2.1 AP Practice Problems (p.171) – Rates of Change and the Derivative

Key

1. The line $x + y = 5$ is tangent to the graph of $y = f(x)$ at the point where $x = 2$. The values $f(2)$ and $f'(2)$ are:

(A) $f(2) = 2;\ f'(2) = -1$ $\boxed{\text{(B)}}$ $f(2) = 3;\ f'(2) = -1$

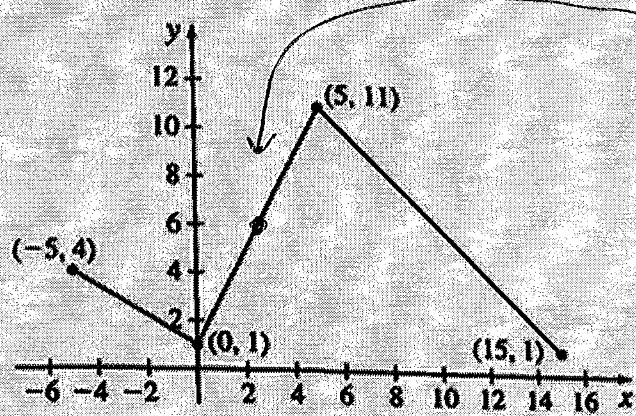
(C) $f(2) = 2;\ f'(2) = 1$ (D) $f(2) = 3;\ f'(2) = 2$

$x+y=5$

$y=5-x$ | $y'=-1$

$\boxed{y(2)=5-2=3 \mid y'(2)=-1}$

2. The graph of the function f, given below, consists of three line segments. Find $f'(3)$.



Graph of f

(A) 1 $\boxed{\text{(B)}}$ 2   (C) 3   (D) $f'(3)$ does not exist

$$f'(3) = \frac{11-1}{5-0} = \frac{10}{5} = 2$$

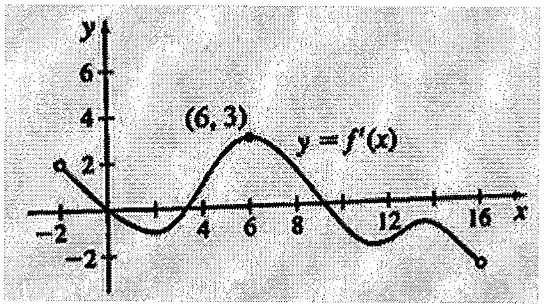
3. What is the instantaneous rate of change of the function $f(x) = 3x^2 + 5$ at $x = 2$?

(A) 5   (B) 7 $\boxed{\text{(C)}}$ 12   (D) 17

$f'(x) = 6x$

$f'(2) = 6(2) = \boxed{12}$

4. The function f is defined on the closed interval $[-2, 16]$. The graph of the derivative of f, $y = f'(x)$, is given below.



The point $(6, -2)$ is on the graph of $y = f(x)$. An equation of the tangent line to the graph of f at $(6, -2)$ is

(A) $y = 3$ (B) $y + 2 = 6(x + 3)$

(C) $y + 2 = 6x$ $\boxed{\text{(D)}}$ $y + 2 = 3(x - 6)$

$f'(6) = 3$

$m = 3$

point: $(6, -2)$

slope: $m = 3$

$y - y_1 = m(x - x_1)$

$\boxed{y + 2 = 3(x - 6)}$

perpendicular (opposite reciprocal)

5. If $x - 3y = 13$ is an equation of the normal line to the graph of f at the point $(2, 6)$, then $f'(2) =$

(A) $-\frac{1}{3}$ (B) $\frac{1}{3}$ $\boxed{\text{(C)}}$ -3 (D) $-\frac{13}{3}$

$x - 3y = 13$ | $y = \frac{-1x}{-3} + \frac{13}{-3}$ | $m = 1/3$ | $\boxed{f'(2) = -3}$

$-3y = -1x + 13$ | $y = \frac{1}{3}x - \frac{13}{3}$ | $m_2 = \frac{-3}{1}$

6. If f is a function for which $\lim_{x \to -3} \frac{f(x) - f(-3)}{x + 3} = 0$, then which of the following statements must be true?

(A) $x = -3$ is a vertical asymptote of the graph.

$\boxed{\text{(B)}}$ The derivative of f at $x = -3$ exists.

(C) The function f is continuous at $x = 3$.

(D) f is not defined at $x = -3$.

* Alternate limit definition of derivative

$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$ | $f'(-3) = \frac{f(x) - f(-3)}{x - (-3)} = 0$ | $\boxed{f'(-3) = 0}$

*slope of the graph at $x = -3$ is 0

7. If the position of an object on the x-axis at time t is $4t^2$, then the average velocity of the object over the interval $0 \le t \le 5$ is

(A) 5 $\boxed{\text{(B)}}$ 20 (C) 40 (D) 100

position function $s(t) = 4t^2$

Avg. velocity $= \frac{\text{change in position}}{\text{change in time}}$

Avg. velocity $= \frac{s(5) - s(0)}{5 - 0}$

$s(5) = 100$ $s(0) = 0$

Avg. velocity $= \frac{100 - 0}{5 - 0} = \frac{100}{5}$

$= \boxed{20}$

$m_{sec} \approx \frac{-5}{3}$ or -3 or $-11/5$

8. A tank is filled with 80 liters of water at 7 a.m. ($t = 0$). Over the next 12 hours the water is continuously used and no water is added to replace it. The table below gives the amount of water $A(t)$ (in liters) remaining in the tank at selected times t, where t measures the number of hours after 7 a.m.

t	0	2	5	7	9	12
$A(t)$	80	71	66	60	54	50

Use the table to approximate $A'(5)$.

* Any of the 3 approximations are acceptable

$A'(5) \approx \frac{A(7) - A(5)}{7 - 5} = \frac{60 - 66}{2} = -3$ liters/hr

OR

$\frac{A(5) - A(2)}{5 - 2} = \frac{66 - 71}{5 - 2} = \frac{-5}{3}$ liters/hr

OR

$\frac{A(7) - A(2)}{7 - 2} = \frac{60 - 71}{7 - 2} = \frac{-11}{5}$ liters/hr

$$f'(x) = \begin{cases} 2x-a & \text{if } x \le 1 \\ a & \text{if } x > 1 \end{cases}$$

Key

2.2 AP Practice Problems (p.182) – Derivative as a function & differentiability

1. The function $f(x) = \begin{cases} x^2 - ax & \text{if } x \le 1 \\ ax + b & \text{if } x > 1 \end{cases}$, where a and b are constants. If f is differentiable at $x = 1$, then $a + b =$

   (A) -3 (B) -2 $\boxed{\text{(C)}}$ 0 (D) 2

* $f(x)$ is continuous at $x = 1$ (set equations equal)

$x^2 - ax = ax + b$

$(1)^2 - a(1) = a(1) + b$

$1 - a = a + b$

$1 = 2a + b \rightarrow 1 = 2(1) + b \rightarrow \boxed{b = -1}$

* $f(x)$ is differentiable at $x = 1$ (set derivatives equal)

$2x - a = a + 0$

$2(1) - a = a + 0$

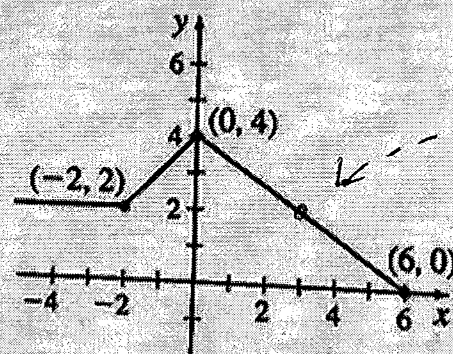
$2 = 2a$

$\boxed{1 = a}$

$a + b \rightarrow$

$1 + (-1) = \boxed{0}$

2. The graph of the function f, given below, consists of three line segments. Find $\lim\limits_{h\to 0} \dfrac{f(3+h) - f(3)}{h}$.



(A) -1 $\boxed{\text{(B)}}$ $-\dfrac{2}{3}$ (C) $-\dfrac{3}{2}$ (D) does not exist

$f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$

$f'(3) = \lim\limits_{h\to 0} \dfrac{f(3+h) - f(3)}{h}$

slope of graph at $x = 3$

$f'(3) = \dfrac{0-4}{6-0} = \dfrac{-4}{6}$

$\boxed{f'(3) = -\dfrac{2}{3}}$

3. If $f(x) = \begin{cases} \dfrac{x^2 - 25}{x - 5} & \text{if } x \ne 5 \\ 5 & \text{if } x = 5 \end{cases}$

   which of the following statements about f are true?

   ✓ I. $\lim\limits_{x\to 5} f$ exists.

   × II. f is continuous at $x = 5$.

   × III. f is differentiable at $x = 5$.

   $\boxed{\text{(A)}}$ I only (B) I and II only

   (C) I and III only (D) I, II, and III

* step thru continuity conditions:

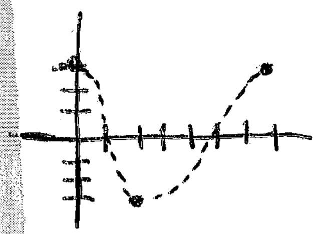
i) $f(5) = 5$

ii) $\lim\limits_{x\to 5} \dfrac{x^2 - 25}{x - 5} \rightarrow \dfrac{0}{0} \rightarrow \lim\limits_{x\to 5} \dfrac{(x+5)\cancel{(x-5)}}{\cancel{(x-5)}} = 10$

iii) $f(5) \ne \lim\limits_{x\to 5} f(x)$

Removable Discontinuity at $x = 5$ (hole at $x = 5$)

*sketch graph first



4. Suppose f is a function that is differentiable on the open interval $(-2, 8)$. If $f(0) = 3$, $f(2) = -3$, and $f(7) = 3$, which of the following must be true?

   ✓ I. f has at least 2 zeros.

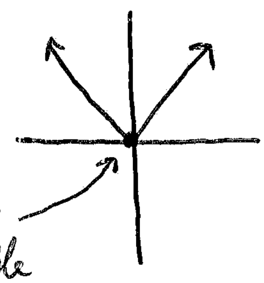
   ✓ II. f is continuous on the closed interval $[-1, 7]$.

   ✓ III. For some c, $0 < c < 7$, $f(c) = -2$.

   (A) I only (B) I and II only

   (C) II and III only $\boxed{\text{(D)}}$ I, II, and III

$f(x) = |x|$



$f(x)$ not differentiable at $x=0$

5. If $f(x) = |x|$, which of the following statements about f are true?

   ✓ I. f is continuous at 0.

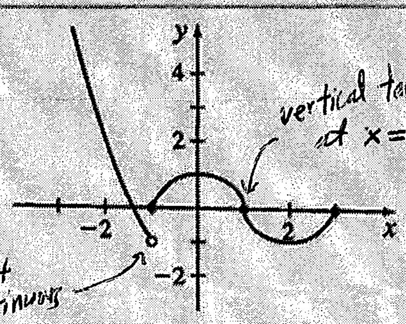
   ✗ II. f is differentiable at 0.

   ✓ III. $f(0) = 0$.

   (A) I only (B) III only

   $\boxed{\text{(C)}}$ I and III only (D) I, II, and III

6. The graph of the function f shown in the figure has horizontal tangent lines at the points $(0, 1)$ and $(2, -1)$ and a vertical tangent line at the point $(1, 0)$. For what numbers x in the open interval $(-2, 3)$ is f not differentiable?



vertical tangent at $x=1$ (not differentiable)

not continuous at $x=-1$

   (A) -1 only $\boxed{\text{(B)}}$ -1 and 1 only

   (C) $-1, 0$, and 2 only (D) $-1, 0, 1$, and 2

7. Let f be a function for which $\lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = -3$.

Which of the following must be true?

✓ I. f is continuous at 1.

II. f is differentiable at 1.

III. f' is continuous at 1.

(A) I only (B) II only

$\boxed{\text{(C)}}$ I and II only (D) I, II, and III

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = -3$$

$$\boxed{f'(1) = -3}$$

slope of the graph at $x=1$ is -3

8. At what point on the graph of $f(x) = x^2 - 4$ is the tangent line parallel to the line $6x - 3y = 2$?

$\boxed{\text{(A)}}$ $(1, -3)$ (B) $(1, 2)$ (C) $(2, 0)$ (D) $(2, 4)$

line: $6x - 3y = 2$

$-3y = -6x + 2$

$y = \frac{-6}{-3}x + \frac{2}{-3}$

$y = 2x - \frac{2}{3}$ → line has slope of $m = 2$

$f(x) = x^2 - 4$

$f'(x) = 2x$

$2 = 2x$

$x = 1$

$f(1) = (1)^2 - 4$

$\boxed{f(1) = -3}$

9. At $x = 2$, the function $f(x) = \begin{cases} 4x + 1 & \text{if } x \le 2 \\ 3x^2 - 3 & \text{if } x > 2 \end{cases}$ is

(A) Both continuous and differentiable.

$\boxed{\text{(B)}}$ Continuous but not differentiable.

(C) Differentiable but not continuous.

(D) Neither continuous nor differentiable.

Continuity conditions:

i) $f(2) = 4(2) + 1 = 9$

ii) $\lim_{x \to 2^-} 4x + 1 = 9$ $\lim_{x \to 2^+} 3x^2 - 3 = 9$

iii) $f(2) = \lim_{x \to 2} f(x) = 9$

$f(x)$ continuous at $x = 2$

$$f'(x) = \begin{cases} 4 & \text{if } x \le 2 \\ 6x & \text{if } x < 2 \end{cases}$$

$\lim_{x \to 2^-} 4 = 4$ $\lim_{x \to 2^+} 6x = 12$

Since $\lim_{x \to 2^-} f'(x) \neq \lim_{x \to 2^+} f'(x)$, $f(x)$ not differentiable at $x = 2$ (no consistent slope at $x = 2$)

10. Oil is leaking from a tank. The amount of oil, in gallons, in the tank is given by $G(t) = 4000 - 3t^2$, where t, $0 \le t \le 24$ is the number of hours past midnight.

(a) Find $G'(5)$ using the definition of the derivative.

(b) Using appropriate units, interpret the meaning of $G'(5)$ in the context of the problem.

a) -30

b)

a) $G'(t) = 0 - 6t$

$G'(5) = -6(5) = -30$

$G'(5) = -30$ gallons per hour

b) $G'(5)$ means oil is leaking at rate of -30 gallons per hour when $t = 5$ hrs after midnight.

11. A rod of length 12 cm is heated at one end. The table below gives the temperature $T(x)$ in degrees Celsius at selected numbers x cm from the heated end.

x	0	2	5	7	9	12
$T(x)$	80	71	66	60	54	50

(a) Use the table to approximate $T'(8)$.

(b) Using appropriate units, interpret $T'(8)$ in the context of the problem.

a) -3

a) $T'(8) \approx \dfrac{T(9) - T(7)}{9 - 7} = \dfrac{54 - 60}{9 - 7} = \dfrac{-6}{2} \rightarrow -3\ {}^\circ C/cm$

b) $T'(8)$ is the rate of change of temperature per cm from one end when $x = 8$ cm

Key

2.3 AP Practice Problems (p. 193) – Derivative Power Rule & exponential e^x

1. If $g(x) = x$, then $g'(7) =$

(A) 0 $\boxed{\text{(B)}}$ 1 (C) 7 (D) $\frac{49}{2}$

$g(x) = x$

$g'(x) = 1$

$\boxed{g'(7) = 1}$

2. The line $x + y = k$, where k is a constant, is a tangent line to the graph of the function $f(x) = x^2 - 5x + 2$. What is the value of k?

(A) -1 (B) 2 $\boxed{\text{(C)}}$ -2 (D) -4

* find slope of the line, set equal to $f'(x)$

$x + y = k$

$y = -1x + k$

$m = -1$

$f'(x) = 2x - 5$

$-1 = 2x - 5$

$4 = 2x$

$x = 2$

set equations equal at $x = 2$

$-1x + k = x^2 - 5x + 2$

$-2 + k = 2^2 - 10 + 2$

$-2 + k = -4$

$\boxed{k = -2}$

3. An object moves along the x-axis so that its position at time t is $x(t) = 3t^2 - 9t + 7$. For what time t is the velocity of the object zero?

(A) -3 (B) 3 $\boxed{\text{(C)}}$ $\frac{3}{2}$ (D) 7

$x(t) = 3t^2 - 9t + 7$

$v(t) = 6t - 9$

$0 = 6t - 9$

$9 = 6t$

$6t = 9$

$t = 9/6 = 3/2$

$\boxed{t = 3/2}$

4. If $f(x) = e^x$, then $\ln(f'(3)) =$

$\boxed{\text{(A)}}$ 3 (B) 0 (C) e^3 (D) $\ln 3$

$f'(x) = e^x$ $f'(3) = e^3$

$\ln(f'(3))$

$\ln(e^3) \to 3 \ln e = \boxed{3}$

5. An equation of the normal line to the graph of $g(x) = x^3 + 2x^2 - 2x + 1$ at the point where $x = -2$ is

(A) $x + 2y = 12$ (B) $x - 2y = 8$

(C) $2x + y = -9$ $\boxed{\text{(D)}}$ $x + 2y = 8$

$g'(x) = 3x^2 + 4x - 2$

$g'(-2) = 3(-2)^2 + 4(-2) - 2 = 2$

* slope of tangent line is $m = 2$

* slope of normal line (perpendicular) is $m_2 = \frac{-1}{2}$

$g(-2) = (-2)^3 + 2(-2)^2 - 2(-2) + 1$

$g(-2) = -8 + 8 + 4 + 1 = 5$

point: $(-2, 5)$

slope: $m = -1/2$

$y - y_1 = m(x - x_1)$

$y - 5 = \frac{-1}{2}(x + 2)$

$2(y - 5 = \frac{-1}{2}x - 1)$

$2y - 10 = -1x - 2$

$\boxed{x + 2y = 8}$

6. The line $9x - 16y = 0$ is tangent to the graph of $f(x) = 3x^3 + k$, where k is a constant, at a point in the first quadrant. Find k.

$\boxed{\text{(A)}}$ $\frac{3}{32}$ (B) $\frac{3}{16}$ (C) $\frac{3}{64}$ (D) $\frac{9}{64}$

$9x - 16y = 0$

$-16y = -9x$

$y = \frac{9}{16}x$

slope is $9/16$

$f'(x) = 9x^2 + 0$

$\frac{9}{16} = 9x^2$

$\frac{1}{16} = x^2$

$x = \sqrt{\frac{1}{16}} = 1/4$

$9(1/4) - 16y = 0$

$\frac{9}{4} = 16y$

$y = 9/64$

$y = 3x^3 + k$

$9/64 = 3(1/4)^3 + k$

$\frac{9}{64} = \frac{3}{64} + k$

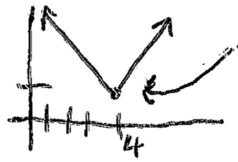
$\boxed{k = \frac{6}{64} = \frac{3}{32}}$

7. If $f(x) = 1 + |x - 4|$, find $f'(4)$.

(A) -1 (B) 0 (C) 1 $\boxed{\text{(D)}}$ $f'(4)$ does not exist.

$y = |x-4| + 1$

vertex at $(4, 1)$



$f(x)$ not differentiable at $x = 4$

8. The cost C (in dollars) of manufacturing x units of a product is $C(x) = 0.3x^2 + 4.02x + 3500$. What is the rate of change of C when $x = 1000$ units?

(A) 307.52 (B) 0.60402 $\boxed{\text{(C)}}$ 604.02 (D) 1020

$C'(x) = 2(0.3)x + 4.02 + 0$

$C'(x) = 0.6x + 4.02$

$C'(1000) = 0.6(1000) + 4.02$

$C'(1000) = 600 + 4.02 = 604.02$

9. $\frac{d}{dx}(5 \ln x) =$

(A) $\frac{1}{5x}$ (B) $5e^x$ (C) $-\frac{5}{\ln x}$ $\boxed{\text{(D)}}$ $\frac{5}{x}$

$* \frac{d}{dx} \ln x = \frac{1}{x}$

$\frac{d}{dx} 5 \ln x \to 5\left(\frac{1}{x}\right) \to \boxed{\frac{5}{x}}$

10. For the function $f(x) = x^2 + 4$

(a) Find $f'(1)$.

(b) Find an equation of the tangent line to the graph of f at $x = 1$.

(c) Find $f'(-4)$.

(d) Find an equation of the tangent line to the graph of f at $x = -4$.

(e) Find the point of intersection of the two tangent lines found in (b) and (d).

a) 2
b) $2x+3$
c) -8
d) $-8x+12$
e) $\left(-\frac{3}{2}, 0\right)$

a) $f'(x) = 2x \to f'(1) = 2(1) = 2$

c) $f'(-4) = 2(-4) = -8$

d) $f(-4) = (-4)^2 + 4 = 20$ $\boxed{y - 20 = -8(x+4)}$

point: $(-4, 20)$ slope: $m = -8$

b) $f(1) = (1)^2 + 4 = 5$

point: $(1, 5)$

slope: $m = 2$

$\boxed{y - 5 = 2(x-1)}$ or $y = 2x + 3$

11. Which is an equation of the tangent line to the graph of $f(x) = x^4 + 3x^2 + 2$ at the point where $f'(x) = 2$?

(A) $y = 2x + 2$ (B) $y = 2x + 2.929$

$\boxed{\text{(C)}}$ $y = 2x + 1.678$ (D) $y = 2x - 2.929$

$*$set $f'(x) = 2$

$f'(x) = 4x^3 + 6x + 0$

$2 = 4x^3 + 6x$

$0 = 4x^3 + 6x + 2$

$0 = 2(2x^3 + 3x + 1)$

$0 = 2x^3 + 3x + 1$

Key

2.4 AP Practice Problems (p. 206) – Product & Quotient Rule & Higher order derivatives

1. What is the instantaneous rate of change at $x = -2$ of the function $f(x) = \dfrac{x-1}{x^2+2}$?

$\boxed{(A)}$ $-\frac{1}{6}$ (B) $\frac{1}{9}$ (C) $\frac{1}{2}$ (D) -1

*quotient Rule

$f'(x) = \dfrac{\overbrace{(1)}^{f'}\overbrace{(x^2+2)}^{g} - \overbrace{(x-1)}^{f}\overbrace{(2x)}^{g'}}{\underbrace{(x^2+2)^2}_{g^2}}$

$f'(x) = \dfrac{x^2+2-2x^2+2x}{(x^2+2)^2}$

$f'(x) = \dfrac{-1x^2+2x+2}{(x^2+2)^2}$

$f'(-2) = \dfrac{-1(2)^2+2(-2)+2}{(2^2+2)^2}$

$f'(2) = \dfrac{-6}{36} = \boxed{\dfrac{-1}{6}}$

2. An equation of the tangent line to the graph of $f(x) = \dfrac{5x-3}{3x-6}$ at the point (3, 4) is

(A) $7x + 3y = 37$ $\boxed{(B)}$ $7x + 3y = 33$
(C) $7x - 3y = 9$ (D) $13x + 3y = 51$

$f'(x) = \dfrac{(5)(3x-6)-(5x-3)(3)}{(3x-6)^2} \rightarrow \dfrac{15x-30-15x+9}{(3x-6)^2}$

$f'(x) = \dfrac{-21}{(3x-6)^2}$

$f'(3) = \dfrac{-21}{(9-6)^2}$

$f'(3) = \dfrac{-21}{9} \rightarrow \dfrac{-7}{3}$

point: (3,4)

slope: $m = -7/3$

$y - 4 = -7/3(x-3)$

$(y - 4 = -7/3 x + 7)3$

$3y - 12 = -7x + 21$

$\boxed{7x + 3y = 33}$

3. If f, g, and h are nonzero differentiable functions of x, then $\dfrac{d}{dx}\left(\dfrac{gh}{f}\right) =$

$\boxed{(A)}$ $\dfrac{fgh' + fg'h - f'gh}{f^2}$ (B) $\dfrac{g'h' - ghf'}{f^2}$

(C) $\dfrac{gh' + g'h}{f'}$ (D) $\dfrac{fgh' + fg'h + f'gh}{f^2}$

$\dfrac{\overbrace{(g')(h)+(g)(h')}^{f'} \cdot \overbrace{f}^{g} - \overbrace{(gh)}^{f} \cdot \overbrace{f'}^{g'}}{\underbrace{f^2}_{g^2}}$

$\dfrac{(g'h + gh')f - f'gh}{f^2} \rightarrow \boxed{\dfrac{fg'h + fgh' - f'gh}{f^2}}$

4. If $y = x^3e^x$, then $\dfrac{dy}{dx} =$

(A) $3x^2e^x$ (B) $3x^2 + e^x$
(C) $3x^2e^x(x+1)$ $\boxed{(D)}$ $x^2e^x(x+3)$

$y' = \overbrace{3x^2}^{f'} \cdot \overbrace{e^x}^{g} + \overbrace{x^3}^{f} \cdot \overbrace{e^x}^{g'}$

$\boxed{y' = x^2e^x(3+x)}$

5. $\dfrac{d}{dt}\left(t^2 - \dfrac{1}{t^2} + \dfrac{1}{t}\right)$ at $t = 2$ is

(A) $\frac{7}{2}$ (B) $\frac{9}{2}$ (C) $\frac{9}{4}$ $\boxed{(D)}$ 4

$f(t) = t^2 - t^{-2} + t^{-1}$

$f'(t) = 2t - (-2t^{-3}) - t^{-2}$

$f'(t) = 2t + \dfrac{2}{t^3} - \dfrac{1}{t^2}$

$f'(2) = 2(2) + \dfrac{2}{2^3} - \dfrac{1}{2^2}$

$f'(2) = 4 + \dfrac{1}{4} - \dfrac{1}{4}$

$\boxed{f'(2) = 4}$

6. The position of an object moving along a straight line at time t, in seconds, is given by $s(t) = 16t^2 - 5t + 20$ meters. What is the acceleration of the object when $t = 2$?

(A) 32 m/s (B) 0 m/s² $\boxed{\text{(C)}}$ 32 m/s² (D) 64 m/s²

$s'(t) = 32t - 5$

$s''(t) = 32$

$\boxed{s''(2) = 32\ m/s^2}$

7. If $y = \frac{x-3}{x+3}$, $x \neq -3$, the instantaneous rate of change of y with respect to x at $x = 3$ is

(A) $-\frac{1}{6}$ $\boxed{\text{(B)}}$ $\frac{1}{6}$ (C) $\frac{1}{36}$ (D) 1

* quotient Rule

$y' = \frac{\overbrace{(1)}^{f'}\overbrace{(x+3)}^{g} - \overbrace{(x-3)}^{f}\overbrace{(1)}^{g'}}{\underbrace{(x+3)^2}_{g^2}}$

$y' = \frac{x+3-x+3}{(x+3)^2}$

$y' = \frac{6}{(x+3)^2}$

$y'(3) = \frac{6}{6^2} = \frac{1}{6}$

$\boxed{y'(3) = 1/6}$

8. Find an equation of the normal line to the graph of the function $f(x) = \frac{x^2}{x+1}$ at $x = 1$.

$\boxed{\text{(A)}}$ $8x + 6y = 11$ (B) $-8x + 6y = -5$

(C) $-3x + 4y = -1$ (D) $3x + 4y = 5$

$f'(x) = \frac{(2x)(x+1) - x^2(1)}{(x+1)^2}$

$f'(x) = \frac{2x^2+2x-x^2}{(x+1)^2}$

$f'(x) = \frac{x^2+2x}{(x+1)^2}$

$f'(1) = \frac{1+2}{2^2} = \frac{3}{4}$

slope of normal line:

$m_2 = -\frac{4}{3}$

$f(1) = \frac{1^2}{1+1} = \frac{1}{2}$

point: $(1, 1/2)$

slope: $m = -4/3$

$y - y_1 = m(x - x_1)$

$\left(y - 1/2 = -4/3(x-1)\right)6$

$6y - 3 = -8x + 8$

$\boxed{8x + 6y = 11}$

9. If $y = xe^x$, then the nth derivative of y is

(A) e^x $\boxed{\text{(B)}}$ $(x+n)e^x$ (C) ne^x (D) $x^n e^x$

$y' = 1e^x + xe^x = (x+1)e^x$

$y'' = (1)e^x + (x+1)e^x = (x+2)e^x$

$y''' = 1e^x + (x+2)e^x = (x+3)e^x$

$y^4(x) = 1e^x + (x+3)e^x \rightarrow (x+4)e^x$

$\boxed{y^n(x) = (x+n)e^x}$

Key

2.5 AP Practice Problems (p. 214) – Derivatives of Trig Functions

1. If $y = \underbrace{x}_{f}\underbrace{\sin x}_{g}$, then $\frac{dy}{dx} =$

   $\boxed{\text{(A)}}$ $x\cos x + \sin x$ (B) $x\cos x - \sin x$

   (C) $\cos x + \sin x$ (D) $(x+1)\cos x$

$y' = \underbrace{(1)}_{f'}\underbrace{\sin x}_{g} + \underbrace{(x)}_{f}\underbrace{\cos x}_{g'}$

$\boxed{y' = \sin x + x\cos x}$

2. What is $\lim_{h\to 0} \frac{\cos\left(\frac{\pi}{3}+h\right) - \cos\frac{\pi}{3}}{h}$?

   (A) 0 (B) $\frac{1}{2}$ (C) $\frac{\sqrt{3}}{2}$ $\boxed{\text{(D)}}$ $-\frac{\sqrt{3}}{2}$

* Limit definition of derivative: $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$f'\left(\frac{\pi}{3}\right) = \lim_{h\to 0} \frac{f\left(\frac{\pi}{3}+h\right) - f(\pi/3)}{h}$

$f(x) = \cos x$

$f'(x) = -\sin x$

$f'(\pi/3) = -\sin(\pi/3)$

$\boxed{f'(\pi/3) = \frac{-\sqrt{3}}{2}}$

3. If $f(x) = \tan x$, then $f'\left(\frac{\pi}{3}\right)$ equals

   (A) $2\sqrt{3}$ $\boxed{\text{(B)}}$ 4 (C) 2 (D) $\frac{1}{4}$

$f'(x) = \sec^2 x$

$f'(\pi/3) = [\sec(\pi/3)]^2$

$f'(\pi/3) = \frac{1}{(\cos(\pi/3))^2} \rightarrow$

$f'(\pi/3) = \frac{1}{(1/2)^2} = \boxed{4}$

4. The position s (in meters) of an object moving along a horizontal line at time t, $0 \le t \le \frac{\pi}{2}$, (in seconds) is given by $s(t) = 6\sin t + \frac{3}{2}t^2 + 8$. What is the velocity of the object when its acceleration is zero?

   (A) 6 m/s (B) $3+\pi$ m/s

   $\boxed{\text{(C)}}$ $\frac{6\sqrt{3}+\pi}{2}$ m/s (D) $\left(3\sqrt{3} - \frac{\pi}{2}\right)$ m/s

$s'(t) = 6\cos(t) + \frac{3}{2}\cdot 2t$

$s''(t) = -6\sin(t) + 3$

$0 = -6\sin(t) + 3$

$6\sin(t) = 3$

$\sin(t) = 1/2$

$t = \pi/6$

$v(t) = 6\cos(t) + 3t$

$v(\pi/6) = 6\cos(\pi/6) + 3(\pi/6)$

$v(\pi/6) = 6\left(\frac{\sqrt{3}}{2}\right) + \frac{\pi}{2}$

$\boxed{v(\pi/6) = \frac{6\sqrt{3}+\pi}{2}}$

5. If $y = \sin x$, then $\frac{d^{50}}{dx^{50}} \sin x$ equals

   (A) $\sin x$ $\boxed{\text{(B)}}$ $-\sin x$ (C) $\cos x$ (D) $-\cos x$

$4\overline{)50}$ = 12 R2

$\;\;48$

$\;\;\;2$

$\boxed{y^{50}(x) = -\sin x}$

(R1) $y' = \cos x$

(R2) $y'' = -\sin x$

(R3) $y''' = -\cos x$

$y^4(x) = \sin x$ (R0)

6. If $f(x) = \dfrac{x}{\cos x}$, find $f'\left(\dfrac{\pi}{3}\right)$.

(A) $2 - \dfrac{2\sqrt{3}}{3}\pi$ (B) $1 + \dfrac{\sqrt{3}}{3}\pi$

(C) $1 - \dfrac{\sqrt{3}}{3}\pi$ $\boxed{\text{(D)}}$ $2 + \dfrac{2\sqrt{3}}{3}\pi$

$$f'(x) = \frac{(1)(\cos x) - x(-\sin x)}{\cos^2 x}$$

$$f'(\pi/3) = \frac{\cos(\pi/3) + \frac{\pi}{3}\sin(\pi/3)}{(\cos(\pi/3))^2} \to \frac{\frac{1}{2} + \frac{\pi}{3}\left(\frac{\sqrt{3}}{2}\right)}{\left(\frac{1}{2}\right)^2}$$

$$f'(\pi/3) = \frac{\frac{1}{2} + \frac{\pi\sqrt{3}}{6}}{1/4} \to \frac{\frac{3}{6} + \frac{\pi\sqrt{3}}{6}}{1/4} \to \frac{3+\pi\sqrt{3}}{6} \cdot \frac{4}{1} \to \frac{(3+\sqrt{3}\pi)2}{3} \to \frac{6}{3} + \frac{2\sqrt{3}\pi}{3}$$

$$\to \boxed{2 + \frac{2\sqrt{3}\pi}{3}}$$

7. If $y = x - \tan x$, then $\dfrac{dy}{dx}$ equals

(A) $1 - \sec x \tan x$ $\boxed{\text{(B)}}$ $-\tan^2 x$

(C) $\tan^2 x$ (D) $-\sec^2 x$

$y' = 1 - \sec^2 x$

* trig identity: $1 + \tan^2 x = \sec^2 x$

$\sec^2 x = 1 + \tan^2 x$

$y' = 1 - (1 + \tan^2 x)$

$y' = 1 - 1 - \tan^2 x$

$\boxed{y = -\tan^2 x}$

8. If $g(x) = e^x \cos x + 2\pi$, then $g'(x) =$

(A) $e^x - \sin x$ (B) $e^x \cos x - e^x \sin x + 3\pi$

$\boxed{\text{(C)}}$ $e^x \cos x - e^x \sin x$ (D) $e^x \cos x + e^x \sin x$

$$g'(\pi) = \underbrace{e^x}_{f'} \cdot \underbrace{\cos x}_{g} + \underbrace{e^x}_{f}\underbrace{(-\sin x)}_{g'} + 0$$

$\boxed{g'(x) = e^x\cos x - e^x \sin x}$

$\boxed{x = \pi/2}$

9. At which of the following numbers x, $0 \le x \le 2\pi$, does the graph of $y = x + \cos x$ have a horizontal tangent line?

(A) 0 only $\boxed{\text{(B)}}$ $\dfrac{\pi}{2}$ only

(C) $\dfrac{3\pi}{2}$ only (D) 0 and $\dfrac{\pi}{2}$ only

* to find horizontal tangent → set $y'(x) = 0$

$y' = 1 + (-\sin x)$

$0 = 1 - \sin x$

$\sin x = 1$

10. An equation of the tangent line to the graph of $f(x) = \sin x$ at $x = \dfrac{2\pi}{3}$ is

(A) $3x + 6y = 4\pi - 3\sqrt{3}$ $\boxed{\text{(B)}}$ $3x + 6y = 2\pi + 3\sqrt{3}$

(C) $6y - 3x = 2\pi - 3\sqrt{3}$ (D) $6y - 3x = 4\pi - 3\sqrt{3}$

point: $f\left(\frac{2\pi}{3}\right) = \sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$

$f'(x) = \cos x$ $f'\left(\frac{2\pi}{3}\right) = \cos\left(\frac{2\pi}{3}\right) = \frac{-1}{2}$

point: $\left(\frac{2\pi}{3}, \frac{\sqrt{3}}{2}\right)$ slope: $m = -1/2$

$\left[y - \frac{\sqrt{3}}{2} = \frac{-1}{2}\left(x - \frac{2\pi}{3}\right)\right]6$

$6y - 3\sqrt{3} = -3x + 2\pi$

$\boxed{3x + 6y = 2\pi + 3\sqrt{3}}$

Key

Ch.2 Review AP Practice Problems (p. 220) – Derivative definition and properties

1. If $f(x) = \sec x$, then $f'\left(\frac{\pi}{4}\right) =$

(A) $\frac{\sqrt{2}}{2}$ (B) 2 (C) 1 $\boxed{\text{(D)}}$ $\sqrt{2}$

$f'(x) = \sec x \tan x$

$f'(\pi/4) = \sec(\pi/4)\tan(\pi/4)$

$f'(\pi/4) = \frac{1}{\cos(\pi/4)} \cdot \tan(\pi/4)$

$= \frac{2}{\sqrt{2}} \cdot 1$

$\frac{2}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{2\sqrt{2}}{2} = \boxed{\sqrt{2}}$

2. If a function f is differentiable at c, then $f'(c)$ is given by

✓ I. $\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$ ← Alt. definition of derivative at a point

II. $\lim_{x \to c} \frac{f(x+h) - f(x)}{h}$

✓ III. $\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$ ← Limit definition of derivative at a point

(A) I only (B) III only

(C) I and II only $\boxed{\text{(D)}}$ I and III only

3. If $y = \frac{3}{x^2 - 5}$, then $\frac{dy}{dx} =$

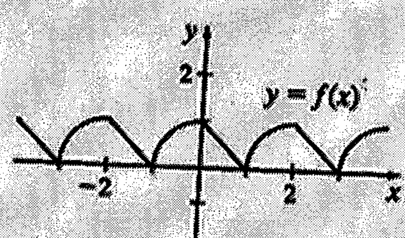
(A) $\frac{6x}{(x^2-5)^2}$ $\boxed{\text{(B)}}$ $-\frac{6x}{(x^2-5)^2}$

(C) $\frac{6x}{x^2-5}$ (D) $\frac{2x}{(x^2-5)^2}$

$y' = \frac{0(x^2-5) - 3(2x)}{(x^2-5)^2}$

$\boxed{y' = \frac{-6x}{(x^2-5)^2}}$

4. The graph of the function f is shown below. Which statement about the function is true?



(A) f is differentiable everywhere. (false)

(B) $0 \le f'(x) \le 1$, for all real numbers. ($f'(x)$ not differentiable everywhere

$\boxed{\text{(C)}}$ f is continuous everywhere.

(D) f is an even function. (false)

5. The table displays select values of a differentiable function f. What is an approximate value of $f'(2)$?

x	1.996	1.998	2	2.002	2.004
$f(x)$	3.168	3.181	3.194	3.207	3.220

$\boxed{\text{(A)}}$ 6.5 (B) 1.154 (C) 0.013 (D) 0.0016

$f'(2) \approx \frac{3.194 - 3.181}{2 - 1.998}$

$\boxed{f'(2) \approx 6.5}$

*use product Rule

6. If $y = \sin x + xe^x + 6$, what is the instantaneous rate of change of y with respect to x at $x = 5$?

(A) $\cos 5 + 6e^5$ (B) 2

(C) $\cos 5 + 5e^5$ (D) $6e^5 - \cos 5$

$y' = \cos x + \underbrace{1 \cdot e^x}_{f'g} + \underbrace{x \cdot e^x}_{fg'} + 0$

$y'(5) = \cos 5 + e^5 + 5e^5$

$\boxed{y'(5) = \cos 5 + 6e^5}$

perpendicular to line parallel (opposite reciprocal)

7. An equation of the normal line to the graph of $f(x) = 3xe^x + 5$ at $x = 0$ is

product Rule

(A) $y = 3x + 5$ (B) $y = -\frac{1}{3}x + 5$

(C) $y = \frac{1}{3}x + 5$ (D) $y = -3x + 5$

$f'(x) = \underbrace{3}_{f'} \cdot \underbrace{e^x}_{g} + \underbrace{3x}_{f} \cdot \underbrace{e^x}_{g'} + 0$

$f'(x) = 3e^x + 3xe^x$

$f'(0) = 3e^0 + 3(0)e^0 = 3$

normal line slope : $m_2 = -1/3$

point : $f(0) = 3(0)e^0 + 5 = 5$

point : $(0, 5)$

slope : $m = -1/3$

$y - 5 = -\frac{1}{3}(x - 0)$

$y - 5 = -\frac{1}{3}x$

$\boxed{y = -\frac{1}{3}x + 5}$

8. An object moves along a horizontal line so that its position at time t is $s(t) = t^4 - 6t^3 - 2t - 1$. At what time t is the acceleration of the object zero?

(A) at 0 only (B) at 1 only

(C) at 3 only (D) at 0 and 3 only

$s'(t) = 4t^3 - 18t^2 - 2$

$s''(t) = 12t^2 - 36t$

$0 = 12t^2 - 36t$

$0 = 12t(t - 3)$

$12t = 0$ | $t - 3 = 0$

$\boxed{t = 0}$ | $\boxed{t = 3}$

9. If $f(x) = e^x(\sin x + \cos x)$, then $f'(x) =$

(A) $2e^x(\cos x + \sin x)$ (B) $e^x \cos x$

(C) $2e^x \cos x$ (D) $e^x(\cos^2 x - \sin^2 x)$

$f'(x) = \underbrace{e^x}_{f'}\underbrace{(\sin x + \cos x)}_{g} + \underbrace{e^x}_{f}\underbrace{(\cos x - \sin x)}_{g'}$

$f'(x) = \cancel{e^x \sin x} + e^x \cos x + e^x \cos x - \cancel{e^x \sin x}$

$\boxed{f'(x) = 2e^x \cos x}$

10. Find an equation of the tangent line to the graph of $f(x) = \dfrac{x+3}{x^2+2}$ at $x = 1$.

(A) $5x + 9y = 17$ (B) $9y - 5x = 7$

(C) $5x + 3y = 9$ (D) $5x + 9y = 7$

$f'(x) = \dfrac{(1)(x^2+2) - (x+3)(2x)}{(x^2+2)^2}$

$f'(1) = \dfrac{3 - 8}{9} = \dfrac{-5}{9}$

$f(1) = \dfrac{1+3}{1+2} = \dfrac{4}{3}$

$y - 4/3 = \frac{-5}{9}(x - 1)$

$\left(y - 4/3 = \frac{-5}{9}x + \frac{5}{9}\right)9$

$9y - 12 = -5x + 5$

$\boxed{5x + 9y = 17}$

11. $\displaystyle\lim_{x \to \frac{\pi}{4}} \frac{\tan x - 1}{x - \frac{\pi}{4}} =$

(A) 0 (B) -1 (C) 2 (D) Does not exist.

$* f'(c) = \lim_{x \to c} \dfrac{f(x) - f(c)}{x - c}$

$f'(\pi/4) = \dfrac{f(x) - f(\pi/4)}{x - \pi/4}$

$f(x) = \tan x$

$f'(x) = \sec^2 x$

$f'(\pi/4) = \sec^2(\pi/4)$

$= [\sec(\pi/4)]^2 = \left(\frac{2}{\sqrt{2}}\right)^2 = \boxed{2}$