Chapter Review

THINGS TO KNOW

2.1 Rates of Change and the Derivative

- **Definition** (Form 1) Derivative of a function f at a number c

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

provided the limit exists. (p. 167)

Three Interpretations of the Derivative

- *Geometric* If $y = f(x)$, the derivative $f'(c)$ is the slope of the tangent line to the graph of f at the point $(c, f(c))$. (p. 167)
- *Rate of change of a function* If $y = f(x)$, the derivative $f'(c)$ is the rate of change of f with respect to x at c. (p. 167)
- *Physical* If the signed distance s from the origin at time t of an object in rectilinear motion is given by the position function $s = f(t)$, the derivative $f'(t_0)$ is the velocity of the object at time t_0. (p. 167)

2.2 The Derivative as a Function

- **Definition of a derivative function** (Form 2)

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

provided the limit exists. (p. 172)

- **Theorem** If a function f has a derivative at a number c, then f is continuous at c. (p. 177)
- **Corollary** If a function f is discontinuous at a number c, then f has no derivative at c. (p. 177)

2.3 The Derivative of a Polynomial Function; The Derivative of $y = e^x$

- **Leibniz notation** $\dfrac{dy}{dx} = \dfrac{d}{dx}y = \dfrac{d}{dx}f(x)$ (p. 183)
- **Basic derivatives**

$\dfrac{d}{dx}A = 0$ A is a constant (p. 184) $\qquad \dfrac{d}{dx}x = 1$ (p. 184)

$\dfrac{d}{dx}e^x = e^x$ (p. 190) $\qquad \dfrac{d}{dx}\ln x = \dfrac{1}{x}$ (p. 190)

- *Simple Power Rule* (p. 185) $\dfrac{d}{dx}x^n = nx^{n-1}$, $n \geq 1$, an integer

Properties of Derivatives

- *Sum Rule* (pp. 186, 187) $\dfrac{d}{dx}[f+g] = \dfrac{d}{dx}f + \dfrac{d}{dx}g$

$$(f+g)' = f' + g'$$

- *Difference Rule* (p. 187) $\dfrac{d}{dx}[f-g] = \dfrac{d}{dx}f - \dfrac{d}{dx}g$

$$(f-g)' = f' - g'$$

- *Constant Multiple Rule* (p. 186) If k is a constant,

$$\frac{d}{dx}[kf] = k\frac{d}{dx}f$$

$$(kf)' = k \cdot f'$$

2.4 Differentiating the Product and the Quotient of Two Functions; Higher-Order Derivatives

Properties of Derivatives

- *Product Rule* (p. 195) $\dfrac{d}{dx}(fg) = f\left(\dfrac{d}{dx}g\right) + \left(\dfrac{d}{dx}f\right)g$

$$(fg)' = fg' + f'g$$

- *Quotient Rule* (p. 196) $\dfrac{d}{dx}\left(\dfrac{f}{g}\right) = \dfrac{\left(\dfrac{d}{dx}f\right)g - f\left(\dfrac{d}{dx}g\right)}{g^2}$

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

provided $g(x) \neq 0$

- *Reciprocal Rule* (p. 197) $\dfrac{d}{dx}\left(\dfrac{1}{g}\right) = -\dfrac{\dfrac{d}{dx}g}{g^2}$

$$\left(\frac{1}{g}\right)' = -\frac{g'}{g^2}$$

provided $g(x) \neq 0$

- *Power Rule* $\dfrac{d}{dx}x^n = nx^{n-1}$, n an integer (p. 198)
- *Higher-order derivatives* See Table 3 (p. 199)
- *Position Function* $s = s(t)$ (p. 200)
- *Velocity* $v = v(t) = \dfrac{ds}{dt}$ (p. 200)
- *Acceleration* $a = a(t) = \dfrac{dv}{dt} = \dfrac{d^2s}{dt^2}$ (p. 200)

2.5 The Derivative of the Trigonometric Functions

Basic Derivatives

$\dfrac{d}{dx}\sin x = \cos x$ (p. 207) $\qquad \dfrac{d}{dx}\sec x = \sec x \tan x$ (p. 210)

$\dfrac{d}{dx}\cos x = -\sin x$ (p. 208) $\qquad \dfrac{d}{dx}\csc x = -\csc x \cot x$ (p. 210)

$\dfrac{d}{dx}\tan x = \sec^2 x$ (p. 210) $\qquad \dfrac{d}{dx}\cot x = -\csc^2 x$ (p. 210)

OBJECTIVES

Section	You should be able to ...	Examples	Review Exercises	Preparing for the AP® Exam: AP® Review Problems
2.1	1 Find equations for the tangent line and the normal line to the graph of a function (p. 162)	1	67–70	
	2 Find the rate of change of a function (p. 163)	2, 3	1, 2, 73 (a)	
	3 Find average velocity and instantaneous velocity (p. 164)	4, 5	71(a), (b); 72(a), (b)	
	4 Find the derivative of a function at a number (p. 166)	6–8	3–8, 75	5
2.2	1 Define the derivative function (p. 171)	1–3	9–12, 77	2
	2 Graph the derivative function (p. 173)	4, 5	9–12, 15–18	
	3 Identify where a function is not differentiable (p. 175)	6–10	13, 14, 75	4
2.3	1 Differentiate a constant function (p. 184)	1		
	2 Differentiate a power function (p. 184)	2, 3	19–22	
	3 Differentiate the sum and the difference of two functions (p. 186)	4–6	23–26, 33, 34, 40, 51, 52, 67	
	4 Differentiate the exponential function $y=e^x$ (p. 189)	7	44, 45, 53, 54, 56, 59, 69	7
2.4	1 Differentiate the product of two functions (p. 194)	1, 2	27, 28, 36, 46, 48–50, 53–56, 60	
	2 Differentiate the quotient of two functions (p. 196)	3–6	29–35, 37–43, 47, 57–59, 68, 73, 74	3, 10
	3 Find higher-order derivatives (p. 198)	7, 8	61–66, 71, 72, 76	8
	4 Find the acceleration of an object in rectilinear motion (p. 200)	9	71, 72, 76	
2.5	1 Differentiate trigonometric functions (p. 207)	1–6	49–60, 70	1, 6, 9

REVIEW EXERCISES

In Problems 1 and 2, use a definition of the derivative to find the rate of change of f at the indicated numbers.

1. $f(x)=\sqrt{x}$ at **(a)** $c=1$ **(b)** $c=4$ **(c)** c any positive real number

2. $f(x)=\dfrac{2}{x-1}$ at **(a)** $c=0$ **(b)** $c=2$ **(c)** c any real number, $c\neq 1$

In Problems 3–8, use a definition of the derivative to find the derivative of each function at the given number.

3. $F(x)=2x+5$ at 2 **4.** $f(x)=4x^2+1$ at -1

5. $f(x)=3x^2+5x$ at 0 **6.** $f(x)=\dfrac{3}{x}$ at 1

7. $f(x)=\sqrt{4x+1}$ at 0 **8.** $f(x)=\dfrac{x+1}{2x-3}$ at 1

In Problems 9–12, use a definition of the derivative to find the derivative of each function. Graph f and f′ on the same set of axes.

9. $f(x)=x-6$ **10.** $f(x)=7-3x^2$

11. $f(x)=\dfrac{1}{2x^3}$ **12.** $f(x)=\pi$

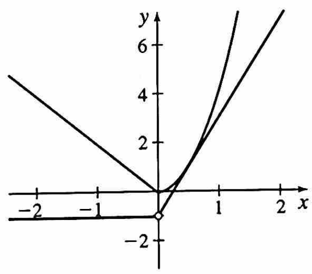
In Problems 13 and 14, determine whether the function f has a derivative at c. If it does, find the derivative. If it does not, explain why. Graph each function.

13. $f(x)=|x^3-1|$ at $c=1$

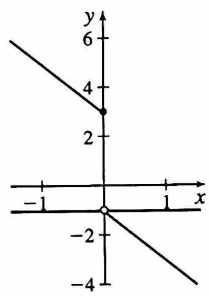
14. $f(x)=\begin{cases}4-3x^2 & \text{if } x\le -1\\ -x^3 & \text{if } x>-1\end{cases}$ at $c=-1$

In Problems 15 and 16, determine whether the graphs represent a function f and its derivative f′. If they do, indicate which is the graph of f and which is the graph of f′.

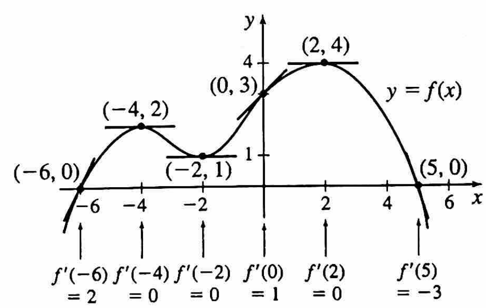
15.



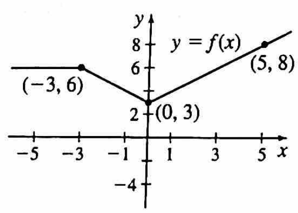
16.

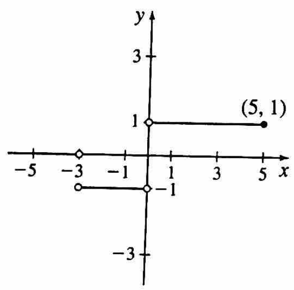


17. Use the information in the graph of $y=f(x)$ to sketch the graph of $y=f'(x)$.

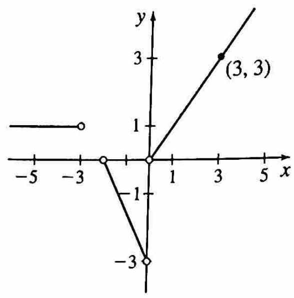


18. Match the graph of $y=f(x)$ with the graph of its derivative.





(A)



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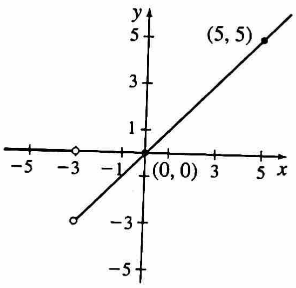

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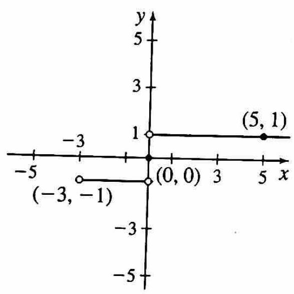


(D)

In Problems 19–60, find the derivative of each function. Treat a and b, if present, as constants.

19. $f(x)=x^5$

20. $f(x)=ax^3$

21. $f(x)=\dfrac{x^4}{4}$

22. $f(x)=-6x^2$

23. $f(x)=2x^2-3x$

24. $f(x)=3x^3+\dfrac{2}{3}x^2-5x+7$

25. $F(x)=7(x^2-4)$

26. $F(x)=\dfrac{5(x+6)}{7}$

27. $f(x)=5(x^2-3x)(x-6)$

28. $f(x)=(2x^3+x)(x^2-5)$

29. $f(x)=\dfrac{6x^4-9x^2}{3x^3}$

30. $f(x)=\dfrac{2x+2}{5x-3}$

31. $f(x)=\dfrac{7x}{x-5}$

32. $f(x)=2x^{-12}$

33. $f(x)=2x^2-5x^{-2}$

34. $f(x)=2+\dfrac{3}{x}+\dfrac{4}{x^2}$

35. $f(x)=\dfrac{a}{x}-\dfrac{b}{x^3}$

36. $f(x)=(x^3-1)^2$

37. $f(x)=\dfrac{3}{(x^2-3x)^2}$

38. $f(x)=\dfrac{x^2}{x+1}$

39. $s(t)=\dfrac{t^3}{t-2}$

40. $f(x)=3x^{-2}+2x^{-1}+1$

41. $F(z)=\dfrac{1}{z^2+1}$

42. $f(v)=\dfrac{v-1}{v^2+1}$

43. $g(z)=\dfrac{1}{1-z+z^2}$

44. $f(x)=3e^x+x^2$

45. $s(t)=1-e^t$

46. $f(x)=ae^x(2x^2+7x)$

47. $f(x)=\dfrac{1+x}{e^x}$

48. $f(x)=(2xe^x)^2$

49. $f(x)=x\sin x$

50. $s(t)=\cos^2 t$

51. $G(u)=\tan u+\sec u$

52. $g(v)=\sin v-\dfrac{1}{3}\cos v$

53. $f(x)=e^x\sin x$

54. $f(x)=e^x\csc x$

55. $f(x)=2\sin x\cos x$

56. $f(x)=(e^x+b)\cos x$

57. $f(x)=\dfrac{\sin x}{\csc x}$

58. $f(x)=\dfrac{1-\cot x}{1+\cot x}$

59. $f(\theta)=\dfrac{\cos\theta}{2e^\theta}$

60. $f(\theta)=4\theta\cot\theta\tan\theta$

In Problems 61–66, find the first derivative and the second derivative of each function.

61. $f(x)=(5x+3)^2$

62. $f(x)=xe^x$

63. $g(u)=\dfrac{u}{2u+1}$

64. $F(x)=e^x(\sin x+2\cos x)$

65. $f(u)=\dfrac{\cos u}{e^u}$

66. $F(x)=\dfrac{\sin x}{x}$

In Problems 67–70, for each function:

(a) Find an equation of the tangent line to the graph of the function at the indicated point.

(b) Find an equation of the normal line to the function at the indicated point.

(c) Graph the function, the tangent line, and the normal line on the same screen.

67. $f(x)=2x^2-3x+7$ at $(-1, 12)$

68. $y=\dfrac{x^2+1}{2x-1}$ at $\left(2, \dfrac{5}{3}\right)$

69. $f(x)=x^2-e^x$ at $(0, -1)$

70. $s(t)=1+2\sin t$ at $(\pi, 1)$

71. Rectilinear Motion As an object in rectilinear motion moves, its signed distance s (in meters) from the origin at time t (in seconds) is given by the position function

$$s=f(t)=t^2-6t$$

(a) Find the average velocity of the object from 0 to 5 s.
(b) Find the velocity at $t=0$, at $t=5$, and at any time t.
(c) Find the acceleration at any time t.

72. Rectilinear Motion As an object in rectilinear motion moves, its signed distance s from the origin at time t is given by the position function $s(t)=t-t^2$, where s is in centimeters and t is in seconds.

(a) Find the average velocity of the object from 1 to 3 s.
(b) Find the velocity of the object at $t=1$ s and $t=3$ s.
(c) What is its acceleration at $t=1$ and $t=3$?

73. **Business** The price p in dollars per pound when x pounds of a commodity are demanded is modeled by the function

$$p(x) = \frac{10{,}000}{5x + 100} - 5$$

when between 0 and 90 lb are demanded (purchased).

(a) Find the rate of change of price with respect to demand.

(b) What is the revenue function R? (Recall, revenue R equals price times amount purchased.)

(c) What is the marginal revenue R' at $x = 10$ and at $x = 40$ lb?

74. If $f(x) = \dfrac{x-1}{x+1}$ for all $x \neq -1$, find $f'(1)$.

75. If $f(x) = 2 + |x - 3|$ for all x, determine whether the derivative f' exists at $x = 3$.

76. **Rectilinear Motion** An object in rectilinear motion moves according to the position function $s = 2t^3 - 15t^2 + 24t + 3$, where t is measured in minutes and s in meters.

(a) When is the object at rest?

(b) Find the object's acceleration when $t = 3$.

77. Find the value of the limit below and specify the function f for which this is the derivative.

$$\lim_{\Delta x \to 0} \frac{[4 - 2(x + \Delta x)]^2 - (4 - 2x)^2}{\Delta x}$$

AP® REVIEW PROBLEMS: CHAPTER 2

1. If $f(x) = \sec x$, then $f'\left(\frac{\pi}{4}\right) =$

(A) $\frac{\sqrt{2}}{2}$ (B) 2 (C) 1 (D) $\sqrt{2}$

2. If a function f is differentiable at c, then $f'(c)$ is given by

I. $\lim\limits_{x \to c} \frac{f(x) - f(c)}{x - c}$

II. $\lim\limits_{x \to c} \frac{f(x+h) - f(x)}{h}$

III. $\lim\limits_{h \to 0} \frac{f(c+h) - f(c)}{h}$

(A) I only (B) III only

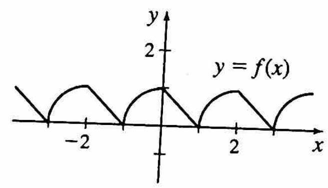
(C) I and II only (D) I and III only

3. If $y = \frac{3}{x^2 - 5}$, then $\frac{dy}{dx} =$

(A) $\frac{6x}{(x^2-5)^2}$ (B) $-\frac{6x}{(x^2-5)^2}$

(C) $\frac{6x}{x^2-5}$ (D) $\frac{2x}{(x^2-5)^2}$

4. The graph of the function f is shown below. Which statement about the function is true?



(A) f is differentiable everywhere.

(B) $0 \le f'(x) \le 1$, for all real numbers.

(C) f is continuous everywhere.

(D) f is an even function.

5. The table displays select values of a differentiable function f. What is an approximate value of $f'(2)$?

x	1.996	1.998	2	2.002	2.004
$f(x)$	3.168	3.181	3.194	3.207	3.220

(A) 6.5 (B) 1.154 (C) 0.013 (D) 0.0016

6. If $y = \sin x + xe^x + 6$, what is the instantaneous rate of change of y with respect to x at $x = 5$?

(A) $\cos 5 + 6e^5$ (B) 2

(C) $\cos 5 + 5e^5$ (D) $6e^5 - \cos 5$

7. An equation of the normal line to the graph of $f(x) = 3xe^x + 5$ at $x = 0$ is

(A) $y = 3x + 5$ (B) $y = -\frac{1}{3}x + 5$

(C) $y = \frac{1}{3}x + 5$ (D) $y = -3x + 5$

8. An object moves along a horizontal line so that its position at time t is $s(t) = t^4 - 6t^3 - 2t - 1$. At what time t is the acceleration of the object zero?

(A) at 0 only (B) at 1 only

(C) at 3 only (D) at 0 and 3 only

9. If $f(x) = e^x(\sin x + \cos x)$, then $f'(x) =$

(A) $2e^x(\cos x + \sin x)$ (B) $e^x \cos x$

(C) $2e^x \cos x$ (D) $e^x(\cos^2 x - \sin^2 x)$

10. Find an equation of the tangent line to the graph of $f(x) = \frac{x+3}{x^2+2}$ at $x = 1$.

(A) $5x + 9y = 17$ (B) $9y - 5x = 7$

(C) $5x + 3y = 9$ (D) $5x + 9y = 7$

11. $\lim\limits_{x \to \frac{\pi}{4}} \frac{\tan x - 1}{x - \frac{\pi}{4}} =$

(A) 0 (B) −1 (C) 2 (D) Does not exist.

Preparing for the AP® Exam

AP® CUMULATIVE REVIEW PROBLEMS: CHAPTERS 1–2

1. $\lim_{x \to 4} \frac{x-4}{4-x} =$

(A) -4 (B) -1 (C) 0 (D) does not exist

2. $\lim_{x \to 0} \frac{3x + \sin x}{2x} =$

(A) 0 (B) 1 (C) 2 (D) does not exist

3. Let h be defined by

$$h(x) = \begin{cases} f(x) \cdot g(x) & \text{if } x \le 1 \\ k + x & \text{if } x > 1 \end{cases}$$

where f and g are both continuous at all real numbers. If $\lim_{x \to 1} f(x) = 2$ and $\lim_{x \to 1} g(x) = -2$, then for what number k is h continuous?

(A) -5 (B) -4 (C) -2 (D) 2

4. Which function has the horizontal asymptotes $y = 1$ and $y = -1$?

(A) $f(x) = \frac{2}{\pi} \tan^{-1} x$ (B) $f(x) = e^{-x} + 1$

(C) $f(x) = \frac{1 - x^2}{1 + x^2}$ (D) $f(x) = \frac{2x^2 - 1}{2x^2 + x}$

5. Suppose the function f is continuous at all real numbers and $f(-2) = 1$ and $f(5) = -3$. Suppose the function g is also continuous at all real numbers and $g(x) = f^{-1}(x)$ for all x. The Intermediate Value Theorem guarantees that

(A) $g(c) = 2$ for at least one c between -3 and 1.

(B) $g(c) = 0$ for at least one c between -2 and 5.

(C) $f(c) = 0$ for at least one c between -3 and 1.

(D) $f(c) = 2$ for at least one c between -2 and 5.

6. The line $x = c$ is a vertical asymptote to the graph of the function f. Which of the following statements cannot be true?

(A) $\lim_{x \to c} f(x) = \infty$ (B) $\lim_{x \to \infty} f(x) = c$

(C) $f(c)$ is not defined. (D) f is continuous at $x = c$.

7. The position function of an object moving along a straight line is $s(t) = \frac{1}{15}t^3 - \frac{1}{2}t^2 + 5t^{-1}$. What is the object's acceleration at $t = 5$?

(A) $-\frac{27}{25}$ (B) $-\frac{1}{5}$ (C) $\frac{1}{5}$ (D) $\frac{27}{25}$

8. If the function $f(x) = \begin{cases} 2ax^2 + bx - 1 & \text{if } x \le 3 \\ bx^2 + bx - a & \text{if } x > 3 \end{cases}$ is continuous for all real numbers x, then

(A) $19a - 15b = 1$ (B) $18a - 9b = 1$

(C) $19a - 9b = 1$ (D) $19a + 15b = 1$

9. Find the slope of the tangent line to the graph of $f(x) = xe^x$ at the point $(1, e)$.

(A) 1 (B) e (C) $2e$ (D) e^2

10. An object in rectilinear motion is modeled by the position function

$$s(t) = 3t^4 - 8t^3 - 6t^2 + 24t \qquad t > 0$$

where s is in feet (ft) and t is in seconds (s). Find the acceleration of the object when its velocity is zero.

(A) -24 ft/s^2, 36 ft/s^2, and 72 ft/s^2 only

(B) 36 ft/s^2 only

(C) 36 ft/s^2 and 72 ft/s^2 only

(D) -24 ft/s^2 and 36 ft/s^2 only