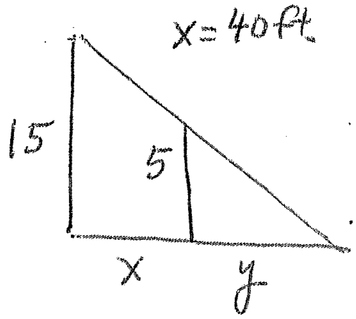
a) $\frac{dy}{dt} = \boxed{\frac{15}{7}\ ft/s}$

b) $\frac{dx}{dt}+\frac{dy}{dt} = 5+\frac{15}{7}$

$= \boxed{\frac{50}{7} \text{ or } 7.14\ ft/s}$

2. A street light is mounted at the top of a 15 ft pole. A man 5 ft tall walks towards the pole at a rate of 5ft per second. A) How fast is the tip of his shadow moving when he is 40 ft from the pole? B) How fast is the length of the shadow changing when he is 40 ft from the pole?

$x = 40 ft$



$\frac{5}{15} = \frac{y}{x+y}$

$\frac{dx}{dt} = -5\ ft/s$

$\frac{1}{3} = \frac{y}{x+y}$

$3y = x+y$

$2y = x$

$2\left(\frac{dy}{dt}\right) = \frac{dx}{dt}$

$2\left(\frac{dy}{dt}\right) = -5$

$\frac{dy}{dt} = \frac{-5}{2}\ ft/s$

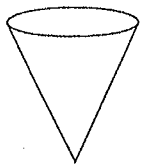
a) $\frac{dx}{dt}+\frac{dy}{dt} = -5-\frac{5}{2}$

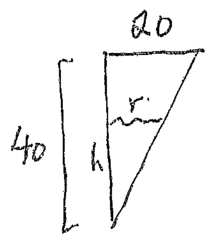
$= \boxed{-7.5\ ft/s}$

b) $\frac{dy}{dt} = \boxed{\frac{-5}{2}\ ft/s \text{ or } -2.5\ ft/s}$

10

3. A conical tank (vertex down) is 40 feet across the top and 40 feet deep. If water is leaking out of the tank at a rate of 80 cubic feet per minute, find the rate of change of the radius of the water when the water is 8 feet deep. ($V = \frac{1}{3}\pi r^2 h$)





$\frac{dV}{dt} = -80 \text{ ft}^3/\text{min}$

$h = 8 \text{ ft}$

$\frac{dr}{dt} = \_\_\_$

$\frac{r}{20} = \frac{h}{40}$

$20h = 40r$

$h = \frac{40}{20} r$

$h = 2r$, $\underline{r = 4}$

* Since $h = 2r$ and $h = 8$, $\underline{r = 4}$

$V = \frac{\pi}{3} r^2 h = \frac{\pi}{3} r^2 (2r)$

$V = \frac{2\pi}{3} r^3$

$\frac{dV}{dt} = \frac{2\pi}{3} \cdot 3r^2 \left(\frac{dr}{dt}\right)$

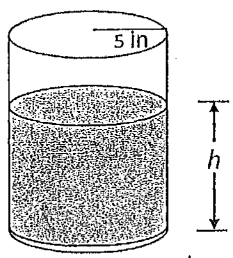
$\frac{dV}{dt} = 2\pi r^2 \left(\frac{dr}{dt}\right)$

$-80 = 2\pi (4)^2 \left(\frac{dr}{dt}\right)$

$\frac{-80}{32\pi} = \frac{dr}{dt}$

$$\boxed{\frac{dr}{dt} = \frac{-5}{2\pi} \text{ ft/min}}$$

4. **2003 AB problem #5**



A coffeepot has the shape of a cylinder with radius 5 inches, as shown in the figure above. Let *h* be the depth of the coffee in the pot, measured in inches, where *h* is a function of time, *t*, measured in seconds. The volume, *V*, of coffee in the pot is changing at the rate of $-5\pi\sqrt{h}$ cubic inches per second. (The volume of a cylinder with radius *r* and height *h* is $V = \pi r^2 h$.) Find $\frac{dh}{dt}$ as a function of *h*. (This means your answer will contain the variable *h*)

$\frac{dV}{dt} = -5\pi\sqrt{h}$

$r = 5 \text{ in.}$

$V = \pi r^2 h$

$V = \pi (5)^2 h$

$V = 25\pi h$

$\frac{dV}{dt} = 25\pi \left(\frac{dh}{dt}\right)$

$-5\pi\sqrt{h} = 25\pi \frac{dh}{dt}$

$\frac{-5\pi\sqrt{h}}{25\pi} = \frac{dh}{dt}$

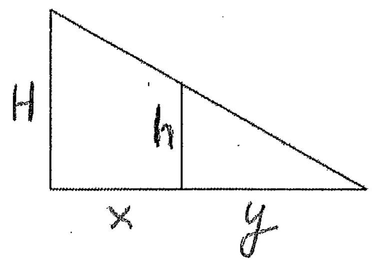
$$\boxed{\frac{dh}{dt} = \frac{-\sqrt{h}}{5} \text{ in/sec.}}$$

Ch. 2.6b More Similar Triangle Related Rates , Linear Approximation, L'Hopital's Rule

A. More Similar Triangles (Shadow Problem)

$\frac{dy}{dt}$ = R.O.C. of length of shadow;

$\frac{dy}{dt} + \frac{dx}{dt}$ = R.O.C. of tip of shadow



$$\frac{h}{H} = \frac{y}{x+y}$$

$$\frac{dx}{dt} = -5 \text{ ft/s}$$

Example 1: A man 6 ft tall walks at a rate of 5 ft/s towards the lamp post that is 15 ft above ground. When he is 10 ft from the base of the light

$x = 10$

a) At what rate is the length of his shadow changing?

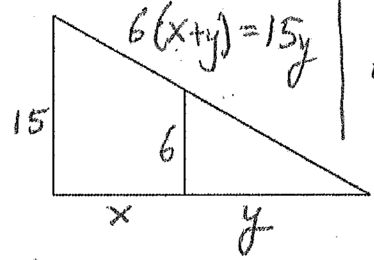
$$\frac{dy}{dt} = \frac{-10}{3} \text{ ft/s}$$

b) At what rate is the tip of his shadow changing?

$$\frac{dy}{dx} + \frac{dx}{dt} = -5 - \frac{10}{3} = \frac{-25}{3} \text{ ft/s}$$

a) $\frac{6}{15} = \frac{y}{x+y}$

$6(x+y) = 15y$



$6x + 6y = 15y$

$6x = 9y$

$6\frac{dx}{dt} = 9\frac{dy}{dt}$

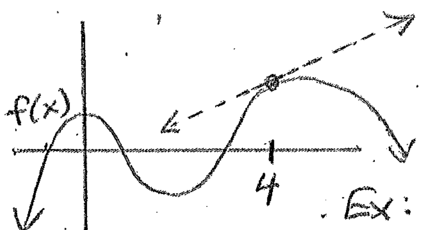
$6(-5) = 9\frac{dy}{dt}$

$\frac{-30}{9} = \frac{dy}{dt}$

$\frac{dy}{dt} = \frac{-10}{3} \text{ ft/s}$

$\frac{dx}{dt} + \frac{dy}{dt} = -5 - \frac{10}{3} = -\frac{25}{3} \text{ ft/s}$

B. Linear Approximation – Approximating nearby y-values on a graph using the tangent line equation



Linear approximation steps:
a) Identify equation and
b) Identify ordered pair of nearest integer x-value
c) Find derivative to determine slope
d) Find equation of tangent line $y - y_1 = m(x - x_1)$
e) Plug decimal value into tangent line equation to approximate function value: $y(x) \approx mx + b$

Ex: Use tangent line to approximate $f(4.2)$ instead of plugging $x = 4.2$ into the function.

Example 2: Use linearization to approximate $\sqrt{9.2}$

$y = \sqrt{x}$ $x = 9$, $y(9) = \sqrt{9} = 3$

point $(9, 3)$

$y' = \frac{1}{2}x^{-1/2}$

$y' = \frac{1}{2\sqrt{x}}$

$y'(9) = \frac{1}{2\sqrt{9}} = \frac{1}{2(3)} = \frac{1}{6}$

point $(9, 3)$

slope: $m = 1/6$

$y - 3 = \frac{1}{6}(x - 9)$

$y = \frac{1}{6}(x - 9) + 3$

$y(9.2) \approx \frac{1}{6}(9.2 - 9) + 3 = \boxed{3.033}$

12

Linear approximation steps:

a) Identify equation and
b) Identify ordered pair of nearest integer x-value
c) Find derivative to determine slope
d) Find equation of tangent line $y - y_1 = m(x - x_1)$
e) Plug decimal value into tangent line equation to approximate function value: $y(x) \approx mx + b$

Example 3: Use linearization to approximate $\sqrt[3]{-8.1}$

$y = \sqrt[3]{x}$ $x = -8$ $y = \sqrt[3]{-8} = -2$ point: $(-8, -2)$

$y = x^{1/3}$ $y'(-8) = \frac{1}{3(-8)^{2/3}} = \frac{1}{3(-2)^2} = \frac{1}{12}$ slope: $m = 1/12$

$y' = \frac{1}{3}x^{-2/3}$ $y + 2 = \frac{1}{12}(x+8)$ $y(-8.1) \approx \frac{1}{12}(-8.1+8) - 2 \approx \boxed{-2.008}$

$y' = \frac{1}{3x^{2/3}}$ $y = \frac{1}{12}(x+8) - 2$ $\sqrt[3]{-8.1} \approx -2.008$

L'Hopital's rule

L'Hopital's rule says that if $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{0}{0}$ or $\pm\frac{\infty}{\infty}$, then $\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$

Example 4: Find $\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x}$

$f'(x) = \frac{1}{2}(x+1)^{-1/2}$ $\lim_{x\to 0} \frac{\frac{1}{2\sqrt{x+1}}}{1} = \frac{1}{2\sqrt{1}} = \boxed{\frac{1}{2}}$

$g'(x) = 1$

Example 5: Find $\lim_{x\to 2} \frac{4x-8}{3x-2}$ $\frac{0}{4} = \boxed{0}$

Example 6: Find $\lim_{x\to 2} \frac{105x^2-420x+420}{108x^2-432x+432}$ $= \frac{0}{0}$

$f'(x) = 210x - 420$ $\lim_{x\to 2} \frac{210x-420}{216x-432} = \frac{0}{0}$

$g'(x) = 216x - 432$

$\lim_{x\to 2} \frac{210}{216} = \frac{210}{216} = \boxed{\frac{35}{36}}$

Example 7: Find $\lim_{x\to 5} \frac{x^2-10}{x-5}$ $\frac{25-10}{5-5} = \frac{15}{0}$ undefined

DNE

Example 8: Find $\lim_{x\to 0} \frac{\frac{1}{5x+2}-\frac{1}{2}}{x}$ $\frac{0}{0}$ $\to (5x+2)^{-1}$

$f'(x) = -1(5x+2)^{-2}(5)$ $\lim_{x\to 0} \frac{-5}{(5x+2)^2} = \frac{-5}{2^2} = \boxed{\frac{-5}{4}}$

$g'(x) = 1$

Example 9: Find $\lim_{x\to\infty} \frac{3x+1}{2x^2-5}$ $\frac{\infty}{\infty} = \frac{3}{4x} \to \frac{0}{4} = \boxed{0}$

Example 10: Find $\lim_{x\to\infty} \frac{3x^2+1}{2x-5}$ $\frac{\infty}{\infty}$

$\lim_{x\to\infty} \frac{6x}{2} \to \infty$

Example 11: Find $\lim_{x\to\infty} \frac{3x+1}{5-2x}$ $= \frac{\infty}{-\infty}$

$\lim_{x\to\infty} \frac{3}{-2} = \boxed{\frac{-3}{2}}$

AP Calculus AB **Linear Approximation Homework**

1. Find the local linear approximation of $f(x) = \sqrt{x}$ and at $a = 1$ and use it to approximate $\sqrt{0.9}$ and $\sqrt{1.1}$.

$y = \sqrt{x}$, use $x = 1$

$y(1) = \sqrt{1} = 1 \quad (1,1)$

$y' = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$

$y'(1) = \frac{1}{2\sqrt{1}} = \frac{1}{2}$

point: $(1,1)$

slope: $m = 1/2$

$y - 1 = \frac{1}{2}(x-1)$

$y = \frac{1}{2}(x-1)+1$

$\sqrt{0.9} \approx \frac{1}{2}(0.9-1)+1 = 0.95$

$\sqrt{1.1} \approx \frac{1}{2}(1.1-1)+1 = 1.05$

2. Find the local linear approximation of $f(x) = \frac{1}{\sqrt{x}}$ at a = 4 and use it to approximate $\frac{1}{\sqrt{3.9}}$ and $\frac{1}{\sqrt{4.1}}$.

$y = \frac{1}{\sqrt{x}} \quad y(4) = \frac{1}{\sqrt{4}} = \frac{1}{2} \quad (4, 1/2)$

$y = x^{-1/2}$

$y' = \frac{-1}{2}x^{-3/2}$

$y' = \frac{-1}{2x^{3/2}}$

$y'(4) = \frac{-1}{2(4)^{3/2}} = \frac{-1}{2(2)^3} = \frac{-1}{16} \quad m = -1/16$

point: $(4, 1/2)$ slope: $m = -1/16$

$y - 1/2 = \frac{-1}{16}(x-4)$

$y = \frac{-1}{16}(x-4) + 1/2$

$\frac{1}{\sqrt{3.9}} \approx \frac{-1}{16}(3.9-4) + 1/2 = \boxed{0.506}$

$\frac{1}{\sqrt{4.1}} \approx \frac{-1}{16}(4.1-4) + 1/2 = \boxed{0.4938}$

3. Use an appropriate local linear approximation to estimate the value of the given quantity.

a. $\sqrt{36.1}$ $\quad y = \sqrt{x}$ at $x = 36 \quad y(36) = \sqrt{36} = 6 \quad$ point: $(36, 6)$

$y = x^{1/2}$

$y' = \frac{1}{2}x^{-1/2}$

$y' = \frac{1}{2\sqrt{x}}$

$y'(36) = \frac{1}{2\sqrt{36}} = \frac{1}{2(6)}$

$y'(36) = \frac{1}{12}$

point: $(36, 6)$

slope: $m = 1/12$

$y - 6 = \frac{1}{12}(x-36)$

$y = \frac{1}{12}(x-36) + 6$

$\sqrt{36.1} \approx \frac{1}{12}(36.1-36)+6 \approx \boxed{6.008}$

b. $\frac{1}{10.1}$ $\quad y(10) = \frac{1}{10} \quad (10, \frac{1}{10})$

$y = \frac{1}{x} = x^{-1}$

$y'(x) = -1x^{-2}$

$y'(x) = \frac{-1}{x^2}$

$y'(10) = \frac{-1}{10^2} = \frac{-1}{100}$

slope: $m = \frac{-1}{100}$

$y - \frac{1}{10} = \frac{-1}{100}(x-10)$

$y = \frac{-1}{100}(x-10) + \frac{1}{10}$

$\frac{1}{10.1} \approx \frac{-1}{100}(10.1-10) + \frac{1}{10}$

$\approx \boxed{0.099}$

(14)

Use an appropriate local linear approximation to estimate the value of the given quantity.

c. $(1.97)^6$

$y = x^6$

$y(2) = 2^6 = 64$

point $(2, 64)$

$y' = 6x^5$

$y'(2) = 6(2)^5$

$y'(2) = 192$

$y - 64 = 192(x-2)$

$y = 192(x-2) + 64$

$(1.97)^6 \approx 192(1.97-2) + 64$

$\approx \boxed{58.24}$

d. $\sqrt[4]{15.8}$

$y = \sqrt[4]{x}$

$y(16) = \sqrt[4]{16} = 2$

point: $(16, 2)$

$y(x) = x^{1/4}$

$y' = \frac{1}{4}x^{-3/4}$

$y' = \frac{1}{4x^{3/4}}$

$y'(16) = \frac{1}{4(16)^{3/4}}$

$y'(16) = \frac{1}{32}$

point: $(16, 2)$

slope: $m = \frac{1}{32}$

$y - 2 = \frac{1}{32}(x-16)$

$y = \frac{1}{32}(x-16) + 2$

$\sqrt[4]{15.8} \approx \frac{1}{32}(15.8-16) + 2$

$\approx \boxed{1.99375}$

e. $\frac{1}{(3.9)^2}$

$y = \frac{1}{x^2} = x^{-2}$

$y(4) = \frac{1}{4^2} = \frac{1}{16}$

point: $(4, \frac{1}{16})$

$y' = -2x^{-3}$

$y' = \frac{-2}{x^3}$

$y'(4) = \frac{-2}{4^3}$

$y' = \frac{-2}{64} = \frac{-1}{32}$

point: $(4, 1/16)$

slope: $m = \frac{-1}{32}$

$y - \frac{1}{16} = \frac{-1}{32}(x-4)$

$y = \frac{-1}{32}(x-4) + \frac{1}{16}$

$\frac{1}{(3.9)^2} \approx \frac{-1}{32}(3.9-4) + \frac{1}{16}$

$\approx \boxed{0.066}$

f. $\frac{1}{1.9\sqrt{1.9}}$

$y = \frac{1}{x\sqrt{x}} = \frac{1}{x \cdot x^{1/2}} = x^{-3/2}$

$y' = \frac{-3}{2}x^{-5/2} = \frac{-3}{2x^{5/2}}$

$y(2) = \frac{1}{2\sqrt{2}}$

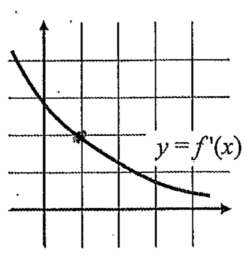
$y'(2) = \frac{-3}{2(2)^{5/2}} = \frac{-3}{8\sqrt{2}}$

$y - \frac{1}{2\sqrt{2}} = \frac{-3}{8\sqrt{2}}(x-2)$

$y = \frac{-3}{8\sqrt{2}}(x-2) + \frac{1}{2\sqrt{2}}$

$\frac{1}{1.9\sqrt{1.9}} \approx \frac{-3}{8\sqrt{2}}(1.9-2) + \frac{1}{2\sqrt{2}} \approx \boxed{0.3805}$

4. Suppose the only information we have about a function f is that $f(1) = 5$ and the graph of its **derivative** is as shown below.
   a. Use a linear approximation to estimate $f(0.9)$ and $f(1.1)$.
   b. Are your estimates in part a too large or too small? Explain why.



$f'(1) = 2$

point: $(1, 5)$

slope: $m = 2$

$y - 5 = 2(x-1)$

$y = 2(x-1) + 5$

$f(0.9) \approx 2(0.9-1) + 5 \approx \boxed{4.8}$

$f(1.1) \approx 2(1.1-1) + 5 \approx \boxed{5.2}$