Derivatives and Limits FRQ Day Problems

1. Let $f(x) = \begin{cases} x^2 & x \le 1 \\ 2x & x > 1 \end{cases}$

a) Find f'(x) for x < 1

b) Find f'(x) for x > 1

c) Find $\lim_{x \to 1^-} f'(x)$:

d) Find $\lim_{x \to 1^+} f'(x)$:

e) Does f'(1) exist? Explain

2. Let f be the function given by $f(x) = \frac{ax+b}{x^2-c}$ and that has the following properties

   i. The graph of f is symmetric to the y-axis

   ii. $\lim_{x \to 2^+} f(x) = \infty$

   iii. f'(1) = -2

a) Determine the values of a, b, and c

b) Write an equation for each vertical and horizontal asymptote of the graph of f.

c) Sketch the graph of f.

3. Consider the equation $x^2 - 2xy + 4y^2 = 64$

   a. Write an expression for the slope of the curve at any point(x,y)

   b. Find the equation of the tangent lines to the curve at the point $x = 2$

   c. Find $\frac{d^2y}{dx^2}$ at (0, 4)

4.

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Let f be a function defined by $f(x) = \begin{cases} 2x+1, & \text{for } x \leq 2 \\ \frac{1}{2}x^2 + k, & \text{for } x > 2 \end{cases}$.

a. For what values of k will f be continuous at $x = 2$? Justify your answer.

b. Using the value of k found in part a, determine whether f is differentiable at $x = 2$. Use the definition of the derivative to justify your answer.

c. Let $k = 4$. Determine whether f is differentiable at $x = 4$. Justify your answer.

Solution

Derivatives and Limits FRQ Day Problems

$f'(x) = \begin{cases} 2x, & x < 1 \\ 2, & x > 1 \end{cases}$

1. Let $f(x) = \begin{cases} x^2 & x \leq 1 \\ 2x & x > 1 \end{cases}$

a) Find f'(x) for x < 1

$f'(x) = 2x$

b) Find f'(x) for x > 1

$f'(x) = 2$

c) Find $\lim_{x \to 1^-} f'(x)$:

$\lim_{x \to 1^-} f'(x) = 2(1) = 2$

d) Find $\lim_{x \to 1^+} f'(x)$:

$\lim_{x \to 1^+} f'(x) = 2$

e) Does f'(1) exist? Explain

f(x) is differentiable at $x = 1$ since $\lim_{x \to 1^-} f'(x) = \lim_{x \to 1^+} f'(x) = 2$

2. Let f be the function given by $f(x) = \frac{ax+b}{x^2-c}$ and that has the following properties

i. The graph of f is symmetric to the y-axis   even function: $f(-x) = f(x)$

ii. $\lim_{x \to 2^+} f(x) = \infty$ Vertical Asymptote at $x = 2$

iii. $f'(1) = -2$ slope at $x = 1$ is -2

a) Determine the values of a, b, and c

b) Write an equation for each vertical and horizontal asymptote of the graph of f.

c) Sketch the graph of f.

a) * Each variable must have an even exponent $(x^0, x^2, x^4, \dots)$, $f(x) = \frac{0x + b}{x^2 - c}$

$\boxed{a = 0}$ $f(x) = \frac{b}{x^2 - c}$

b) vertical asymptote at $x = 2, x = -2$

$x^2 - c = 0$

$2^2 - c = 0$

$\underline{C = 4}$

$f(x) = \frac{b}{x^2 - 4}$

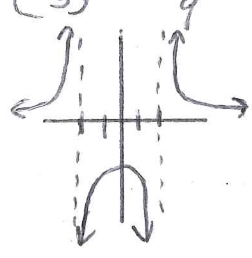
c) $f'(x) = \frac{0(x^2-4) - b(2x)}{(x^2-4)^2} = \frac{-2xb}{(x^2-4)^2}$

$f'(1) = \frac{-2xb}{(x^2-4)^2} = \frac{-2(1)b}{(1^2-4)^2} = \frac{-2b}{(-3)^2} = \frac{-2b}{9}$

$-2 = -\frac{2b}{9}$

$-18 = -2b$ $\underline{b = 9}$

$y = \frac{9}{x^2-4}$



3. Consider the equation $x^2 - 2xy + 4y^2 = 64$

   a. Write an expression for the slope of the curve at any point(x,y)

   b. Find the equation of the tangent lines to the curve at the point x = 2

   c. ~~Find $\frac{d^2y}{dx^2}$ at (0, 4)~~

$2x - \left(2y + 2x\frac{dy}{dx}\right) + 8y\left(\frac{dy}{dx}\right) = 0$

$2x - 2y - 2x\frac{dy}{dx} + 8y\frac{dy}{dx} = 0$

$-2x\frac{dy}{dx} + 8y\frac{dy}{dx} = 2y - 2x$

$\frac{dy}{dx}(-2x + 8y) = 2y - 2x$

$\frac{dy}{dx} = \frac{2y - 2x}{-2x + 8y} = \frac{y - x}{-x + 4y}$

b) * find y-value $2^2 - 2(2)y + 4y^2 = 64$

$4 - 4y + 4y^2 = 64 \quad 4y^2 - 4y - 60 = 0$

$4(y^2 - y - 15) = 0 \quad \frac{1 \pm \sqrt{1^2 - 4(1)(-15)}}{2(1)} = \frac{1 \pm \sqrt{61}}{2}$

$y = 4.41,\ y = -3.41$

points: $(2, 4.41)$ and $(2, -3.41)$

$\frac{dy}{dx}_{(2,4.41)} = \frac{4.41 - 2}{4(4.41) - 2} = 0.15$

point: $(2, 4.41)$

slope: $m = 0.15$

$\boxed{y - 4.41 = 0.15(x - 2)}$

$\frac{dy}{dx}_{(2,-3.41)} = \frac{-3.41 - 2}{-2 + 4(-3.41)}$

$m = 0.35$

point: $(2, -3.41)$

slope: $m = 0.35$

$\boxed{y + 3.41 = 0.35(x - 2)}$

4.

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Let f be a function defined by $f(x) = \begin{cases} 2x + 1, & \text{for } x \le 2 \\ \frac{1}{2}x^2 + k, & \text{for } x > 2 \end{cases}$.

a. For what values of k will f be continuous at x = 2? Justify your answer.

b. Using the value of k found in part a, determine whether f is differentiable at x = 2. ~~Use the definition of the derivative to justify your answer.~~

c. Let k = 4. Determine whether f is differentiable at x = 4. Justify your answer.

a) $\overset{x=2}{2x + 1} = \frac{1}{2}x^2 + k$

$2(2) + 1 = \frac{1}{2}(2)^2 + k$

$4 + 1 = 2 + k$

$5 - 2 = k$

$k = 3$

$\lim_{x \to 2^-} 2x + 1 = \lim_{x \to 2^+} \frac{1}{2}x^2 + \overset{k}{3} = 5$

and $f(2) = 5$

b) Since $f'(x) = \begin{cases} 2 & \text{for } x \le 2 \\ x & \text{for } x > 2 \end{cases}$ and

$\lim_{x \to 2^-} f'(x) = \lim_{x \to 2^+} f'(x) = 2, \quad f'(2) = 2$

c) when $k = 4$, $f(x) = \begin{cases} 2x + 1 & \text{for } x \le 2 \\ \frac{1}{2}x^2 + 4 & \text{for } x > 2 \end{cases}$

$\lim_{x \to 2^-} 2x + 1 = 5$

$\lim_{x \to 2^+} \frac{1}{2}x^2 + 4 = 6$

Since $\lim_{x \to 2} f(x) = \text{DNE}$, f(x) is not continuous, and therefore not differentiable.