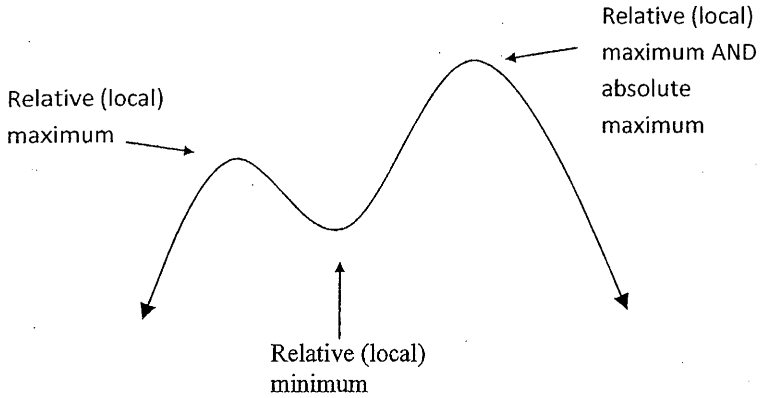
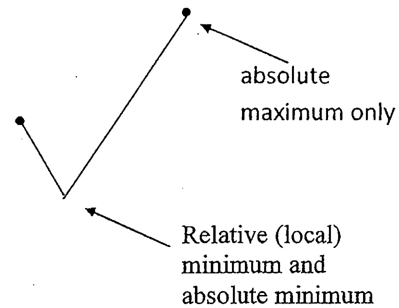
Calculus Ch. 3.1 Notes: Extrema and Critical Points

Key

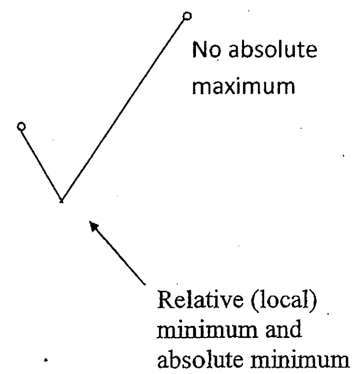
Extrema : maximums and minimums



Closed interval



Open Interval



Relative (local) extrema: any "hills and valleys" of graph

Absolute (global) extrema: highest or lowest points on the entire graph

*holes and $\pm\infty$ can **not** be considered as absolute extrema.

Extreme Value Theorem: If a function is continuous on a closed interval, then it has **both** an (absolute) minimum and an (absolute) maximum on that interval.

Fermat's Theorem: If a function is continuous on a closed interval, then the absolute extreme will **either** be at the **a)** critical numbers or **b)** at an endpoint.

Critical numbers (values) : x-values in the domain of a function where the derivative of a function is either 0 or undefined.

*Relative extrema ONLY occur at critical numbers, but not all critical numbers are where relative extrema occur.

*Maximum and minimum values refer to the **y-values** of the point.

Steps:

1. Find critical points
   a. Set $f'(x) = 0$
   b. Find where $f'(x)$ is undefined (Set denominator of $f'(x) = 0$)

2. Plug all critical points and endpoints into f(x)

3. Compare y-values to determine absolute maximum(s) and absolute minimum(s)

Find all critical numbers for each. What are the values of the absolute extrema?

Example 1: $f(x) = 3x^4 - 4x^3$ on [0, 2]

$f'(x) = 12x^3 - 12x^2$

$0 = 12x(x-1)$

$x = 0, 1$

$f(0) = 0$

$f(1) = -1$ (min)

$f(2) = 16$ (max)

Example 2: $f(x) = (x-1)^{\frac{2}{3}}$ on [-1, 0] f(x) continuous on [-1, 0]

$f'(x) = \frac{2}{3}(x-1)^{-1/3}(1)$

$f'(x) = \frac{2}{3(x-1)^{1/3}}$

$x = 1$ (critical pt.)

$f(-1) = \sqrt[3]{4}$ (Abs. max is $\sqrt[3]{4}$ at $x = -1$)

$f(0) = 1$ (Abs. min is 1 at $x = 0$)

f(x) continuous on [0, 3]

Example 3: $f(x) = \frac{4}{3}x\sqrt{3-x}$ on [0, 3] $f(x) = \frac{4}{3}x(3-x)^{1/2}$

$f'(x) = \frac{4}{3}(3-x)^{1/2} + \frac{4}{3}x \cdot \frac{1}{2}(3-x)^{-1/2}(-1)$

$0 = \frac{4\sqrt{3-x}}{3} - \frac{2x}{3\sqrt{3-x}}$

$= \frac{4(3-x) - 2x}{3\sqrt{3-x}} = \frac{12 - 4x - 2x}{3\sqrt{3-x}}$

$f'(x) = \frac{12 - 6x}{3\sqrt{3-x}}$

$12 - 6x = 0$ $6x = 12$ $x = 2$

$3\sqrt{3-x} = 0$ $x = 3$

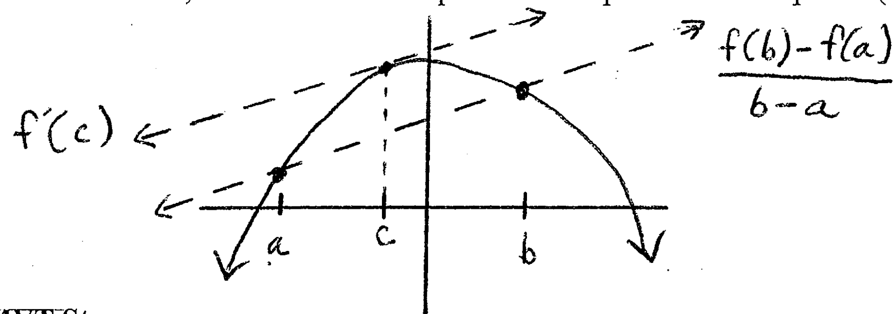
$f(0) = 0$

$f(3) = 0$ (Abs. min)

$f(2) = \frac{4}{3}(2)\sqrt{1} = \frac{8}{3}$ (Abs. max)

Mean Value Theorem (MVT): If a function, f(x), is **continuous** on [a, b] and **differentiable** on (a, b), then there must be at least one point, c in (a, b) where the slope of the tangent (derivative) is equal to the slope of the secant. $f'(c) = \dfrac{f(b) - f(a)}{b - a}$

*In other words, set the derivative equal to the slope between endpoints ($m_{avg.}$) *



MVT Steps:

1. Check Continuity (no breaks between endpoints)
   a. Does f(x) have variables in the denominator? (V.A. or holes)
   b. If so, then look to see if the x-value lies in the **closed** interval [a, b]
   c. If the x lies between the interval, then function is not continuous on the interval, MVT fails
2. Check Differentiability (smooth curve between endpoints)
   a. Does f'(x) have variables in the denominator? (sharp points, slope undefined)
   b. If yes, then look to see if the x-value lies in the **open** interval (a, b)
   c. If the x lies between the interval, then function is not differentiable on the interval, MVT fails

Note, all polynomials are continuous and differentiable everywhere

3. Find m_{avg}. (This is the slope between your endpoints, slope of secant line)
4. Set $f'(x) = m_{avg}$ and solve for x

Example 1: Determine if the mean value theorem can be applied to $f(x) = 2x^3 + x + 4$ on the interval [-2, 1]. If so, find the value of c based on the theorem.

f(x) is continuous, differentiable on [-2, 1]

$f(-2) = -14$

$f(1) = 7$

$m_{Avg} = \dfrac{-14-7}{-2-1}$

$m_{Avg} = \dfrac{-21}{-3} = 7$

$f'(x) = 6x^2 + 1$

set $f'(x) = m_{Avg}$

$6x^2 + 1 = 7$

$6x^2 = +6$

$x = \pm 1$

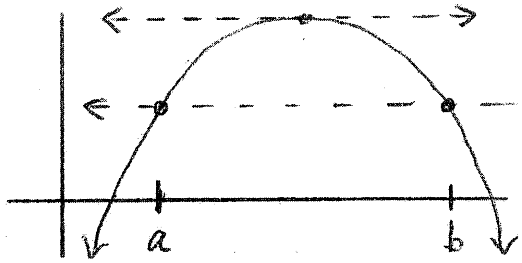
* we are only looking for c-value in the open interval (-2, 1)

$\cancel{c = 1}$, $\boxed{c = -1}$

Rolle's Theorem: If a function, f(x) is continuous on [a, b], differentiable on (a, b), and f(a) = f(b), then there must be at least one point on the function where the slope of the tangent (derivative) is 0.

*In other words, if the endpoints have the same y-values, then we can guarantee a relative maximum or relative minimum somewhere between the endpoints

*Rolle's Theorem is just a specific case of the Mean Value Theorem



$$\frac{f(b)-f(a)}{b-a} = 0$$

Rolle's Theorem Steps:

1. Check Continuity (no breaks between endpoints)
2. Check Differentiability (smooth curve between endpoints)

Note, all polynomials are continuous and differentiable everywhere

3. Test endpoints. Does f(a) = f(b)? If not ,then Rolle's fails / does not apply
4. If yes, then set f'(x) = 0 and solve for x

Example 2: Determine if Rolle's theorem can be applied to $f(x) = x^2 - 3x + 2$ on the interval [1, 2]. If so, find the value of c such that f '(c) = 0.

f(x) is continuous and differentiable on [1, 2]

$f(1) = 0$

$f(2) = 0$

$m_{Avg} = \frac{0-0}{2-1} = 0$

set $f'(c) = m_{Avg}$

$f'(x) = 2x - 3$

$2x - 3 = 0$

$x = 3/2$

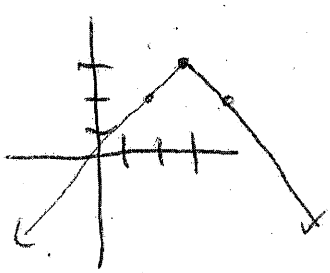
$\boxed{c = 3/2}$

Example 3: Determine if Rolle's theorem can be applied for f(x) = 3 – |x – 3| on [0, 6]

V(3, 3)

f(x) not differentiable at x = 3, not differentiable on interval [0, 6]

Rolle's theorem does not apply.



Calculus Ch. 3.3 **Notes: First Derivative Test**

What does the derivative represent? slope of tangent line to function (curve)

When the function is **increasing**, what is common about the derivatives at those points? $f'(x) > 0$

When the function is **decreasing**, what is common about the derivatives at those points? $f'(x) < 0$

When $f'(x) > 0$, f(x) is increasing

When $f'(x) < 0$, f(x) is decreasing

When $f'(x) = 0$, f(x) is constant (horizontal or changing direction)

First Derivative Test Steps (Finds inc/dec and relative max/min)

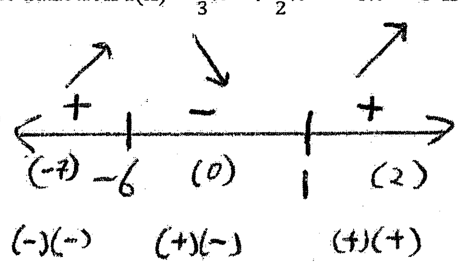
1. Find f '(x), set equal to zero
   a. Find critical points from BOTH numerator and denominator
   b. Remember, critical points also exist where function is not differentiable (sharp point)
2. Put all critical points on sign line
3. Test intervals
   a. Plug values into f '(x) to determine slope
      i. Positive (+) means increasing slope
      ii. Negative(–) means decreasing slope
4. Write Because Statements
   a. f(x) increasing in interval (a,b) b/c f '(x) > 0
   b. f(x) decreasing in interval (a,b) b/c f '(x) < 0
   c. Relative max at (**a, f(a)**) b/c f '(x) changes from + to –
   d. Relative min at (**a, f(a)**) b/c f '(x) changes from – to +

Example 1: Determine the intervals at which the function $f(x) = \frac{1}{3}x^3 + \frac{5}{2}x^2 - 6x - 3$ is increasing and decreasing. Locate the relative extrema.

$f'(x) = x^2 + 5x - 6$

$0 = (x+6)(x-1)$

$x = 1, \; x = -6$



f(x) is increasing on $(-\infty, -6) \cup (1, \infty)$ b/c $f'(x) > 0$

f(x) is decreasing on $(-6, 1)$ b/c $f'(x) < 0$

Relative max at $(-6, 51)$ b/c $f'(x)$ changes from + to –

Relative min at $(1, -6.167)$ b/c $f'(x)$ changes from – to +

First Derivative Test Steps (Finds inc/dec and relative max/min)

1. Find f '(x), set equal to zero
   a. Find critical points from BOTH numerator and denominator
   b. Remember, critical points also exist where function is not differentiable (sharp point)
2. Put all critical points on sign line
3. Test intervals
   a. Plug values into f '(x) to determine slope
      i. Positive (+) means increasing slope
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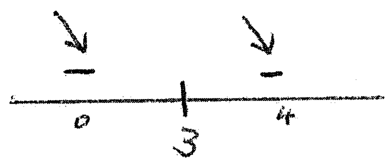
4. Write Because Statements
   a. f(x) increasing in interval (a,b) b/c f '(x) > 0
   b. f(x) decreasing in interval (a,b) b/c f '(x) < 0
   c. Relative max at (**a, f(a)**) b/c f '(x) changes from + to –
   d. Relative min at (**a, f(a)**) b/c f '(x) changes from – to +

Example 2: Determine the intervals at which the function $f(x) = \dfrac{5x+2}{x-3}$ is increasing and decreasing. Locate the relative extrema.

$$f'(x) = \frac{5(x-3)-(5x+2)(1)}{(x-3)^2} = \frac{5x-15-5x-2}{(x-3)^2} = \frac{-17}{(x-3)^2}$$

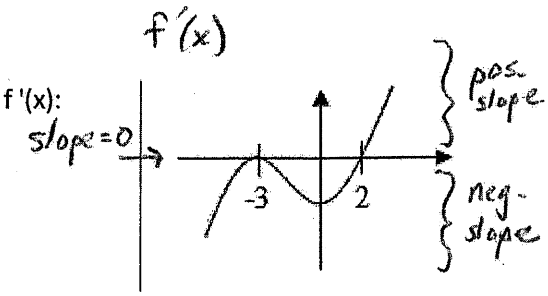
critical pt: $x = 3$



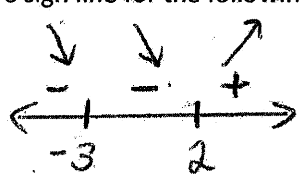
f(x) is decreasing on $(-\infty, 3) \cup (3, \infty)$ b/c $f'(x) < 0$

No relative extrema for f(x)

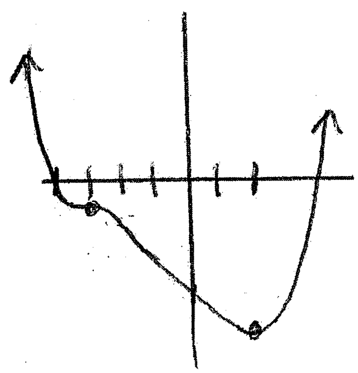
Example 3: Make a first derivative sign line for the following graph of f '(x):



1) Make sign line:



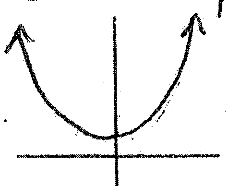
2) sketch graph using sign line



Calculus Ch. 3.4 **Notes: "Concavity Test" and the 2nd Derivative Test**

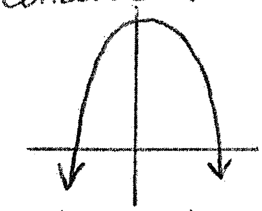

Are both of these functions increasing? yes What do we know about their derivatives? rate of slope is changing

concave up



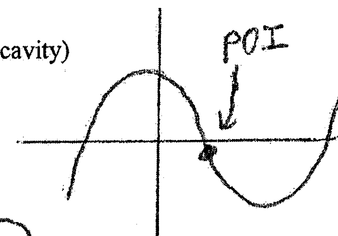
slope is becoming more positive

concave down



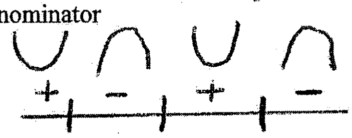
slope is becoming more negative

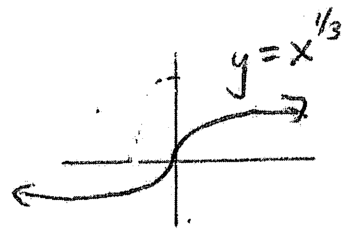
1) If f "(x) is > 0, then f '(x) is increasing and f(x) is concave up.
2) If f "(x) is < 0, then f '(x) is decreasing and f(x) is concave down.
3) A Point of Inflection (POI) occurs whenever f "(x) changes sign. (f(x) changes concavity)



"Concavity Test" Steps (Finding interval Concave Up/Down and POI)

1. Find $f''(x)$, set equal to zero
   a. Find critical points from BOTH numerator and denominator
2. Put all critical points on sign line
3. Test intervals
   a. Plug values into $f''(x)$ to determine concavity
      i. Positive (+) means concave up
      ii. Negative(–) means concave down
4. Write Because Statements
   a. Concave up in interval (a,b) b/c f''(x) > 0
   b. Concave down in interval (a,b) b/c f''(x) < 0
5. Point of Inflection at (**a, f(a)**) b/c $f''(x)$ changes signs





*Note: POI may exist on graph where $f''(x)$ does not exist (sharp point). POI exists as long as graph is continuous and $f''(x)$ changes concavity (change of signs) or vertical tangent line

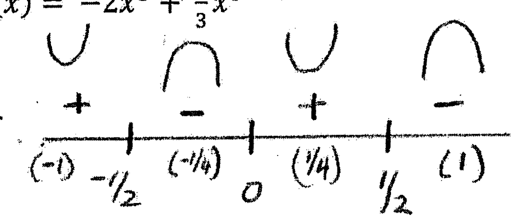
Example 1: Find the points of inflection if $f(x) = -2x^5 + \frac{5}{3}x^3$ * and find intervals concave up/down

$f'(x) = -10x^4 + 5x^2$

$f''(x) = -40x^3 + 10x$

$0 = -10x(4x^2 - 1)$

$x = 0, \quad x = \pm \frac{1}{2}$



$f''(x)$ concave up on $(-\infty, -\frac{1}{2}) \cup (0, \frac{1}{2})$ b/c $f''(x) > 0$

$f''(x)$ concave down on $(-\frac{1}{2}, 0) \cup (\frac{1}{2}, \infty)$ b/c $f''(x) < 0$

POI at $(-\frac{1}{2}, -0.146), (0,0), (\frac{1}{2}, 0.146)$ b/c $f''(x)$ changes signs.

The 2^{nd} Derivative Test

The 2^{nd} derivative test is a test for relative extrema (max/min) and NOT for Point of Inflection

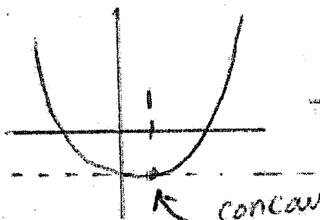
*The 2^{nd} derivative test achieves the same as the 1^{st} derivative test.

1) If you plug a critical number from f'(x) into $f''(x)$ and if $f''(x) > 0$, then that is the x-value of the relative **minimum**

2) If you plug a critical number from f'(x) into $f''(x)$ and if $f''(x) < 0$, then that is the x-value of the relative **maximum**

3) If you plug a critical number from f'(x) into $f''(x)$ and if $f''(x) = 0$, then the test is inconclusive. We need the first derivative test to determine if critical number is a relative extrema.

*critical numbers from $f'(x)$ indicate candidates for relative max/min.



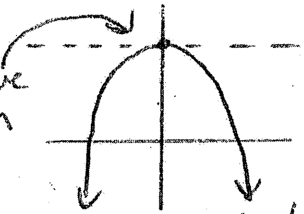
$f'(1)=0$

$f''(1)>0$

so the graph has relative min at $x=1$

concave up

concave down



$f'(0)=0$

$f''(0)<0$

* graph has relative max at $x=0$

2^{nd} Derivative Test Steps (Test for Relative Extrema, **NOT** Point of Inflection)

1. Find f'(x) , set equal to zero
   a. Find critical points. Set numerator and denominator of f '(x) = 0. (These are candidates for relative max/min)
2. Find $f''(x)$
3. Plug the critical points (from step #1) into $f''(x)$.
   a. If result is positive value, then f''(x) > 0, concave up, and therefore relative minimum exists at x-value
   b. If result is negative value, then f''(x) < 0, concave down, and therefore relative maximum exists at x-value
   c. f result is zero, then since f''(x) = 0, then this test is inconclusive. We cannot determine whether relative extrema exists. (Use First Derivative Test)

Example 2: Find the relative extrema of $f(x) = x^3 - 4x^2 - 3x$ *using 2nd derivative test.

$f'(x) = 3x^2 - 8x - 3$

$0 = (3x+1)(x-3)$

$x = -1/3, 3$

$f''(x) = 6x - 8$

*Test critical points using $f''(x)$.

$f''(x) = 6x - 8$

$f''(-\frac{1}{3}) = 6(-\frac{1}{3}) - 8 < 0$

$\boxed{\text{Relative max at } x = -\frac{1}{3} \text{ b/c } f'(-1/3) = 0 \text{ and } f''(-\frac{1}{3}) < 0}$

$f''(3) = 6(3) - 8 = 10 > 0$

$\boxed{\text{Relative min at } x = 3 \text{ b/c } f'(3) = 0 \text{ and } f''(3) > 0}$