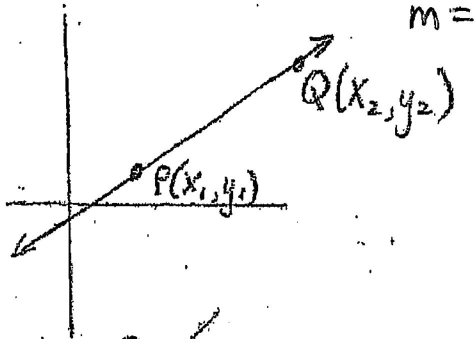
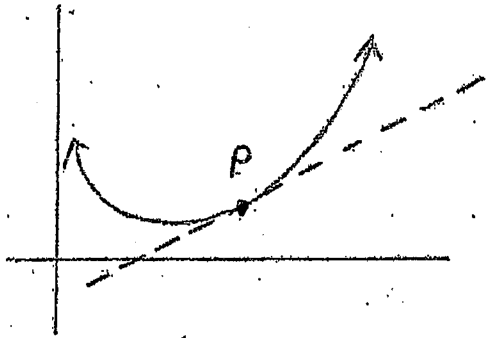
Ch. 2.1 Notes: The Derivative and Tangent Line Problem

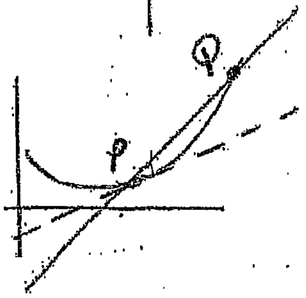
Answer Key ①

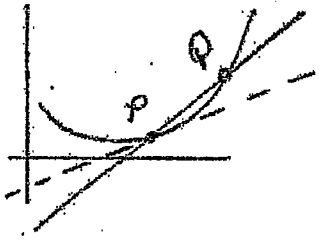
Goal: To find a formula to calculate the slope of all tangent lines to a curve.
(steepness)

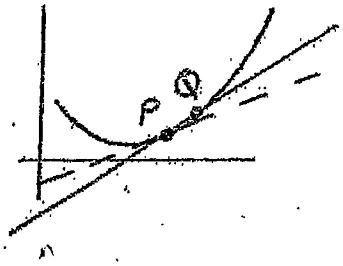
$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

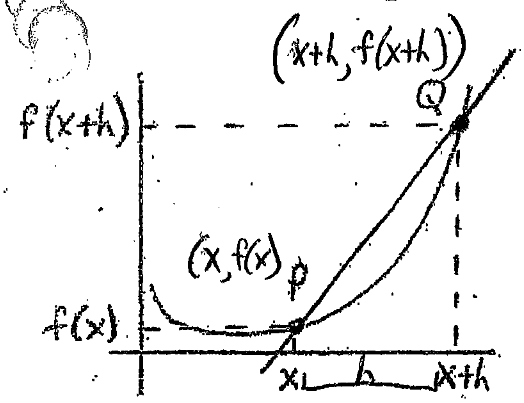












slope: $\frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x+h) - f(x)}{\cancel{x} + h - \cancel{x}}$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

A. General (Limit) Definition of the Derivative

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

"f prime of x": This is the notation for the derivative function

Derivative: the slope or steepness of a curve at a single point.

* The Derivative is a slope-finding formula for a curved function, where the slope is ever-changing.

B. Alternative Derivative Definition

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

Ex. 1 Find the general derivative of $f(x)=x^2$. Then write the equation of the line tangent to $f(x)$ at $x=1$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

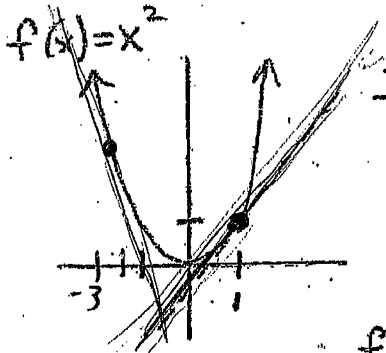
$$f'(x)=\lim_{h\to 0}\frac{(x+h)^2-x^2}{h}$$

$$f'(x)=\lim_{h\to 0}\frac{(x+h)(x+h)-x^2}{h}$$

$$=\lim_{h\to 0}\frac{\cancel{x^2}+2xh+h^2-\cancel{x^2}}{h}$$

$$=\lim_{h\to 0}\frac{\cancel{h}(2x+h)}{\cancel{h}}=2x+0$$

$$\boxed{f'(x)=2x}$$



$f'(-3)=2(-3)=-6$

* Therefore, the derivative (slope-finding formula) for $f(x)=x^2$

$f(x)=x^2$
- $f(x)$ is the height-finding formula
- Since $f(1)=1^2=1$, this tells us that when $x=1$, the height of graph has a y-value of 1

$f'(x)=2x$
- $f'(x)$ is the slope-finding formula for the $f(x)$ graph
- Since $f'(1)=2(1)=2$, this tells us that when $x=1$ the slope of tangent line to $f(x)$ has slope of 2 (steepness)

Find Tangent-line equation:

* $y-y_1=m(x-x_1)$ point-slope

point: $(1,1)$

slope: 2

$y-1=2(x-1)$

Ex. 2 Find equation of tangent line to $f(x)=x^2$ at $x=-5$

$f'(x)=2x$

$f(-5)=(-5)^2=25$

$f'(-5)=2(-5)=-10$

point $(-5,25)$

$m=-10$

$y-y_1=m(x-x_1)$

$$\boxed{y-25=-10(x+5)}$$

Ex.3 ⓐ Find derivative of $\boxed{f(x)=\sqrt{x}}$.

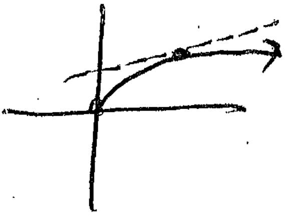
ⓑ Find the slope ~~of function~~ at $x=2$

$$f'(x)=\lim_{h\to0}\frac{f(x+h)-f(x)}{h}$$

$$f'(x)=\lim_{h\to0}\frac{\sqrt{x+h}-\sqrt{x}}{h}\cdot\frac{(\sqrt{x+h}+\sqrt{x})}{(\sqrt{x+h}+\sqrt{x})}$$

$$\lim_{h\to0}\frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})}=\lim_{h\to0}\frac{\cancel{h}}{\cancel{h}(\sqrt{x+h}+\sqrt{x})}=\frac{1}{\sqrt{x}+\sqrt{x}}=\frac{1}{2\sqrt{x}}$$

$$\boxed{f'(x)=\frac{1}{2\sqrt{x}}}\qquad f'(2)=\boxed{\frac{1}{2\sqrt{2}}}$$



Ex.4 Use the alternative derivative definition to find slope of $f(x)=\sqrt{x}$ at $x=2$ $\quad f'(c)=\lim_{x\to c}\frac{f(x)-f(c)}{x-c}$

$c=2$ $\quad f(2)=\sqrt{2}$

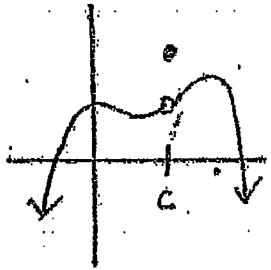
$$f'(2)=\lim_{x\to2}\frac{f(x)-f(2)}{x-2}$$

$$f'(2)=\lim_{x\to2}\frac{\sqrt{x}-\sqrt{2}}{x-2}$$

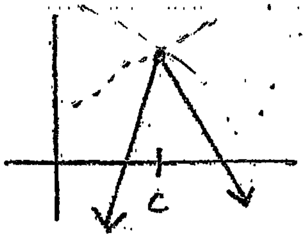
$$f'(2)=\lim_{x\to2}\frac{(\sqrt{x}-\sqrt{2})}{x-2}\cdot\frac{(\sqrt{x}+\sqrt{2})}{\sqrt{x}+\sqrt{2}}$$

$$f'(2)=\lim_{x\to2}\frac{\cancel{(x-2)}}{\cancel{(x-2)}(\sqrt{x}+\sqrt{2})}=\lim_{x\to2}\frac{1}{\sqrt{x}+\sqrt{2}}=\boxed{\frac{1}{2\sqrt{2}}}$$

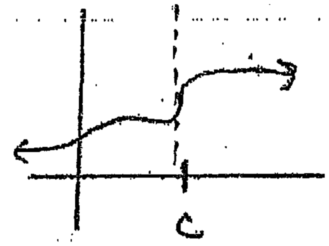
<u>Differentiability</u>: In order for a function to be <u>differentiable</u> (smooth curve) at a point, c, it must be continuous at that point, cannot contain a sharp point, cannot have vertical tangent



Graph not continuous
$f'(c)=$ DNE



Sharp point at $f(c)$
$f'(c)=$ DNE



vertical tangent at $f(c)$
$f'(c)=$ DNE

Ex-5 Use General Definition of Derivative to find

a) $f'(x)$ when $f(x) = x^2 - 5x + 2$

b) Find $f'(-3)$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - 5(x+h) + 2 - (x^2 - 5x + 2)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 \cancel{-5x} - 5h \cancel{+2} \cancel{-x^2} \cancel{+5x} \cancel{-2}}{h}$$

$$\frac{d}{dx} f(x) = 2x - 5$$

$$= \lim_{h \to 0} \frac{\cancel{h}(2x + h - 5)}{\cancel{h}} = 2x + 0 - 5 \quad \boxed{a)\ f'(x) = 2x - 5}$$

$$b)\ f'(-3) = 2(-3) - 5 = \boxed{-11}$$

Ex. 6 Alternative definition $f(x) = \sqrt{x+1}$ $c = 2$

$$f(2) = \sqrt{2+1} = \sqrt{3}$$

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

$$f'(2) = \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2}$$

$$= \lim_{x \to 2} \frac{\sqrt{x+1} - \sqrt{3}}{x - 2} \cdot \frac{(\sqrt{x+1} + \sqrt{3})}{(\sqrt{x+1} + \sqrt{3})}$$

$$= \lim_{x \to 2} \frac{x + 1 - 3}{(x-2)(\sqrt{x+1} + \sqrt{3})}$$

$$= \lim_{x \to 2} \frac{\cancel{(x-2)}}{\cancel{(x-2)}(\sqrt{x+1} + \sqrt{3})}$$

$$\lim_{x \to 2} \frac{1}{\sqrt{x+1} + \sqrt{3}} = \frac{1}{\sqrt{3} + \sqrt{3}}$$

$$\boxed{f'(2) = \frac{1}{2\sqrt{3}}}$$

b) point: $(2, \sqrt{3})$ $m = \frac{1}{2\sqrt{3}}$

$$\boxed{y - \sqrt{3} = \frac{1}{2\sqrt{3}}(x - 2)}$$

Ex. 7 $f(x)=\frac{1}{x^2}$ find $f'(x)$ and $f'(-2)$

General definition

$$f'(x)=\lim_{h\to 0}\frac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h}=\lim_{h\to 0}\frac{\frac{x^2-(x+h)^2}{x^2(x+h)^2}}{h}$$

$$f'(x)=\lim_{h\to 0}\frac{\frac{x^2-(x^2+2xh+h^2)}{x^2(x+h)^2}\cdot\frac{1}{h}}{h\cdot\frac{1}{h}} \quad \lim_{h\to 0}\frac{\cancel{x^2}-\cancel{x^2}-2xh-h^2}{h(x^2)(x+h)^2}$$

$$\lim_{h\to 0}\frac{\cancel{h}(-2x-h)}{\cancel{h}(x^2)(x+h)^2}=\frac{-2x}{(x^2)(x+0)^2}=\frac{-2x}{x^4}=\frac{-2}{x^3}$$

$$\boxed{f'(x)=\frac{-2}{x^3}} \quad f'(-2)=\frac{-2}{(-2)^3}=\frac{-2}{-8}=\boxed{\frac{1}{4}}$$

c) Tangent line: point $(-2,\frac{1}{4})$

slope: $m=\frac{1}{4}$

$$\boxed{y-\frac{1}{4}=\frac{1}{4}(x+2)}$$

4c

$\boxed{\text{Ex 8}}$ Alt. def. $f(x) = \frac{3}{x}$ $c = 4$

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

$$f'(4) = \lim_{x \to 4} \frac{f(x) - f(4)}{x - 4}$$

$$= \lim_{x \to 4} \frac{\frac{3}{x} - \frac{3}{4}}{x - 4}$$

$$= \lim_{x \to 4} \frac{\frac{12 - 3x}{4x}}{x - 4}$$

$$f'(4) = \lim_{x \to 4} \frac{\frac{12 - 3x}{4x} \cdot \frac{1}{x-4}}{\cancel{x-4} \cdot \cancel{\frac{1}{x-4}}}$$

$$= \lim_{x \to 4} \frac{12 - 3x}{4x(x-4)}$$

$$= \lim_{x \to 4} \frac{3\overset{-1}{\cancel{(4-x)}}}{4x\cancel{(x-4)}} = \lim_{x \to 4} \frac{-3}{4x}$$

$$\boxed{f'(4) = \frac{-3}{16}}$$

Key ⑤

Non-AP Calculus 2.1 Definition of Derivative Practice Worksheet

1) Use the Limit Definition of a derivative to find f'(x) if f(x) = $2x^2 - 3x + 1$

$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ $f(\ ) = 2(\ )^2 - 3(\ ) + 1$

$$f'(x) = \lim_{h\to 0}\frac{2(x+h)^2-3(x+h)+1-(2x^2-3x+1)}{h}$$

$$f'(x) = \lim_{h\to 0}\frac{2(x^2+2xh+h^2)-3x-3h+1-2x^2+3x-1}{h}$$

$$f'(x) = \lim_{h\to 0}\frac{\cancel{2x^2}+4xh+2h^2\cancel{-3x}-3h\cancel{+1}\cancel{-2x^2}\cancel{+3x}\cancel{-1}}{h}$$

$$f'(x) = \lim_{h\to 0}\frac{4xh+2h^2-3h}{h}$$

$$f'(x) = \lim_{h\to 0}\frac{\cancel{h}(4x+2h-3)}{\cancel{h}} = 4x+2(0)-3$$

$$\boxed{f'(x) = 4x-3}$$

2) Use the Alternative definition of the derivative to find $H'(2)$ if $H(x) = \sqrt{3-x}$

$f'(c) = \lim_{x\to c}\frac{f(x)-f(c)}{x-c}$ $c=2$ $h(2) = \sqrt{3-2} = 1$

$$h'(2) = \lim_{x\to 2}\frac{h(x)-h(2)}{x-2}$$

$$h'(2) = \lim_{x\to 2}\frac{\sqrt{3-x}-1}{x-2}$$

$$h'(2) = \lim_{x\to 2}\frac{(\sqrt{3-x}-1)}{(x-2)}\frac{(\sqrt{3-x}+1)}{(\sqrt{3-x}+1)}$$

$$h'(2) = \lim_{x\to 2}\frac{3-x-1}{(x-2)(\sqrt{3-x}+1)}$$

$$h'(2) = \lim_{x\to 2}\frac{\cancel{(2-x)}(-1)}{\cancel{(x-2)}(\sqrt{3-x}+1)}$$

$$h'(2) = \frac{-1}{\sqrt{3-2}+1} = \frac{-1}{1+1} = \boxed{\frac{-1}{2}}$$

3) Use the Limit Definition of a Derivative to find f'(x) if f(x) = $\sqrt{2x-1}$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$f(\ ) = \sqrt{2(\ )-1}$

$f(x+h) = \sqrt{2(x+h)-1}$

$$f'(x) = \lim_{h\to 0}\frac{\sqrt{2x+2h-1}-\sqrt{2x-1}}{h}$$

$$f'(x) = \lim_{h\to 0}\frac{\sqrt{2x+2h-1}-\sqrt{2x-1}}{h}\cdot\frac{(\sqrt{2x+2h-1}+\sqrt{2x-1})}{\sqrt{2x+2h-1}+\sqrt{2x-1}}$$

$$f'(x) = \lim_{h\to 0}\frac{2x+2h-1-(2x-1)}{h[\sqrt{2x+2h-1}+\sqrt{2x-1}]}$$

$$f'(x) = \lim_{h\to 0}\frac{\cancel{2x}+2h\cancel{-1}\cancel{-2x}\cancel{+1}}{h[\sqrt{2x+2h-1}+\sqrt{2x-1}]}$$

$$f'(x) = \lim_{h\to 0}\frac{2\cancel{h}}{\cancel{h}(\sqrt{2x+2h-1}+\sqrt{2x-1})}$$

$$= \frac{2}{\sqrt{2x-1}+\sqrt{2x-1}} = \frac{2}{2\sqrt{2x-1}} = \boxed{\frac{1}{\sqrt{2x-1}}}$$

⑥

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$f(\ ) = \frac{2}{5-(\ )}$$

4) Use the Limit Definition of a derivative to find f'(3) if $f(x) = \frac{2}{5-x}$

$$f(x+h) = \frac{2}{5-(x+h)}$$

$$f'(x) = \lim_{h\to 0}\frac{\frac{2}{5-(x+h)} - \frac{2}{5-x}}{h}$$

$$f'(x) = \lim_{h\to 0}\frac{\frac{2}{5-x-h} - \frac{2}{5-x}}{h}\cdot(5-x)(5-x-h)$$

$$\lim_{h\to 0}\frac{2(5-x)-2(5-x-h)}{h(5-x)(5-x-h)}$$

$$\lim_{h\to 0}\frac{\cancel{10}-\cancel{2x}-\cancel{10}+\cancel{2x}+2h}{h(5-x)(5-x-h)}$$

$$\lim_{h\to 0}\frac{2\cancel{h}}{\cancel{h}(5-x)(5-x-h)}$$

$$f'(x) = \lim_{h\to 0}\frac{2}{(5-x)(5-x-h)} = \frac{2}{(5-x)(5-x)}$$

$$\boxed{f'(x) = \frac{2}{(5-x)^2}}$$

$$f'(3) = \frac{2}{(5-3)^2} = \frac{2}{(2)^2}$$

$$\boxed{f'(3) = \frac{1}{2}}$$

$f(-1) = 2(-1)-3 = -5$

5) Use either general or alternative method above to find the equation of the tangent line to $f(x) = 2x - 3x^2$ at $x = -1$. $y - y_1 = m(x - x_1)$

$$f(\ ) = 2(\ ) - 3(\ )^2$$

$$f(x+h) = 2(x+h) - 3(x+h)^2$$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$f'(x) = \lim_{h\to 0}\frac{2(x+h)-3(x+h)^2-(2x-3x^2)}{h}$$

$$f'(x) = \lim_{h\to 0}\frac{2x+2h-3(x^2+2xh+h^2)-2x+3x^2}{h}$$

$$f'(x) = \lim_{h\to 0}\frac{\cancel{2x}+2h-\cancel{3x^2}-6xh-3h^2-\cancel{2x}+\cancel{3x^2}}{h}$$

$$\lim_{h\to 0}\frac{2h-6xh-3h^2}{h} \rightarrow \lim_{h\to 0}\frac{\cancel{h}(2-6x-3h)}{\cancel{h}}$$

$$f'(x) = 2-6x-0$$

$$\boxed{f'(x) = 2-6x}$$

$$f'(-1) = 2-6(-1)$$

$$= 2+6$$

$$\underline{f'(-1) = 8}$$

Alt. method $f'(c) = \lim_{x\to c}\frac{f(x)-f(c)}{x-c}$

$$f'(-1) = \lim_{x\to -1}\frac{f(x)-f(-1)}{x-(-1)}$$

$$f'(-1) = \lim_{x\to -1}\frac{2x-3x^2-(-5)}{x+1} = \frac{2x-3x^2+5}{x+1}$$

$$= \lim_{x\to -1}\frac{-1(3x^2-2x-5)}{x+1}$$

$$f'(-1) = \lim_{x\to -1}\frac{-1(3x-5)\cancel{(x+1)}}{\cancel{(x+1)}}$$

$$\underline{f'(-1) = -1(-3-5) = 8}$$

point: $f(-1) = -2-3 = -5$

slope: $m = 8$

point: $(-1, -5)$

slope: $m = 8$

$$\boxed{y+5 = 8(x+1)}$$

Non-AP Calculus 2.1 **Definition of Derivative Quiz Review WS #1**

$\boxed{\text{Key}}$ ⑦

1) Use the Limit Definition of a derivative to find G'(x) if G(x) = $3x^2 - 4x + 5$

$$G'(x) = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$

$$g(x+h) = 3(x+h)^2 - 4(x+h) + 5$$

$$G'(x) = \lim_{h\to 0} \frac{3(x+h)^2 - 4(x+h) + 5 - (3x^2 - 4x + 5)}{h}$$

$$G'(x) = \lim_{h\to 0} \frac{3(x^2 + 2xh + h^2) - 4x - 4h + 5 - 3x^2 + 4x - 5}{h}$$

$$G'(x) = \lim_{h\to 0} \frac{\cancel{3x^2} + 6xh + 3h^2 \cancel{-4x} - 4h \cancel{+5} \cancel{-3x^2} \cancel{+4x} \cancel{-5}}{h}$$

$$G'(x) = \lim_{h\to 0} \frac{\cancel{h}(6x + 3h - 4)}{\cancel{h}}$$

$$\boxed{G'(x) = 6x - 4}$$

2) Use the Alternative definition of the derivative to find $H'(2)$ if $H(x) = \sqrt{5-x}$

$$h'(c) = \lim_{x\to c} \frac{h(x) - h(c)}{x - c}$$

$$h'(2) = \lim_{x\to 2} \frac{h(x) - h(2)}{x - 2}$$

$$h'(2) = \lim_{x\to 2} \frac{\sqrt{5-x} - \sqrt{3}}{x - 2}$$

$$h(2) = \sqrt{5-2} = \sqrt{3}$$

$$h'(2) = \lim_{x\to 2} \frac{\sqrt{5-x} - \sqrt{3}}{x-2} \cdot \frac{\sqrt{5-x} + \sqrt{3}}{\sqrt{5-x} + \sqrt{3}}$$

$$= \lim_{x\to 2} \frac{(5-x) - 3}{(x-2)(\sqrt{5-x} + \sqrt{3})}$$

$$= \lim_{x\to 2} \frac{5 - x - 3}{(x-2)(\sqrt{5-x} + \sqrt{3})}$$

$$\lim_{x\to 2} \frac{(\cancel{2-x})\ \enclose{circle}{-1}}{(\cancel{x-2})(\sqrt{5-x} + \sqrt{3})}$$

$$\lim_{x\to 2} \frac{-1}{\sqrt{5-2} + \sqrt{3}} = \boxed{\frac{-1}{2\sqrt{3}}}$$

$$\boxed{H'(2) = \frac{-1}{2\sqrt{3}}}$$

3) Use the Limit Definition of a Derivative to find H'(x) if $H(x) = \sqrt{x-3}$

$$h'(x) = \lim_{h\to 0} \frac{h(x+h) - h(x)}{h}$$

$$h(x+h) = \sqrt{x+h-3}$$

$$h'(x) = \lim_{h\to 0} \frac{\sqrt{x+h-3} - \sqrt{x-3}}{h}$$

$\swarrow \frac{0}{0}$

$$h'(x) = \lim_{h\to 0} \left(\frac{\sqrt{x+h-3} - \sqrt{x-3}}{h}\right) \cdot \left(\frac{\sqrt{x+h-3} + \sqrt{x-3}}{\sqrt{x+h-3} + \sqrt{x-3}}\right)$$

$$h'(x) = \lim_{h\to 0} \frac{x + h - 3 - (x - 3)}{h(\sqrt{x+h-3} + \sqrt{x-3})}$$

$$h'(x) = \lim_{h\to 0} \frac{\cancel{x} + h \cancel{-3} \cancel{-x} \cancel{+3}}{h(\sqrt{x+h-3} + \sqrt{x-3})}$$

$$h'(x) = \lim_{h\to 0} \frac{1}{\sqrt{x+h-3} + \sqrt{x-3}}$$

$$= \frac{1}{\sqrt{x-3} + \sqrt{x-3}}$$

$$\boxed{h'(x) = \frac{1}{2\sqrt{x-3}}}$$

⑧

4) Use the Limit Definition of a derivative to find f'(3) if $f(x) = \frac{5}{x-2}$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f(x+h) = \frac{5}{x+h-2}$$

$\frac{0}{0}$

$$f'(x) = \lim_{h \to 0} \frac{\frac{5}{x+h-2} - \frac{5}{x-2}}{h} \cdot \frac{(x+h-2)(x-2)}{(x+h-2)(x-2)}$$

$$f'(x) = \lim_{h \to 0} \frac{5(x-2) - 5(x+h-2)}{h(x+h-2)(x-2)}$$

$$\lim_{h \to 0} \frac{\cancel{5x} \cancel{-10} - \cancel{5x} - 5h \cancel{+10}}{h(x+h-2)(x-2)}$$

$$f'(x) = \lim_{h \to 0} \frac{-5\cancel{h}}{\cancel{h}(x+h-2)(x-2)}$$

$$f'(x) = \lim_{h \to 0} \frac{-5}{(x-2)(x-2)} = \frac{-5}{(x-2)^2}$$

$$\boxed{f'(x) = \frac{-5}{(x-2)^2}} \quad f'(3) = \frac{-5}{(3-2)^2} = \boxed{-5}$$

5) Use either general or alternative method above to find the equation of the tangent line to $f(x) = 3x - 4x^2$ at $x = -1$.

General Method

$$\rightarrow f(x+h) = 3(x+h) - 4(x+h)^2$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{3(x+h) - 4(x+h)^2 - (3x - 4x^2)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{3x + 3h - 4(x^2 + 2xh + h^2) - 3x + 4x^2}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{\cancel{3x} + 3h \cancel{- 4x^2} - 8xh - 4h^2 \cancel{- 3x} \cancel{+ 4x^2}}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{\cancel{h}(3 - 8x - 4h)}{\cancel{h}} = 3 - 8x - 4(0)$$

$$\boxed{f'(x) = 3 - 8x}$$

$$\boxed{f(x) = 3x - 4x^2}$$

$$f(-1) = 3(-1) - 4(-1)^2$$

$$= -3 - 4 = -7$$

point: $(-1, -7)$

slope : m = 11

$$f'(-1) = 3 - 8(-1) = 11$$

$$\boxed{y + 7 = 11(x+1)}$$

$$f(-1) = 3(-1) - 4(-1)^2$$

$$= -7$$

Alternative Method

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

$$f'(-1) = \lim_{x \to -1} \frac{f(x) - f(-1)}{x - (-1)}$$

$$f'(-1) = \lim_{x \to -1} \frac{3x - 4x^2 - (-7)}{x + 1}$$

$$= \lim_{x \to -1} \frac{-4x^2 + 3x + 7}{x + 1}$$

$$= \lim_{x \to -1} \frac{-1(4x^2 - 3x - 7)}{x + 1}$$

-28

-7 4

4 4

-3

$$= \lim_{x \to -1} \frac{-1(4x-7)\cancel{(x+1)}}{\cancel{(x+1)}}$$

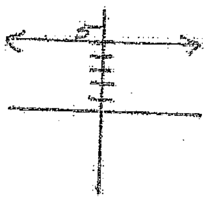
$$= \lim_{x \to -1} -1(4x - 7) = \boxed{11}$$

$$\boxed{f'(-1) = 11}$$

Ch. 2.2a Derivative Rules - Notes

1. Constant Rule: If $f(x)=c$, then $f'(x)=0$

Ex. $f(x)=5 \longrightarrow f'(x)=0$



2. Power Rule: If $f(x)=x^n$, then $f'(x)=n\cdot x^{n-1}$

steps: a) Bring exponent down, in front of variable
b) subtract 1 from original exponent value

*Important Note: Be sure function is in appropriate form before applying power rule.

→ convert any radicals to rational exponents
→ Move all variables from denominator to numerator (if necessar

Ex. 1 Find derivatives of the following:

a) $y=x^7 \longrightarrow y'=7x^6$

b) $g(x)=\sqrt[3]{x} \quad g(x)=x^{1/3} \quad g'(x)=\frac{1}{3}x^{1/3-3/3}=\frac{1}{3}x^{-2/3}=\boxed{\frac{1}{3x^{2/3}} \text{ or } \frac{1}{3\sqrt[3]{x^2}}}$

c) $y=\frac{4}{x^5} \quad y=4x^{-5} \quad y'=-20x^{-5-1}=-20x^{-6}=\boxed{\frac{-20}{x^6}}$

d) $y=8x^{2/3}-\sqrt[5]{x}+\frac{2}{\sqrt{x}}+0.875$

$y=8x^{2/3}-x^{1/5}+2x^{-1/2}+0.875$

$y'=\frac{2}{3}\cdot 8x^{2/3-3/3}-\frac{1}{5}x^{\frac{1}{5}-5/5}+\frac{-1}{2}\cdot 2x^{-1/2-2/2}+0$

$y'=\frac{16}{3}x^{-1/3}-\frac{1}{5}x^{-4/5}-1x^{-3/2}+0$

$\boxed{y'=\frac{16}{3x^{1/3}}-\frac{1}{5x^{4/5}}-\frac{1}{x^{3/2}}}$

Ex. 2 If $f(x) = x^{-2}$, find $f'(2)$

$$f'(x) = -2x^{-3} = \frac{-2}{x^3} \qquad f'(2) = \frac{-2}{2^3} = \frac{-2}{8} = \boxed{\frac{-1}{4}}$$

Ex. 3 If $f(x) = \sqrt[3]{x^2}$, write tangent line equation to $f(x)$ at $x=$

$$f(x) = x^{2/3} \qquad f'(x) = \tfrac{2}{3}x^{-1/3} \qquad f'(8) = \frac{2}{3(8)^{1/3}} = \frac{2}{3\cdot 2} = \frac{1}{3}$$

$$f'(x) = \tfrac{2}{3}x^{\frac{2}{3}-\frac{3}{3}} \qquad f'(x) = \frac{2}{3x^{1/3}} \qquad f(8) = \sqrt[3]{8^2} = 4$$

point: $(8, 4)$

slope: $m = 1/3$

$$y - y_1 = m(x - x_1)$$

$$\boxed{y - 4 = \tfrac{1}{3}(x - 8)}$$

Ex. 4 $f(x) = \dfrac{x^4 - 3x^2 + 4(\sqrt[3]{x})}{\sqrt{x}}$ find $f'(x)$

$$f(x) = \frac{x^4}{x^{1/2}} - \frac{3x^2}{x^{1/2}} + \frac{4x^{1/3}}{x^{1/2}} = x^{7/2} - 3x^{3/2} + 4x^{-1/6}$$

$$f'(x) = \tfrac{7}{2}x^{5/2} - 3\cdot\tfrac{3}{2}x^{1/2} + 4(-1/6)x^{-7/6}$$

$$= \boxed{\frac{7}{2}x^{5/2} - \frac{9}{2}x^{1/2} - \frac{2}{3x^{7/6}}}$$

Ex. 5 $f(x) = 3x(x+1)^2$ find $f'(x)$

$$f(x) = 3x(x^2 + 2x + 1) = 3x^3 + 6x^2 + 3x$$

$$\boxed{f'(x) = 9x^2 + 12x + 3}$$

10b

Ch. 2.2a Homework p.115-116 #1-17 odd, 25-35 odd,
39-49 odd, 53-59 odd, 63, 65

7) $y=\frac{1}{x^7}$ $y=x^{-7}$ $y'=-7x^{-8}$ $\boxed{y'=\frac{-7}{x^8}}$

9) $f(x)=\sqrt[5]{x}$ $f(x)=x^{1/5}$ $f'(x)=\frac{1}{5}x^{-4/5}$ $\boxed{f'(x)=\frac{1}{5x^{4/5}}}$

27) $y=\frac{3}{(2x)^3}=\frac{3}{8x^3}=\frac{3}{8}x^{-3}$ $y'=-3\cdot\frac{3}{8}x^{-3-1}$ $y'=\frac{-9}{8}x^{-4}$

$\boxed{y'=\frac{-9}{8x^4}}$

29) $y=\frac{\sqrt{x}}{x}=\frac{1}{\sqrt{x}}=x^{-1/2}$ $y'=-\frac{1}{2}x^{-1/2-\frac{2}{2}}$ $y'=\frac{-1}{2}x^{-3/2}=\boxed{\frac{-1}{2x^{3/2}}}$

35) $y=(2x+1)^2$ at $(0,1)$

$y=(2x+1)(2x+1)$ $y=4x^2+4x+1$ $y'=8x+4$

$y'(0)=8(0)+4=4$

$\boxed{y'(0)=4}$

43) $f(x)=\frac{x^3-3x^2+4}{x^2}=\frac{x^3}{x^2}-\frac{3x^2}{x^2}+\frac{4}{x^2}$

$f(x)=x-3+4x^{-2}$

$f'(x)=1+0-8x^{-3}$ $\boxed{f'(x)=1-\frac{8}{x^3}}$

(10c)

47) $f(x)=\sqrt{x}-6\sqrt[3]{x} \quad = x^{1/2}-6x^{1/3}$

$f'(x)=\frac{1}{2}x^{-1/2}-\frac{1}{3}\cdot 6x^{-2/3}$ $\quad \boxed{f'(x)=\frac{1}{2x^{1/2}}-\frac{2}{x^{2/3}}}$

49) $h(s)=s^{4/5}-s^{2/3}$

$h'(s)=\frac{4}{5}s^{-1/5}-\frac{2}{3}s^{-1/3}$ $\quad h'(s)=\frac{4}{5s^{1/5}}-\frac{2}{3s^{1/3}}$

55) $f(x)=\frac{2}{\sqrt[4]{x^3}}$ at $(1,2)$ Write equation of tangent line

$f(x)=2x^{-3/4}$

$f'(x)=\frac{-3}{4}\cdot 2x^{-7/4}$

$f'(x)=\frac{-3}{2x^{7/4}}$

$f'(1)=\frac{-3}{2(1)^{9/4}}=\frac{-3}{2}$

point: $(1,2)$

slope: $m=-3/2$

$y-y_1=m(x-x_1)$

$\boxed{y-2=\frac{-3}{2}(x-1)}$

57) $y=x^4-8x^2+2$ *Determine point where f(x) has horizontal tangent

*set $y'(x)=0$, solve for x

slope $=0$

$f'(x)=0$

$y'(x)=4x^3-16x$

$0=4x(x^2-4)$

$0=4x(x+2)(x-2)$

$x=0,-2,2$

points are: $\begin{cases} y(0)=2 \\ y(2)=-14 \\ y(-2)=-14 \end{cases}$ $\quad \underline{\underline{(0,2),(2,-14),(-2,-14)}}$

63) Find k such that line is tangent to graph *set $f'(x)=$ slope

line

*set $f(x)=$ equation

line

$f(x)=x^2-kx$ $\quad$ line: $y=4x-9$

$f'(x)=2x-k$ $\quad$ slope: $m=4$

$2x-k=4$

$k=2x-4$

$x^2-kx=4x-9$

$x^2-(2x-4)x=4x-9$

$x^2-2x^2+4x=4x-9$

$-x^2=-9$

$\boxed{\begin{array}{l} x=\pm 3 \\ \text{when } x=3,\ k=2 \\ \text{when } x=-3,\ k=-10 \end{array}}$

⑪

Key

2.2 Derivative Power Rule Practice/Review Worksheet

Derivative Power Rule:

$$\frac{d}{dx}x^n = n * x^{n-1}$$

Power Rule Conditions:

i) All Radicals converted to Rational Exponents
ii) All denominator variables brought up to the numerator
iii) All parentheses resolved, all terms expanded

Finding a Derivative use the rules of differentiation to find the derivative of the function.

1) $y = x^7$

$\boxed{y' = 7x^6}$

2) $y = \frac{1}{x^5}$

$y = x^{-5}$

$y' = -5x^{-6}$

$\boxed{y' = \frac{-5}{x^6}}$

3) $y = \frac{3}{x^7}$

$y = 3x^{-7}$

$y' = 3 \cdot -7x^{-8}$

$\boxed{y' = \frac{-21}{x^8}}$

4) $f(x) = \sqrt[5]{x}$

$f(x) = x^{1/5}$

$f'(x) = \frac{1}{5}x^{-4/5}$

$\boxed{f'(x) = \frac{1}{5x^{4/5}}}$

5) $f(t) = -2t^2 + 3t - 6$

$\boxed{f'(t) = -4t + 3}$

6) $y = \frac{5}{2x^2}$

$y = \frac{5}{2}x^{-2}$

$y' = \frac{5}{2} \cdot -2x^{-3}$

$\boxed{y' = \frac{-5}{x^3}}$

7) $y = \frac{3}{2x^4}$

$y = \frac{3}{2}x^{-4}$

$y' = \frac{3}{2} \cdot -4x^{-5}$

$y' = \frac{-12}{2}x^{-5}$

$\boxed{y' = \frac{-6}{x^5}}$

8) $y = \frac{6}{(5x)^3}$

$y = \frac{6}{5^3x^3}$

$y = \frac{6}{125}x^{-3}$

$y' = \frac{6}{125} \cdot -3x^{-4}$

$y' = \frac{-18}{125}x^{-4}$

$\boxed{y' = \frac{-18}{125x^4}}$

⑫

Derivative Power Rule:

$$\frac{d}{dx}x^n = n * x^{n-1}$$

Power Rule Conditions:

- i) All Radicals converted to Rational Exponents
- ii) All denominator variables brought up to the numerator
- iii) All parentheses resolved, all terms expanded

Find the derivative of the functions below:

10) $g(t) = t^2 - \frac{4}{t^3}$

$g(t)=t^2-4t^{-3}$

$g'(t)=2t-4(-3t^{-4})$

$\boxed{g'(t)=2t+\frac{12}{t^4}}$

11) $f(x) = \frac{4x^3 + 3x^2}{x}$

$f(x)=(4x^3+3x^2)x^{-1}$

$f(x)=4x^2+3x$

$\boxed{f'(x)=8x+3}$

12) $f(x) = \frac{2x^4 - x}{x^3}$

$f(x)=(2x^4-x)x^{-3}$

$f(x)=2x^1-x^{-2}$

$f'(x)=2-(-2x^{-3})$

$\boxed{f'(x)=2+\frac{2}{x^3}}$

13) $y = x^2(2x^2 - 3x)$

$y=2x^4-3x^3$

$\boxed{y'=8x^3-9x^2}$

14) $f(x) = \sqrt{x} - 6\sqrt[3]{x}$

$f(x)=x^{1/2}-6x^{1/3}$

$f'(x)=\frac{1}{2}x^{-1/2}-6\cdot\frac{1}{3}x^{-2/3}$

$\boxed{f'(x)=\frac{1}{2x^{1/2}}-\frac{2}{x^{2/3}}}$

15) $f(t) = t^{2/3} - t^{1/3} + 4$

$f'(t)=\frac{2}{3}t^{-1/3}-\frac{1}{3}t^{-2/3}$

$\boxed{f'(t)=\frac{2}{3t^{1/3}}-\frac{1}{3t^{2/3}}}$

Finding an Equation of a Tangent Line **In Exercises** **(a) find an equation of the tangent line to the graph of f at the given point.**

Equation of tangent line:

- i) Find ordered pair $((x_1, y_1)$ using $f(x)$
- ii) Find slope m using $f'(x)$
- iii) $y - y_1 = m(x - x_1)$

16) $y = x^4 - 3x^2 + 2$ $(1, 0)$

$y'=4x^3-6x$

$y'(1)=4(1)^3-6(1)=-2$

point: $(1,0)$

slope: $m=-2$

$y-0=-2(x-1)$

$\boxed{y=-2(x-1)}$

17) $y = x^3 - 3x$ $(2, 2)$

$y'=3x^2-3$

$y'(2)=3(2)^2-3=9$

point: $(2,2)$

slope: $m=9$

$\boxed{y-2=9(x-2)}$

2.2 Derivative Power Rule   Practice/Review Worksheet #2

Key ⑬

Derivative Power Rule:

$$\frac{d}{dx}x^n = n * x^{n-1}$$

Power Rule Conditions:

- i) All Radicals converted to Rational Exponents
- ii) All denominator variables brought up to the numerator
- iii) All parentheses resolved, all terms expanded

Finding a Derivative In Exercises 3–24, use the rules of differentiation to find the derivative of the function.

1) $f(x) = 3x^5 - 4x + 156$

$$\boxed{f'(x) = 15x^4 - 4}$$

2) $f(x) = \frac{5}{3x^6}$

$f(x) = \frac{5}{3}x^{-6}$

$f'(x) = \frac{5}{3} \cdot -6x^{-7}$

$f'(x) = \frac{-30}{3}x^{-7}$

$$\boxed{f'(x) = \frac{-10}{x^7}}$$

3) $g(x) = 3\sqrt{x^9}$

$g(x) = 3x^{9/2}$

$g'(x) = 3 \cdot \frac{9}{2}x^{7/2}$

$$\boxed{g'(x) = \frac{27}{2}x^{7/2}}$$

4) $f(x) = \frac{\sqrt{x^9}}{3}$

$f(x) = \frac{1}{3}x^{9/2}$

$f'(x) = \frac{1}{3} \cdot \frac{9}{2}x^{7/2} = \frac{9}{6}x^{7/2}$

$$\boxed{f'(x) = \frac{3}{2}x^{7/2}}$$

5) $h(t) = \frac{7}{5(2t)^3}$

$h(t) = \frac{7}{5 \cdot 2^3 t^3}$

$h(t) = \frac{7}{40t^3}$

$h(t) = \frac{7}{40}t^{-3}$

$h'(t) = \frac{7}{40} \cdot -3t^{-4}$

$$\boxed{h'(t) = \frac{-21}{40t^4}}$$

6) $f(t) = \frac{7}{(3t)^3}$

$f(t) = \frac{7}{27t^3}$

$f(t) = \frac{7}{27}t^{-3}$

$f'(t) = \frac{7}{27} \cdot -3t^{-4}$

$f'(t) = \frac{-21}{27}t^{-4}$

$$\boxed{f'(t) = \frac{-7}{9t^4}}$$

7) $f(x) = \frac{7}{x\sqrt{x}}$

$f(x) = \frac{7}{x \cdot x^{1/2}}$

$f(x) = \frac{7}{x^{3/2}}$

$f(x) = 7x^{-3/2}$

$f'(x) = 7 \cdot \frac{-3}{2}x^{-5/2}$

$$\boxed{f'(x) = \frac{-21}{2x^{5/2}}}$$

8) $f(x) = 5\sqrt{x} - 3x^2(2 - 5x)$

$f(x) = 5x^{1/2} - 6x^2 + 15x^3$

$f'(x) = 5 \cdot \frac{1}{2}x^{-1/2} - 12x + 45x^2$

$$\boxed{f'(x) = \frac{5}{2x^{1/2}} - 12x + 45x^2}$$

Derivative Power Rule:

$\frac{d}{dx}x^n = n * x^{n-1}$

Power Rule Conditions:

- i) All Radicals converted to Rational Exponents
- ii) All denominator variables brought up to the numerator
- iii) All parentheses resolved, all terms expanded

Find the derivative of the functions below:

9) $f(x) = x(2-5x)^2$

$f(x)=x(2-5x)(2-5x)$

$f(x)=x(4-20x+25x^2)$

$f(x)=4x-20x^2+25x^3$

$\boxed{f'(x)=4-40+75x^2}$

10) $f(x) = \frac{5x^4-3x+1}{x^2}$

$f(x)=(5x^4-3x+1)x^{-2}$

$f(x)=5x^2-3x^{-1}+x^{-2}$

$f'(x)=10x+3x^{-2}-2x^{-3}$

$\boxed{f'(x)=10x+\frac{3}{x^2}-\frac{2}{x^3}}$

11) $f(x) = \frac{3x^4-2x+1}{\sqrt{x}}$

$f(x)=(3x^4-2x+1)x^{-1/2}$

$f(x)=3x^{7/2}-2x^{1/2}+1x^{-1/2}$

$f'(x)=3\cdot\frac{7}{2}x^{5/2}-2\cdot\frac{1}{2}x^{-1/2}-\frac{1}{2}x^{-3/2}$

$\boxed{f'(x)=\frac{21}{2}x^{5/2}-\frac{1}{x^{1/2}}-\frac{1}{2x^{3/2}}}$

12) $f(x) = \frac{2x^3-4x^2+5}{\sqrt{x}}$

$f(x)=(2x^3-4x^2+5)x^{-1/2}$

$f(x)=2x^{5/2}-4x^{3/2}+5x^{-1/2}$

$f'(x)=2\cdot\frac{5}{2}x^{3/2}-4\cdot\frac{3}{2}x^{1/2}-5\cdot\frac{-1}{2}x^{-3/2}$

$\boxed{f'(x)=5x^{3/2}-6x^{1/2}+\frac{5}{2x^{3/2}}}$

Finding an Equation of a Tangent Line In Exercises 53–56, (a) find an equation of the tangent line to the graph of f at the given point.

Equation of tangent line:

- i) Find ordered pair $((x_1, y_1)$ using $f(x)$
- ii) Find slope m using $f'(x)$
- iii) $y - y_1 = m(x - x_1)$

13) $f(x) = \frac{2}{\sqrt[4]{x^3}}$ $(1, 2)$

$f(x)=2x^{-3/4}$

$f'(x)=2\cdot\frac{-3}{4}x^{-7/4}$

$f'(x)=\frac{-3}{2x^{7/4}}$

$f'(1)=\frac{-3}{2(1)^{7/4}}=\frac{-3}{2}$

point: $(1,2)$

slope: $m=\frac{-3}{2}$

$\boxed{y-2=-\frac{3}{2}(x-1)}$

14) $y = (x-2)(x^2+3x)$ $(1, -4)$

$y=x^3+3x^2-2x^2-6x$

$y=x^3+x^2-6x$

$y'=3x^2+2x-6$

$y'(1)=3(1)^2+2(1)-6$

$y'(1)=-1$

point: $(1,-4)$

slope: $m=-1$

$\boxed{y+4=-1(x-1)}$

AP Calculus PVA (Position-Velocity-Acceleration) Notes

Instantaneous velocity ,v(t), of the object is the derivative of the position function s(t) with respect to time

$v(t) = s'(t)$

Acceleration, a(t), is the derivative of velocity with respect to time

$a(t) = v'(t) = s''(t)$

AVERAGE rate of change of f(x) from a to b = slope of secant = $\dfrac{f(b)-f(a)}{b-a}$

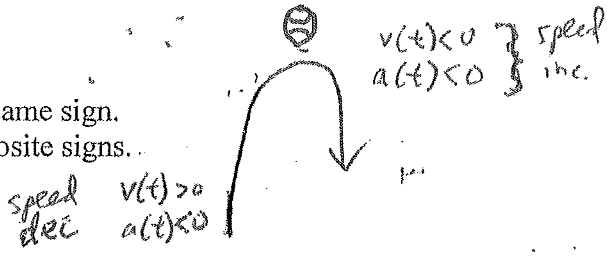
INSTANTANEOUS rate of change of f(x) at x = c = slope of tangent = $f'(c)$

Speed = |velocity|

Displacement = how far you are from where you started
Distance = total amount you have traveled

Ex) If I travel 10 feet to the right and then turn around and travel 3 feet back to the left, my distance is 13 feet but my displacement is 7 feet.

Speed is increasing when velocity and acceleration have the same sign.
Speed is decreasing when velocity and acceleration have opposite signs.



Particle Motion

Particle motion (linear motion) describes the object moving along a line (usually along a horizontal line)

x(t) = Position function
v(t) = velocity function
a(t) = acceleration function

Positive velocity indicates particle moving in positive direction (usually right)

Negative velocity indicates " " " negative direction (usually left)

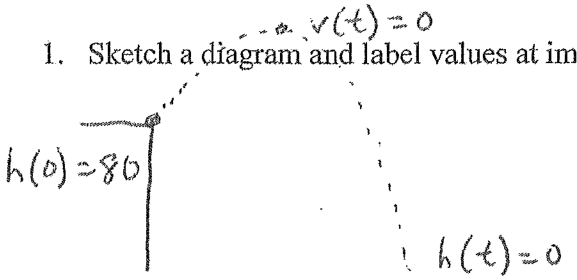
When v(t) = 0, this indicates particle is at rest

⑯

A.P. Calculus PVA **Worksheet 2-2a**

A ball is thrown vertically upwards from the edge of a building and it eventually hits the ground next to the building. If the height of the ball at any given time, $t \geq 0$ (seconds), is $h(t) = -16t^2 + 64t + 80$ (feet), answer the following:

1. Sketch a diagram and label values at important places



$h(t) = -16t^2 + 64t + 80$

$v(t) = -32t + 64$

$a(t) = -32$

2. How tall is the building?

$h(0) = 80$ ft.

3. When does the ball reach maximum height?

find t when $v(t) = 0$

$0 = -32t + 64$

$32t = 64$

$\boxed{t = 2 \text{ sec}}$

4. What is the maximum height?

find $h(2)$

$h(2) = -16(2)^2 + 64(2) + 80 = \boxed{144 \text{ ft}}$

5. How long does it take to hit the ground?

set $h(t) = 0$

$0 = -16t^2 + 64t + 80$

$0 = 16t^2 - 64t - 80$

$0 = t^2 - 4t - 5$

$0 = (t-5)(t+1)$

$t = 5, \cancel{t = -1}$ $t = 5$ secs.

6. What was the initial velocity?

$v(0) = -32(0) + 64 = 64$

7. What is the velocity at $t = 1$ second? At $t = 2$ seconds?

$v(1) = -32 + 64 = 32$ ft/s

$v(2) = -32(2) + 64 = 0$ ft/s

8. What is the height at $t = 3$ seconds?

$h(3) = -16(3)^2 + 64(3) + 80 = 128$ ft.

9. What is the speed when it hits the ground?

$v(5) = -32(5) + 64 = -96 \rightarrow 96$ ft/s

10. What is the acceleration at $t = 1$ second? At $t = 2$ seconds?

$a(1) = -32$ ft/s^2

$a(2) = -32$ ft/s^2

11) Avg. velocity $[0, 2]$

$\frac{h(2) - h(0)}{2 - 0} = \frac{144 - 80}{2 - 0} = 32$ ft/s

12) Avg. acceleration $[1, 2]$

$\frac{v(2) - v(1)}{2 - 1} = \frac{0 - 32}{1} = -32$ ft/s^2

13) inc. or dec. speed at $t = 1$ sec.

$\left.\begin{array}{l} v(1) = 32 \text{ ft/s} \\ a(1) = -32 \text{ ft/s} \end{array}\right]$ dec. speed

⑰

A.P. Calculus PVA **Worksheet 2-2b** **Linear Motion Problems**

1. An object is traveling at 20 m/sec to the left. What is its speed and velocity?

speed = 20m/s

velocity = -20m/s

2. Which has the greater speed and velocity: object A with a velocity of -20 m/sec or object B with a velocity of -10 m/sec?

greater velocity → object B (-10m/s)

greater speed: object A (20 m/s)

3. A billiard ball is hit and travels in a straight line. If x centimeters is the distance of the ball from its initial position at t seconds, then $x(t) = 5t^2 - 4t$. If the ball hits a cushion that is 12 cm from its initial position, at what velocity does it hit the cushion?

$12 = x(t)$

$12 = 5t^2 - 4t$

$5t^2 - 4t - 12 = 0$

$(5t+6)(t-2) = 0 \quad t = -6/5, 2$

$v(t) = 10t - 4$

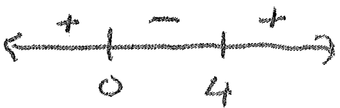
$v(2) = 10(2) - 4 = 16\text{cm/s}$

4. If a particle moves along a line according to the equation $s(t) = t^5 - 5t^4$ for all real numbers, t, then how many times does the particle reverse its direction?

$v(t) = 5t^4 - 20t^3$

$0 = 5t^3(t-4)$

$t = 0, 4$



$\boxed{\text{twice, at } t=0, t=4s}$

5. The position in meters of a particle moving on the x-axis is given by $x(t) = 2t^3 - 2t + 1$ at all times t, $t > 0$. Find the acceleration when the velocity is 4 m/sec.

$v(t) = 6t^2 - 2$

$a(t) = 12t$

$4 = 6t^2 - 2$

$6 = 6t^2$

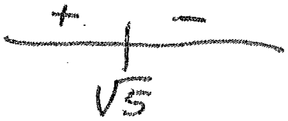
$t = 1$

$a(1) = 12(1) = \boxed{12m/s^2}$

6. If $x(t) = \dfrac{t}{t^2+5}$ is the position function of a moving particle for $t > 0$, at what instant of time will the particle start to reverse its direction of motion, and where is it at that instant?

$$v(t) = \frac{1(t^2+5) - t(2t)}{(t^2+5)^2}$$

$$= \frac{t^2+5-2t^2}{(t^2+5)^2} = \frac{5-t^2}{(t^2+5)^2} = 0 \quad \boxed{t = \sqrt{5}}$$



⑱

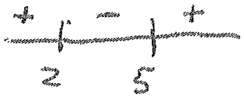
7. The position function of a particle moving on a coordinate line is given by: $x(t) = 2t^3 - 21t^2 + 60t + 3$, where x is in feet and t is in seconds. $v(t) = 6t^2 - 42t + 60 = 6(t^2 - 7t + 10)$

a) When is the particle at rest?

$0 = 6(t-5)(t-2)$

$t = 5, 2$ secs.

b) When does the particle reverse direction?



$t = 2, 5$ secs.

c) What is the velocity when the acceleration is zero?

$a(t) = 12t - 42$

$a(t) = 6(2t-7)$

$t = 7/2$

$v(7/2) = -13.5$ ft/s

d) What is the speed when the acceleration is 6 ft/sec?

$6 = 6(2t-7)$

$1 = 2t - 7$

$2t = 8 \quad t = 4$

$v(4) = -12$

Speed = 12 ft/s

e) What is the displacement from $t = 1$ to $t = 3$?

$x(1) = 44$

$x(3) = 48$

$48 - 44 = 4$ft

f) What is the total distance moved from $t = 1$ to $t = 3$?

$x(1) = 44 > 11$

$x(2) = 55 > 7$

$x(3) = 48$

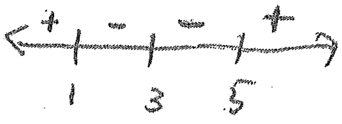
$11 + 7 = 18$ ft

8. If $v(t) = (t-5)(t-3)^2(t-1)$ represents the velocity of a particle moving along a line,

a) When will the particle be at rest?
b) When will the particle move to the left?
c) When will the particle change direction?

g)

a) $t = 5, 3, 1$



b) $(1,3) \cup (3,5)$

c) $t = 1, 5$ sec.

9. A ball is thrown vertically upwards from the edge at the top of a building 160 ft tall with an initial velocity of 24 ft/sec. If the height of the ball (measured from the ground) is given by the function: $h(t) = -16t^2 + bt + c$,

a) Find the values of b and c.

$b = 24$ ft/s $\quad c = 160$

$h(t) = -16t^2 + 24t + 160$

$= -8(2t^2 - 3t - 20)$

$-8(2t+5)(t-4)$

b) How long does it take the ball to reach its maximum height?

$v(t) = -32t + 24 \quad 0 = -32t + 24 \quad t = 3/4$ sec.

c) What is the maximum height of the ball?

$h(3/4) = 169$ ft.

d) How long before the ball passes the top of the building on the way down?

$\cancel{160} = -16t^2 + 24t + \cancel{160} \quad 0 = -8t(2t-3) \quad t = \frac{3}{2}$ sec.

e) How long does it take for the ball to hit the ground?

$h(t) = -8(2t+5)(t-4) \quad t = 4$ sec.

f) What is the speed of the ball when it hits the ground?

$v(4) = -104 \quad 104$ ft/s

g) What is the speed of the ball at $t = 1$ second?

$v(1) = -8 \quad 8$ ft/s

Ch. 2.2 PVA (Position-Velocity-Acceleration) Textbook Problems (Page 116)

Solution Key ⑲

Vertical Motion In Exercises 97 and 98, use the position function $s(t) = -16t^2 + v_0t + s_0$ for free-falling objects.

97. A silver dollar is dropped from the top of a building that is 1362 feet tall.

$V_0 = 0 \qquad S_0 = 1362$

(a) Determine the position and velocity functions for the coin.

(b) Determine the average velocity on the interval [1, 2].

(c) Find the instantaneous velocities when $t = 1$ and $t = 2$.

(d) Find the time required for the coin to reach ground level. * set $s(t)=0$

(e) Find the velocity of the coin at impact.

a) $s(t) = -16t^2 + 0t + 1362$

$v(t) = -32t$

b) $\text{avg. velocity} = \frac{s(2)-s(1)}{2-1} = \frac{1298-1346}{2-1} = \boxed{-48 \text{ ft/s}}$

$s(1) = 1346$

$s(2) = 1298$

c) $v(1) = -32$ ft/s

$v(2) = -64$ ft/s

d) $0 = -16t^2 + 1362 \qquad t^2 = \frac{1362}{16} \qquad t = \sqrt{\frac{1362}{16}} \qquad \boxed{t \approx 9.226 \text{ secs}}$

$16t^2 = 1362$

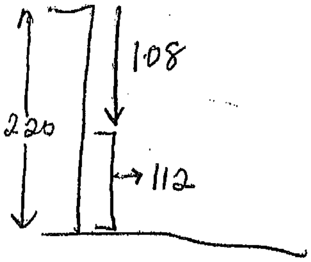
e) $v(9.226) \approx \boxed{-295.242 \text{ ft/s}}$

Vertical Motion In Exercises 97 and 98, use the position function $s(t) = -16t^2 + v_0t + s_0$ for free-falling objects.

$V_0 = -22$ ft/s $\quad S_0 = 220$

98. A ball is thrown straight down from the top of a 220-foot building with an initial velocity of −22 feet per second. What is its velocity after 3 seconds? What is its velocity after falling 108 feet?

height/position = 220 − 108 = 112 ft.
* height is measured from the ground up.



$s(t) = -16t^2 - 22t + 220$

$v(t) = -32t - 22$

$v(3) = -32(3) - 22 = -118$ ft/s

* Find t when $h(t) = 112$, then find $v(t)$

$112 = -16t^2 - 22t + 220$

$0 = -16t^2 - 22t + 108$

$0 = -2(8t^2 + 11t - 54)$

$-2(t-2)(8t+27) = 0$

$\boxed{t=2} \qquad t = \frac{-27}{8}$

$v(2) = -32(2) - 22$

$= \boxed{-86 \text{ ft/s}}$

Vertical Motion **In Exercises 99 and 100, use the position function $s(t) = -4.9t^2 + v_0t + s_0$ for free-falling objects.**

$V_0 = 120$ m/s

$S_0 = 0$

99. A projectile is shot upward from the surface of Earth with an initial velocity of 120 meters per second. What is its velocity after 5 seconds? After 10 seconds?

$s(t) = -4.9t^2 + 120t + 0$

$s'(t) = -9.8t + 120$

$s'(5) = -9.8(5) + 120 = \boxed{71 \text{ m/s}}$

$s'(10) = -9.8(10) + 120 = \boxed{22 \text{ m/s}}$

Vertical Motion **In Exercises 99 and 100, use the position function $s(t) = -4.9t^2 + v_0t + s_0$ for free-falling objects.**

$V_0 = 0$

$S_0 = S_0$

100. To estimate the height of a building, a stone is dropped from the top of the building into a pool of water at ground level. The splash is seen 5.6 seconds after the stone is dropped. What is the height of the building?

$s(t) = -4.9t^2 + 0t + S_0$

* $s(t) = 0$ when $t = 5.6$ sec.

$0 = -4.9(5.6)^2 + S_0$

$S_0 = 4.9(5.6)^2$

$\boxed{S_0 \approx 153.7 \text{m}}$

Ch. 2.3 Notes Product and Quotient Rules

<u>Product Rule</u>: formula used to find the derivatives of products of two or more functions

$$* \ \frac{d}{dx}\left[f(x)\cdot g(x)\right] = f'(x)g(x) + f(x)g(x)$$

"f prime g plus f g prime"

$\boxed{\text{Ex.1}}$ $y = \underbrace{(3x-2x^2)}_{f(x)}\underbrace{(5+4x)}_{g(x)}$ Find $\frac{dy}{dx}$

$$\frac{dy}{dx} = \underbrace{(3-4x)}_{f'}\underbrace{(5+4x)}_{g} + \underbrace{(3x-2x^2)}_{f}\underbrace{(4)}_{g'}$$

<u>Quotient Rule</u>: formula for finding derivative of function that is the quotient of two other functions.

$$* \ \frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

$\boxed{\text{Ex.2}}$ $y = \frac{3x-2x^2}{5+4x}$ Find y'

$$y' = \frac{\overbrace{(3-4x)}^{f'}\overbrace{(5+4x)}^{g} - \overbrace{(3x-2x^2)}^{f}\overbrace{(4)}^{g'}}{\underbrace{(5+4x)^2}_{g^2}}$$

Higher order derivatives

$\boxed{\text{Ex.3}}$ $y = 2x^5 + x^4 - 3x^3 - 8x^2 + 10x - 12$. Find y''''

$$y' = 10x^4 + 4x^3 - 9x^2 - 16x + 10$$
$$y'' = 40x^3 + 12x^2 - 18x - 16$$
$$y''' = 120x^2 + 24x - 18$$
$$y'''' = 240x + 24$$

* Notations

Notations for 1st derivative : $f'(x)$, $g'(x)$, y' , $\frac{dy}{dx}$

Notation for 2nd derivative : $f''(x)$, $y''(x)$, y'' , $\frac{d^2y}{dx^2}$

*Note : This means "2nd derivative", NOT "square the 1st derivative"

Notation for 3rd derivative : $f'''(x)$, y''', $\frac{d^3y}{dx^3}$

Ch. 2.3 Homework p. 126-129 #13, 15, 19-33 odd, 69, 73, 77, 81, 87, 93, 99-103 odd, 105-108, 115, 118, 129-133 odd

19) $y = \frac{x^2+2x}{3} = \frac{x^2}{3} + \frac{2}{3}x = \frac{1}{3}x^2 + \frac{2}{3}x$ $\quad \boxed{y' = \frac{2}{3}x + \frac{2}{3}}$

21) $y = \frac{7}{3x^3} = \frac{7}{3}x^{-3}$ $\quad y' = -3 \cdot \frac{7}{3}x^{-4} = \boxed{\frac{-7}{x^4}}$

23) $y = \frac{4x^{3/2}}{x^1} = 4x^{3/2-1} = 4x^{1/2}$ $\quad y' = \frac{1}{2} \cdot 4x^{-1/2} = \boxed{\frac{2}{x^{1/2}}}$

27) $f(x) = x\left(1 - \frac{4}{x+3}\right) = x - \frac{4x}{x+3}$ $\quad f'(x) = 1 - \frac{4(x+3) - (4x)(1)}{(x+3)^2}$

$= \boxed{1 - \frac{12}{(x+3)^2}}$

29) $f(x) = \frac{2x+5}{\sqrt{x}} = \frac{2x}{x^{1/2}} + \frac{5}{x^{1/2}} = 2x^{1/2} + 5x^{-1/2}$

$f'(x) = x^{-1/2} - \frac{5}{2}x^{-3/2} = \boxed{\frac{1}{x^{1/2}} - \frac{5}{2x^{3/2}}}$

33) $f(x) = \frac{2 - 1/x}{x-3} \cdot \frac{x}{x} = \frac{2x-1}{x^2-3x}$ $\quad f'(x) = \frac{(2)(x^2-3x) - (2x-1)(2x-3)}{(x^2-3x)^2}$

69) Find equation of tangent line: $f(x) = \frac{8}{x^2+4}$ (2,1)

$f'(x) = \frac{(0)(x^2+4) - 8(2x)}{(x^2+4)^2}$

$f'(x) = \frac{-16x}{(x^2+4)^2}$

$f'(2) = \frac{-16(2)}{(2^2+4)^2} = \frac{-32}{64} = \frac{-1}{2}$

point: (2,1)

slope: $m = -1/2$

$\boxed{y - 1 = \frac{-1}{2}(x-2)}$

24c

73) Determine where function has horizontal tangent line

$f(x)=\frac{x^2}{x-1}$ * set $f'(x)=0$

$f'(x)=\frac{(2x)(x-1)-(x^2)(1)}{(x-1)^2}=\frac{2x^2-2x-x^2}{(x-1)^2}=\frac{x^2-2x}{(x-1)^2}=\frac{x(x-2)}{(x-1)^2}$

* when $f'(x)=0$, set just the numerator of $f'(x)=0$:

$\frac{x(x-2)}{(x-1)^2}=0$ when $x(x-2)=0$ $f'(x)=0$ when $x=0, x=2$

$f(x)=\frac{x^2}{x-1}$ $f(0)=\frac{0}{-1}=0$

$f(2)=\frac{4}{2-1}=4$

$\boxed{(0,0) \text{ and } (2,4)}$

77) Find equation of tangent line to $f(x)=\frac{x+1}{x-1}$, parallel to line $2y+x=6$

* set $f'(x)$ = slope of line

$f'(x)=\frac{(1)(x-1)-(x+1)(1)}{(x-1)^2}=\frac{-2}{(x-1)^2}$

line: $2y+x=6$

$2y=-x+6$

$y=\frac{-1}{2}x+3$ so slope $=-1/2$

set $\frac{-2}{(x-1)^2}=\frac{-1}{2}$

$(x-1)^2=4$

$x-1=\pm 2$

$x=-1, 3$ $f(-1)=0$, $f(3)=2$

slope: $m=\frac{-1}{2}$ slope: $m=\frac{-1}{2}$

$$\boxed{\begin{array}{l} y-0=\frac{-1}{2}(x+1) \\ y-2=\frac{-1}{2}(x-3) \end{array}}$$

(25)

2.2-2.3 Review WS #1 (Asynchronous Day)

Key

No negative exponents in answer.

* power Rule conditions
1) variable in numerator
2) radicals to rationals.
3) no parentheses (expand)

1. Find $\frac{dy}{dx}$ if $y = 7x^3(x-1) - \frac{3x^2}{11} + 4\pi x - 5\pi^4 + \sqrt[5]{x^4} + \frac{5}{\sqrt{x^7}}$

$$y = 7x^4 - 7x^3 - \frac{3}{11}x^2 + 4\pi x - 5\pi^4 + x^{4/5} + 5x^{-7/2}$$

$$y' = 28x^3 - 21x^2 - \frac{6}{11}x + 4\pi - 0 + \frac{4}{5}x^{-1/5} - \frac{35}{2}x^{-9/2}$$

$$\boxed{\frac{dy}{dx} = 28x^3 - 21x^2 - \frac{6}{11}x + 4\pi + \frac{4}{5x^{1/5}} - \frac{35}{2x^{9/2}}}$$

2. If $f(x) = \frac{x+4}{x^2-2}$ find $f'(x)$ (simplify fully). Then write the equation of the line tangent to $f(x)$ at $x = 1$ in point-slope form.

*quotient rule

$$f'(x) = \frac{\overbrace{(1)}^{f'}\overbrace{(x^2-2)}^{g} - \overbrace{(x+4)}^{f}\overbrace{(2x)}^{g'}}{\underbrace{(x^2-2)^2}_{g^2}}$$

point: $f(1) = \frac{1+4}{1^2-2} = \frac{5}{-1} = -5$

slope: $f'(1) = \frac{1(1^2-2)-(1+4)(2)}{(1-2)^2} \rightarrow \frac{-1-10}{1} = -11$

point: $(1,-5)$

slope: $m = -11$

$y - y_1 = m(x - x_1)$

$y - (-5) = -11(x-1)$

$\boxed{y + 5 = -11(x-1)}$

3) Find the derivative of f(x) and then evaluate the slope of the graph at x = 1
$f(x) = (3x^5 - 4\sqrt{x})(2x - 5\pi + 9)$

* product rule

$$f(x) = (3x^5 - 4x^{1/2})(2x - 5\pi + 9)$$

$$f'(x) = \overbrace{(15x^2 - 2x^{-1/2})}^{f'}\overbrace{(2x-5\pi+9)}^{g} + \overbrace{(3x^5 - 4x^{1/2})}^{f}\overbrace{(2)}^{g'}$$

$$f'(1) = (15 - 2(1))(2 - 5\pi + 9) + (3-4)(2)$$

$$f'(1) = 13(11 - 5\pi) - 2$$

$$f'(1) = 143 - 65\pi - 2$$

$$\boxed{f'(1) = 141 - 65\pi}$$

㉖

3. Particle moves along the x-axis so that its position at time t is given $x(t) = t^3 - 9t^2 + 15t - 7$ where $x(t)$ is in feet per second and $t \geq 0$. Use this to answer the questions below. **Include units with your answers**

a) Find the velocity and acceleration function

$v(t) = 3t^2 - 18t + 15$

$a(t) = 6t - 18$

b) What is its velocity at $t = 2$ seconds?

$v(2) = 3(2)^2 - 18(2) + 15$

$\boxed{v(2) = -9 \text{ ft/s}}$

c) What is its acceleration at t = 4 seconds?

$a(4) = 6(4) - 18 = \boxed{6 \text{ ft/s}^2}$

$\boxed{\text{Avg. velocity} = \frac{\text{change in position}}{\text{change in time}} \rightarrow \frac{x(8) - x(3)}{8 - 3}}$

d) Find the average velocity of particle in [3, 8]

$x(8) = 49$

$x(3) = -16$

$\text{Avg. velocity} = \frac{49 - (-16)}{8 - 3} = \frac{65}{5}$

$= \boxed{13 \text{ ft/s}}$

e) When is the particle at rest? * set $v(t) = 0$

$v(t) = 3t^2 - 18t + 15$

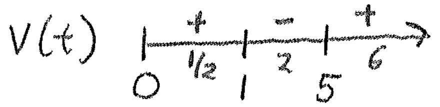
$0 = 3(t^2 - 6t + 5)$

$0 = 3(t - 5)(t - 1)$

$\boxed{t = 1, 5 \text{ seconds b/c } v(t) = 0}$

f) When is the particle moving right? When does particle change directions? (Create Sign Line) Give justification.

$v(t) = 3(t - 5)(t - 1)$



ⓐ particle moves right $[0, 1), (5, \infty)$ b/c $v(t) > 0$

ⓑ particle change directions at $t = 1, 5$ seconds b/c $v(t)$ change signs

g) What is displacement of particle from t = 2 to t = 6? Show work.

*displacement = final position − initial position

$x(6) - x(2)$

$= -25 - (-5)$

$= \boxed{-20 \text{ ft}}$

h) What is the total distance of particle from t = 2 to t = 6? Show work.

$27 + 7 = \boxed{34 \text{ ft}}$

$x(2) = -5$ > 27

$x(5) = -32$ > 7

$x(6) = -25$

i) Is the speed increasing or decreasing at t = 4? Justify.

$v(4) = -9 \text{ ft/s}$

$a(4) = 6 \text{ ft/s}^2$

speed is decreasing at $t = 4$ since $v(t)$ and $a(t)$ have opposite signs.

j) Is velocity increasing or decreasing at t = 2? Justify.

← this is talking about acceleration!

Since $a(2) = -6 \text{ ft/s}^2$

Velocity is decreasing at $t = 2$ because $a(t) < 0$

Ch. 2.4 Notes: The Chain Rule

Chain Rule: Method of computing the derivative of the composition of 2 or more functions (function within a function)

* Rule: $\frac{d}{dx}\left[f(g(x))\right] = f'(g(x)) \cdot g'(x)$

Steps:

1) Take the derivative of the outside while keeping the inside portion unchanged
2) Then multiply by the derivative of the inside function.

$\boxed{\text{Ex.1}}$ $f(x) = (3x^2+2)^5$

$$f'(x) = \underbrace{5(3x^2+2)^4}_{f'(g(x))} \cdot \underbrace{(6x)}_{g'(x)} \qquad \boxed{f'(x) = 30x(3x^2+2)^4}$$

$\boxed{\text{Ex.2}}$ Find all values of x of $f(x) = \sqrt[3]{(x^2-1)^2}$ for which $f'(x) = 0$ and where $f'(x)$ does not exist.

$$f(x) = (x^2-1)^{2/3}$$

$$f'(x) = \tfrac{2}{3}(x^2-1)^{-1/3} \cdot (2x)$$

$$f'(x) = \frac{4x}{3(x^2-1)^{1/3}}$$

To find where $f'(x) = 0$, set numerator $= 0$

$4x = 0 \rightarrow x = 0$ $\quad \underline{f'(x) = 0 \text{ at } x = 0}$

$f'(x) = \text{DNE}$, set denominator $= 0$

$3(x^2-1)^{1/3} = 0 \quad x^2-1 = 0 \quad x = \pm 1$

$\underline{f'(x) = \text{undefined at } x = 1,\ x = -1}$

~~*choose the use Rule that affects larger portion of the problem first~~

$$3(x^2-1)^{1/3} = 0$$

$$\left(\sqrt[3]{x^2-1}\right)^3 = (0)^3$$

$$x^2 - 1 = 0$$

$$x = \pm 1$$

3d

Ex. 3 $y = \frac{4}{(x+2)^2}$ find equation of tangent line to y at $x=-3$

$$y = 4(x+2)^{-2}$$
$$y' = -2 \cdot 4(x+2)^{-3}(1)$$
$$y' = \frac{-8}{(x+2)^3}$$

$$y(-3) = \frac{4}{(-3+2)^2} = 4$$
$$y'(-3) = \frac{-8}{(-3+2)^3} = 8$$

point: $(-3,4)$

slope: $m = 8$

$$y - y_1 = m(x - x_1)$$
$$\boxed{y - 4 = 8(x+3)}$$

Ex. 4 $y = \left(\frac{x-1}{x^2-4}\right)^3$ quotient and chain.

* Apply the Rule that affects the larger portion of the expression first

$f(x) = (\ )^3 \quad g(x) = \frac{x-1}{x^2-4} \quad \frac{f'g - fg'}{g^2}$

$$y' = 3\left(\frac{x-1}{x^2-4}\right)^2 \left[\frac{\overset{x^2-4-2x^2+2x}{(1)(x^2-4) - (x-1)(2x)}}{(x^2-4)^2}\right]$$

$$= \frac{3(x-1)^2(-x^2+2x-4)}{(x^2-4)^2(x^2-4)^2} = \boxed{\frac{3(x-1)^2(-x^2+2x-4)}{(x^2-4)^4}}$$

Ex. 5 $y = \frac{x}{\sqrt{x^2-1}} = \frac{x}{(x^2-1)^{1/2}}$ quotient first, then chain

$\frac{f'g - fg'}{g^2}$

$$y' = \frac{1(x^2-1)^{1/2} - x \cdot \frac{1}{2}(x^2-1)^{-1/2}(2x)}{\left[(x^2-1)^{1/2}\right]^2}$$

$$y' = \frac{\left[(x^2-1)^{1/2} - \frac{x^2}{(x^2-1)^{1/2}}\right]}{\left[\ (x^2-1)^1\ \right]} \cdot \frac{(x^2-1)^{1/2}}{(x^2-1)^{1/2}}$$

$$y' = \frac{x^2-1-x^2}{(x^2-1)^{3/2}} = \boxed{\frac{-1}{(x^2-1)^{3/2}}}$$

Ch. 2.4 Homework p. 137-139 # 7-31 odd, 59-63 od
67, 69, 97, 99

11) $f(t)=\sqrt{1-t} = (1-t)^{1/2}$ $f'(t)=\frac{1}{2}(1-t)^{-1/2}(-1)$

$= \frac{-1}{2\sqrt{1-t}}$ or $\frac{-1}{2(1-t)^{1/2}}$

13) $y = 2\sqrt[4]{4-x^2} = 2(4-x^2)^{1/4}$

$y' = \frac{1}{4}\cdot 2(4-x^2)^{-3/4}(-2x)$ $y' = \frac{-x}{(4-x^2)^{3/4}}$

17) $y = \frac{1}{x-2} = (x-2)^{-1}$ $y' = -1(x-2)^{-2}(1) = \boxed{\frac{-1}{(x-2)^2}}$

21) $y = \frac{1}{\sqrt{x+2}} = (x+2)^{-1/2}$ $y' = -\frac{1}{2}(x+2)^{-3/2}(1) = \frac{-1}{2(x+2)^{3/2}}$

23) $f(x) = \underbrace{x^2}_{f}\underbrace{(x-2)^4}_{g}$ * use product rule, chain rule

$f'g + fg'$

$$f'(x) = \overbrace{2x}^{f'}\overbrace{(x-2)^4}^{g} + \overbrace{x^2}^{f}\cdot\overbrace{4(x-2)^3(1)}^{g'} = 2x(x-2)^4 + 4x^2(x-2)^3$$

$$= 2x(x-2)^3\left[x-2+2x\right] = \boxed{2x(x-2)^3(3x-2)}$$

25) $y = x\sqrt{1-x^2} = x(1-x^2)^{1/2}$ * use product rule, chain rule

$$y' = \underbrace{1}_{f'}\underbrace{(1-x^2)^{1/2}}_{g} + \underbrace{x}_{f}\cdot\underbrace{\tfrac{1}{2}(1-x^2)^{-1/2}(-2x)}_{g'}$$

$$= \boxed{\sqrt{1-x^2} - \frac{x^2}{\sqrt{1-x^2}}}$$

30c

27) $y = \frac{x}{\sqrt{x^2+1}} = \frac{x}{(x^2+1)^{1/2}}$ * use quotient rule, chain rule

$$y' = \frac{1(x^2+1)^{1/2} - x \cdot \frac{1}{2}(x^2+1)^{-1/2}(2x)}{\left[(x^2+1)^{1/2}\right]^2} = \frac{\left(\sqrt{x^2+1} - \frac{x^2}{\sqrt{x^2+1}}\right)}{\left(x^2+1\right)} \cdot \frac{\sqrt{x^2+1}}{\sqrt{x^2+1}}$$

$$= \frac{x^2+1-x^2}{(x^2+1)(x^2+1)^{1/2}} = \boxed{\frac{1}{(x^2+1)^{3/2}}}$$

29) $g(x) = \left(\frac{x+5}{x^2+2}\right)^2$ * Chain rule, quotient rule

$$g'(x) = 2\left(\frac{x+5}{x^2+2}\right)^1 \cdot \left[\frac{1(x^2+2)-(x+5)(2x)}{(x^2+2)^2}\right] = \frac{2(x+5)(x^2+2-2x^2-10x)}{(x^2+2)(x^2+2)^2}$$

$$g'(x) = \frac{2(x+5)(-x^2-10x+2)}{(x^2+2)^3}$$

31) $f(v) = \left(\frac{1-2v}{1+v}\right)^3$ * Chain rule, quotient rule

$$f'(v) = 3\left(\frac{1-2v}{1+v}\right)^2\left[\frac{(1+v)(-2)-(1-2v)}{(1+v)^2}\right] = \boxed{\frac{-9(1-2v)^2}{(1+v)^4}}$$

59) $s(t) = \sqrt{t^2+2t+8}$ at $(2, 4)$ Evaluate derivative at given point

$s(t) = (t^2+2t+8)^{1/2}$

$s'(t) = \frac{1}{2}(t^2+2t+8)^{-1/2}(2t+2)$

$s'(2) = \frac{1}{2}(2^2+2(2)+8)^{-1/2}(4+2)$

$s'(2) = \frac{1}{2}(16)^{-1/2}(6) = \frac{1}{2}\left(\frac{1}{\sqrt{16}}\right)(6)$

$s'(2) = \left(\frac{1}{2}\right)\left(\frac{1}{4}\right)(6) = \frac{3}{4}$

61) $f(x) = \frac{3}{x^3-4}$ at $(-1, -3/5)$

$f(x) = 3(x^3-4)^{-1}$

$f'(x) = -3(x^3-4)^{-2}(3x^2) = \frac{-9x^2}{(x^3-4)^2}$

$f'(-1) = \frac{-9(-1)^2}{(-1-4)^2} = \boxed{\frac{-9}{25}}$

63) $f(t) = \frac{3t+2}{t-1}$ $(0,-2)$

$f'(t) = \frac{3(t-1)-(3t+2)(1)}{(t-1)^2} = \frac{3t-3-3t-2}{(t-1)^2} = \frac{-5}{(t-1)^2}$

$f'(0) = \frac{-5}{(0-1)^2} = \boxed{-5}$

67) $f(x) = \sqrt{3x^2-2}$ at $(3,5)$ Find equation of tangent line

$f(x) = (3x^2-2)^{1/2}$

$f'(x) = \frac{1}{2}(3x^2-2)^{-1/2}(6x)$

$f'(x) = \frac{3x}{\sqrt{3x^2-2}}$ $f'(3) = \frac{9}{\sqrt{25}} = \frac{9}{5}$

$y - y_1 = m(x - x_1)$

$\boxed{y-5 = \frac{9}{5}(x-3)}$

69) $y = (2x^3+1)^2$ at $(-1,1)$

$y' = 2(2x^3+1)^1(6x^2)$

$y' = 12x^2(2x^3+1)$

$y'(-1) = 12(-1) = -12$

$\boxed{y-1 = -12(x+1)}$

30f

97) Given: $g(5)=-3$ $h(5)=3$
$g'(5)=6$ $h'(5)=-2$

Find $f'(5)$

a) $f(x)=g(x)h(x)$ * product rule

$f'(x)=g'(x)h(x)+g(x)h'(x)$

$f'(5)=g'(5)h(5)+g(5)h'(5)$

$=(6)(3)+(-3)(-2) = \boxed{24}$

b) $f(x)=g(h(x))$ * chain rule

$f'(x)=g'(h(x))\cdot h'(x)$

Ch. 2.5 Notes Implicit Differentiation

Explicit equations: Equations where x's and y's are separated on different sides of the equation: (example: $y = 9x^2 + 4\sqrt{x} + 1$)
(solved for y)

Implicit equations: Equations where x's and y's are mixed together on same side(s) of the equation (example: $y^2 = xy - x^2$)
(not solved for y)

Explicit Differentiation

$y = 3x^2 - 9x^3 + 5$

$$\boxed{\frac{dy}{dx} = 6x - 27x^2}$$

Implicit Differentiation

$y^2 - 5x = 4$

$2y\left(\frac{dy}{dx}\right) - 5 = 0$

$$\boxed{\frac{dy}{dx} = \frac{5}{2y}}$$

Steps:

1) Take derivative of each term with respect to x
2) If variable is y, find derivative and attach $\frac{dy}{dx}$ to the derivative
3) Move all terms containing $\frac{dy}{dx}$ to left side of equation.
4) Move all other terms to right side of equation.
5) Factor out $\frac{dy}{dx}$ on left side of equation
6) Solve for $\frac{dy}{dx}$

Ex.1 $x^2 - 2y^3 + 4y = 2$ Find $\frac{dy}{dx}$

$2x - 6y^2\left(\frac{dy}{dx}\right) + 4\left(\frac{dy}{dx}\right) = 0$

$-6y^2\frac{dy}{dx} + 4\frac{dy}{dx} = -2x$

$\frac{dy}{dx}(4 - 6y^2) = -2x$

$$\boxed{\frac{dy}{dx} = \frac{-2x}{4 - 6y^2}}$$

Ex. 2 $3xy^3 - 2y = 7$ Find $\frac{dy}{dx}$ or y'

*product rule
$f'g + fg'$

$\underbrace{3x}_{f}\underbrace{y^3}_{g} - 2y = 7$

$\overbrace{(3)}^{f'}\overbrace{(y^3)}^{g} + \overbrace{(3x)}^{f}\overbrace{(3y^2)\frac{dy}{dx}}^{g'} - 2\left(\frac{dy}{dx}\right) = 0$

$9xy^2\frac{dy}{dx} - 2\frac{dy}{dx} = -3y^3$

$\frac{dy}{dx}(9xy^2 - 2) = -3y^3$

$$\boxed{\frac{dy}{dx} = \frac{-3y^3}{9xy^2 - 2}}$$

Ex. 3 Differentiate $y^2 = 5x$ with respect to t

$2y\left(\frac{dy}{dt}\right) = 5\left(\frac{dy}{dt}\right)$

Ch. 2.5 Implicit Differentiation Worksheet #1

Key

33

Finding a Derivative In Exercises 1–16, find dy/dx by implicit differentiation.

Product Rule:

$$\frac{d}{dx}[f(x)g(x)] = f'g + fg'$$

1. $x^2 + y^2 = 9$

$$2x + 2y\left(\frac{dy}{dx}\right) = 0$$

$$2y\left(\frac{dy}{dx}\right) = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{2y}$$

$$\boxed{\frac{dy}{dx} = \frac{-x}{y}}$$

2. $x^2 - y^2 = 25$

$$2x - 2y\left(\frac{dy}{dx}\right) = 0$$

$$-2y\left(\frac{dy}{dx}\right) = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{-2y}$$

$$\boxed{\frac{dy}{dx} = \frac{x}{y}}$$

4. $2x^3 + 3y^3 = 64$

$$6x^2 + 9y^2\left(\frac{dy}{dx}\right) = 0$$

$$9y^2\left(\frac{dy}{dx}\right) = -6x^2$$

$$\frac{dy}{dx} = \frac{-6x^2}{9y^2}$$

$$\boxed{\frac{dy}{dx} = \frac{-2x^2}{3y^2}}$$

5. $x^3 - xy + y^2 = 7$ *product rule

$$x^3 - \overset{f}{x}\overset{g}{y} + y^2 = 7$$

$$3x^2 - \left(\overset{f'}{(1)}\overset{g}{(y)} + \overset{f}{(x)}\overset{g'}{\left(\frac{dy}{dx}\right)}\right) + 2y\left(\frac{dy}{dx}\right) = 0$$

$$3x^2 - y - x\left(\frac{dy}{dx}\right) + 2y\left(\frac{dy}{dx}\right) = 0$$

$$-x\left(\frac{dy}{dx}\right) + 2y\left(\frac{dy}{dx}\right) = y - 3x^2$$

$$\frac{dy}{dx}\left(-x + 2y\right) = y - 3x^2$$

$$\boxed{\frac{dy}{dx} = \frac{y - 3x^2}{-x + 2y}}$$

6. $x^2y + y^2x = -2$

$$\overset{f}{x^2}\overset{g}{y} + \overset{f}{y^2}\cdot\overset{g}{x} = -2$$

$$\overset{f'}{2x}\cdot\overset{g}{y} + \overset{f}{x^2}\cdot\overset{g'}{\frac{dy}{dx}} + \overset{f'}{2y\left(\frac{dy}{dx}\right)}\cdot\overset{g}{x} + \overset{f}{y^2}\cdot\overset{g'}{(1)} = 0$$

$$2xy + x^2\left(\frac{dy}{dx}\right) + 2xy\left(\frac{dy}{dx}\right) + y^2 = 0$$

$$x^2\left(\frac{dy}{dx}\right) + 2xy\left(\frac{dy}{dx}\right) = -2xy - y^2$$

$$\frac{dy}{dx}\left(x^2 + 2xy\right) = -2xy - y^2$$

$$\boxed{\frac{dy}{dx} = \frac{-2xy - y^2}{x^2 + 2xy}}$$

7. $x^3y^3 - y = x$

$$\overset{f}{x^3}\overset{g}{y^3} - y = x$$

$$\overset{f'}{3x^2}\cdot\overset{g}{y^3} + \overset{f}{x^3}\cdot\overset{g'}{3y^2\left(\frac{dy}{dx}\right)} - 1\left(\frac{dy}{dx}\right) = 1$$

$$3x^2y^3 + 3x^3y^2\left(\frac{dy}{dx}\right) - 1\left(\frac{dy}{dx}\right) = 1$$

$$3x^3y^2\left(\frac{dy}{dx}\right) - 1\left(\frac{dy}{dx}\right) = 1 - 3x^2y^3$$

$$\frac{dy}{dx}\left(3x^3y^2 - 1\right) = 1 - 3x^2y^3$$

$$\boxed{\frac{dy}{dx} = \frac{1 - 3x^2y^3}{3x^3y^2 - 1}}$$

Finding and Evaluating a Derivative In Exercises 21–28, find dy/dx by implicit differentiation and evaluate the derivative at the given point.

21. $xy = 6, \quad (-6, -1)$

$\overset{f}{x}\overset{g}{y} = 6$

$\overset{f'}{(1)}\overset{g}{(y)} + \overset{f}{(x)}\overset{g'}{\left(\frac{dy}{dx}\right)} = 0$

$x\left(\frac{dy}{dx}\right) = -y$

$\frac{dy}{dx} = \frac{-y}{x}$

$\left.\frac{dy}{dx}\right|_{(-6,-1)} = \frac{-(-1)}{(-6)}$

$\boxed{\left.\frac{dy}{dx}\right|_{(-6,-1)} = \frac{-1}{6}}$

22. $y^3 - x^2 = 4, \quad (2, 2)$

$3y^2\left(\frac{dy}{dx}\right) - 2x = 0$

$3y^2\left(\frac{dy}{dx}\right) = 2x$

$\frac{dy}{dx} = \frac{2x}{3y^2}$

$\left.\frac{dy}{dx}\right|_{(2,2)} = \frac{2(2)}{3(2)^2} = \boxed{\frac{1}{3}}$

24. $x^{2/3} + y^{2/3} = 5, \quad (8, 1)$

$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\left(\frac{dy}{dx}\right) = 0$

$\frac{2}{3x^{1/3}} + \frac{2}{3y^{1/3}}\left(\frac{dy}{dx}\right) = 0$

$\frac{2}{3y^{1/3}}\left(\frac{dy}{dx}\right) = \frac{-2}{3x^{1/3}}$

$\frac{dy}{dx} = \frac{-2}{3x^{1/3}} \cdot \frac{3y^{1/3}}{2}$

$\frac{dy}{dx} = \frac{-y^{1/3}}{x^{1/3}}$

$\left.\frac{dy}{dx}\right|_{(8,1)} = \frac{-(1)^{1/3}}{(8)^{1/3}}$

$\left.\frac{dy}{dx}\right|_{(8,1)} = \boxed{\frac{-1}{2}}$

25)

$(x^2 + 4)y = 8$

Point: (2, 1)

$\overset{f}{(x^2+4)}\overset{g}{(y)} = 8$

$\overset{f'}{2x} \cdot \overset{g}{y} + \overset{f}{(x^2+4)}\overset{g'}{\left(\frac{dy}{dx}\right)} = 0$

$2xy + (x^2+4)\left(\frac{dy}{dx}\right) = 0$

$(x^2+4)\left(\frac{dy}{dx}\right) = -2xy$

$\frac{dy}{dx} = \frac{-2xy}{x^2+4}$

$\left.\frac{dy}{dx}\right|_{(2,1)} = \frac{-2xy}{x^2+4}$

$\left.\frac{dy}{dx}\right|_{(2,1)} = \frac{-2(2)(1)}{2^2+4}$

$\left.\frac{dy}{dx}\right|_{(2,1)} = \frac{-4}{8}$

$= \boxed{\frac{-1}{2}}$

Find Equation of tangent line:

51. $\sqrt{x} + \sqrt{y} = 5, \quad (9, 4)$

$x^{1/2} + y^{1/2} = 5$

$\frac{1}{2}x^{-1/2} + \frac{1}{2}y^{-1/2}\left(\frac{dy}{dx}\right) = 0$

$\frac{1}{2x^{1/2}} + \frac{1}{2y^{1/2}}\left(\frac{dy}{dx}\right) = 0$

$\frac{1}{2y^{1/2}}\left(\frac{dy}{dx}\right) = \frac{-1}{2x^{1/2}}$

$\frac{dy}{dx} = \frac{-1}{2x^{1/2}} \cdot \frac{2y^{1/2}}{1}$

$\frac{dy}{dx} = \frac{-2y^{1/2}}{2x^{1/2}}$

$\frac{dy}{dx} = \frac{-y^{1/2}}{x^{1/2}}$

$\left.\frac{dy}{dx}\right|_{(9,4)} = \frac{-(4)^{1/2}}{(9)^{1/2}} = \frac{-2}{3}$

point: (9,4)

slope: $m = \frac{-2}{3}$

$y - y_1 = m(x - x_1)$

$\boxed{y - 4 = \frac{-2}{3}(x - 9)}$

37

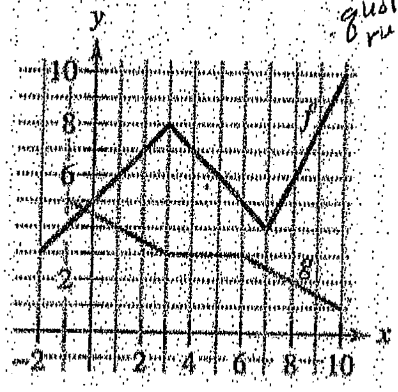
Key

Ch. 2.3 **Product,Quotient Rule** HW Problems Evaluating Derivatives using graphs

Evaluating Derivatives In Exercises 81 and 82, use the graphs of f and g. Let $p(x) = f(x)g(x)$ and $q(x) = f(x)/g(x)$. ↙ *product rule

81. (a) Find $p'(1)$.

(b) Find $q'(4)$.



quotient rule

a) $p'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$

$p'(1) = f'(1) \cdot g(1) + f(1) \cdot g'(1)$

$= (1) \cdot (4) + (6) \cdot (-1/2) = 4 - 3 = \boxed{1}$

b) $$q'(x) = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g(x)^2}$$

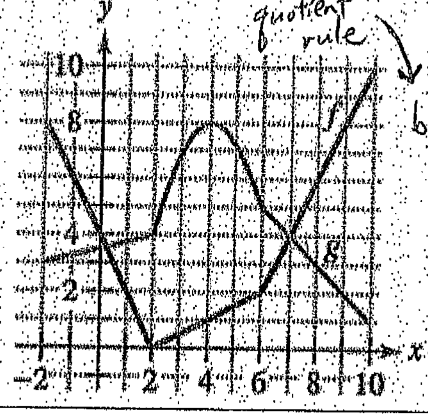
$$q'(4) = \frac{f'(4)g(4) - f(4)g'(4)}{[g(4)]^2}$$

$$q'(4) = \frac{(-1)(3) - (7)(0)}{3^2}$$

$$q'(4) = \frac{-3}{3^2} = \boxed{-\frac{1}{3}}$$

82. (a) Find $p'(4)$.

(b) Find $q'(7)$.



quotient rule

product Rule → a) $p'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$

$p'(4) = f'(4) \cdot g(4) + f(4) \cdot g'(4)$

$= \left(\frac{1}{2}\right) \cdot (8) + (1)(0) = 4$

$\boxed{p'(4) = 4}$

b) $$q'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

$$q'(7) = \frac{f'(7)g(7) - f(7)g'(7)}{g(7)^2}$$

$$q'(7) = \frac{(2)(4) - 4(-1)}{4^2} = \frac{8+4}{16} = \frac{12}{16} = \frac{3}{4} \qquad \boxed{q'(7) = 3/4}$$

Using Relationships In Exercises 103–106, use the given information to find $f'(2)$.

$g(2) = 3$ and $g'(2) = -2$

$h(2) = -1$ and $h'(2) = 4$

103. $f(x) = 2g(x) + h(x)$

$f'(x) = 2g'(x) + h'(x)$

$f'(2) = 2g'(2) + h'(2)$

$= 2(-2) + 4 = 0$

$\boxed{f'(2) = 0}$

Apply quotient rule

105. $f(x) = \dfrac{g(x)}{h(x)}$

$$f'(x) = \frac{g'(x)h(x) - g(x)h'(x)}{h(x)^2}$$

$$f'(2) = \frac{g'(2)h(2) - g(2)h'(2)}{h(2)^2}$$

$$f'(2) = \frac{(-2)(-1) - 3(4)}{(-1)^2}$$

$$f'(2) = \frac{2 - 12}{1} = -10$$

$\boxed{f'(2) = -10}$

Ch.2.4 p.139

In Exercises 99 the graphs of f and g are shown. Let $h(x) = f(g(x))$ and $s(x) = g(f(x))$. Find each derivative, if it exists. If the derivative does not exist, explain why.

$h(x) = f(g(x))$

$s(x) = g(f(x))$

99. (a) Find $h'(1)$.

(b) Find $s'(5)$.

ⓐ $h'(x) = f'[g(x)] \cdot g'(x)$

$h'(1) = f'[g(1)] \cdot g'(1)$

$h'(1) = f'[4] \cdot (-\frac{1}{2})$

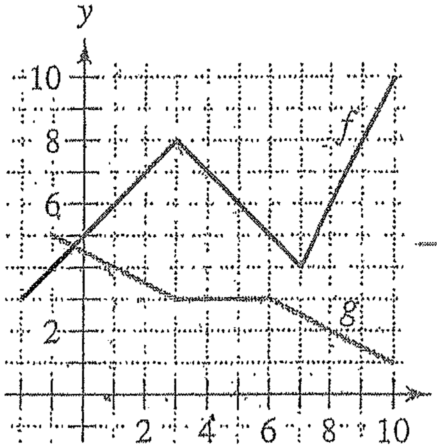
$= (-1)(-1/2)$

$\boxed{h'(1) = \frac{1}{2}}$

$g(1) = 4$

$g'(1) = -\frac{1}{2}$

$f'(4) = -1$



$s'(x) = g'(f(x)) \cdot f'(x)$

$s'(5) = g'(f(5)) \cdot f'(5)$

$s'(5) = g'(6) \cdot (-1)$

$\boxed{s'(5) = \text{DNE}}$

$f(5) = 6$

$f'(5) = -1$

$g'(6) = \text{DNE}$

$h(x) = f(x)g(x)$

$s(x) = \frac{f(x)}{g(x)}$

100. (a) Find $h'(3)$.

(b) Find $s'(9)$.

$h'(x) = f'(x)g(x) + f(x)g'(x)$

$h'(3) = f'(3)g(3) + f(3)g'(3)$

$h'(3) = (\frac{1}{2})(5) + (\frac{1}{2})(1)$

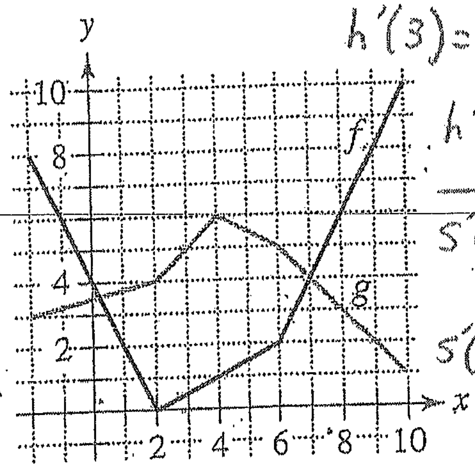
$h'(3) = \frac{5}{2} + \frac{1}{2} = \frac{6}{2} = \boxed{3}$

$f(3) = 1/2$

$f'(3) = 1/2$

$g(3) = 5$

$g'(3) = 1$



$$s'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

$$s'(9) = \frac{f'(9)g(9) - f(9)g'(9)}{[g(9)]^2}$$

$f(9) = 8$

$f'(9) = 2$

$g(9) = 2$

$g'(9) = -1$

$$s'(9) = \frac{(2)(2) - (8)(-1)}{(2)^2} = \frac{4+8}{4} = \frac{12}{4} = 3 \quad \boxed{s'(9) = 3}$$

③ Key

Ch. 2.4 Chain Rule HW Problems #102, #115

102. Using Relationships Given that $g(5) = -3$, $g'(5) = 6$, $h(5) = 3$, and $h'(5) = -2$, find $f'(5)$ for each of the following, if possible. If it is not possible, state what additional information is required.

Recall: product rule : $\frac{d}{dx} f(x)g(x) = f'(x)(g(x)) + f(x)g'(x)$

quotient rule : $\frac{d}{dx} \frac{f(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$

Chain rule : $\frac{d}{dx} f[g(x)] = f'(g(x)) \cdot g'(x)$

(a) $f(x) = g(x)h(x)$ *product rule

$f'(x) = g'(x)h(x) + g(x)h'(x)$

$f'(5) = g'(5)h(5) + g(5)h'(5)$

$f'(5) = 6(3) + (-3)(-2)$

$f'(5) = 18 + 6 = 24$ $\boxed{f'(5) = 24}$

(b) $f(x) = g(h(x))$ ← *chain rule

$f'(x) = g'[h(x)] \cdot h'(x)$

$f'(5) = g'[h(5)] \cdot h'(5)$

$= g'(3) \cdot h'(5)$

$= g'(3) \cdot -2$

$\boxed{f'(5) = -2g'(3)}$

(c) $f(x) = \frac{g(x)}{h(x)}$ *quotient rule

$f'(x) = \frac{g'(x)h(x) - g(x)h'(x)}{h(x)^2}$

$f'(5) = \frac{g'(5)h(5) - g(5)h'(5)}{h(5)^2}$

$= \frac{6(3) - (-3)(-2)}{3^2}$

$= \frac{18 - 6}{9} = \frac{12}{9} = 4/3$

$\boxed{f'(5) = 4/3}$

(d) $f(x) = [g(x)]^3$ ← *chain rule

$f'(x) = 3[g(x)]^2 \cdot g'(x)$

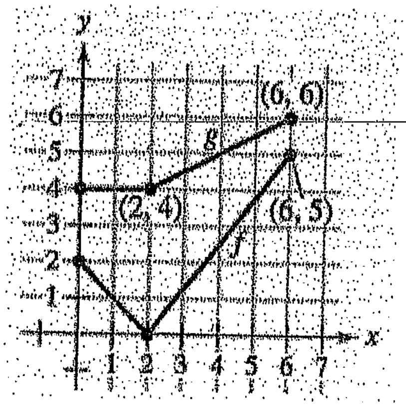
$f'(5) = 3[g(5)]^2 \cdot g'(5)$

$= 3[-3]^2 \cdot 6$

$= 3(9)(6) = 162$

$\boxed{f'(5) = 162}$

115. Think About It Let $r(x) = f(g(x))$ and $s(x) = g(f(x))$, where f and g are shown in the figure. Find (a) $r'(1)$ and (b) $s'(4)$. ← Apply chain rule



a) $r'(x) = f'[g(x)] \cdot g'(x)$

$r'(1) = f'[g(1)] \cdot g'(1)$

$r'(1) = f'[4] \cdot 0$

$r'(1) = \frac{5}{4}(0) = \boxed{0}$

$\boxed{r'(1) = 0}$

b) $s'(x) = g'[f(x)] \cdot f'(x)$

$s'(4) = g'[f(4)] \cdot f'(4)$

$= g'[5/2] \cdot (5/4)$

$= (\frac{1}{2})(5/4) = 5/8$

$\boxed{s'(4) = 5/8}$

㊵ Key

Ch.2.5 **Implicit Differentiation** Vertical, Horizontal Tangent Lines HW Problems #57, #58

Vertical and Horizontal Tangent Lines In Exercises 57 and 58, find the points at which the graph of the equation has a vertical or horizontal tangent line. $\left(dy/dx\right)$

*Find **Horizontal Tangent** lines by setting **numerator of derivative** equal to zero, solve for x

*Find **Vertical Tangent** lines by setting **denominator of derivative** equal to zero, solve for x

57. $25x^2 + 16y^2 + 200x - 160y + 400 = 0$ ← Apply implicit differentiation

$50x + 32y\left(\frac{dy}{dx}\right) + 200 - 160\left(\frac{dy}{dx}\right) + 0 = 0$

$32y\left(\frac{dy}{dx}\right) - 160\left(\frac{dy}{dx}\right) = -50x - 200$

$\frac{dy}{dx}\left[32y - 160\right] = -50x - 200$

$\frac{dy}{dx} = \frac{-50x - 200}{32y - 160}$

horiz. tangent: $-50x - 200 = 0$

$-50x = +200$

plug into equation ← $\boxed{x = -4}$

$25(-4)^2 + 16y^2 + 200(-4) - 160y + 400 = 0$

$400 + 16y^2 - 800 - 160y + 400 = 0$

$16y^2 - 160y = 0$ $16y(y - 10) = 0$ $y = 0, 10$

$\boxed{\text{Horiz. tangents: } (-4,0) \text{ and } (-4,10)}$

vertical tangent: $32y - 160 = 0$ $32y = 160$ $\underline{y = 5}$

$25x^2 + 16(5)^2 + 200x - 160(5) + 400 = 0$

$25x^2 + \underline{400} + 200x - \underline{800} + \underline{400} = 0$

$25x^2 + 200x = 0$

$25x(x + 8) = 0$

$x = 0, -8$

$\boxed{\text{vertical tangents: } (0,5) \text{ and } (-8,5)}$

58. $4x^2 + y^2 - 8x + 4y + 4 = 0$

$8x + 2y\left(\frac{dy}{dx}\right) - 8 + 4\left(\frac{dy}{dx}\right) + 0 = 0$

$2y\left(dy/dx\right) + 4\left(\frac{dy}{dx}\right) = 8 - 8x$

$\frac{dy}{dx}(2y + 4) = 8 - 8x$

$\boxed{\frac{dy}{dx} = \frac{8 - 8x}{2y + 4}}$

horizontal tangent: $8 - 8x = 0$ $\underline{\underline{x = 1}}$

$\underline{4(1)^2} + y^2 - \underline{8(1)} + 4y + \underline{4} = 0$

$y^2 + 4y = 0$

$y(y + 4) = 0$

$y = 0, -4$

$\boxed{\text{horizontal tangents: } (1,0) \text{ and } (1,-4)}$

vertical tangents: $2y + 4 = 0$

$2y = -4$ $\underline{\underline{y = -2}}$

$4x^2 + (-2)^2 - 8x + 4(-2) + 4 = 0$

$4x^2 + \underline{4} - 8x - \underline{8} + \underline{4} = 0$ $4x^2 - 8x = 0$

$4x(x - 2) = 0$

$x = 0, 2$

$\boxed{\text{vertical tangents } (0,-2) \text{ and } (2,-2)}$