Basic Integration Formulas

1. $\int kf(u)\,du = k\int f(u)\,du$
2. $\int [f(u) \pm g(u)]\,du = \int f(u)\,du \pm \int g(u)\,du$
3. $\int du = u + C$
4. $\int a^u\,du = \left(\frac{1}{\ln a}\right)a^u + C$
5. $\int e^u\,du = e^u + C$
6. $\int \sin u\,du = -\cos u + C$
7. $\int \cos u\,du = \sin u + C$
8. $\int \tan u\,du = -\ln|\cos u| + C$
9. $\int \cot u\,du = \ln|\sin u| + C$
10. $\int \sec u\,du = \ln|\sec u + \tan u| + C$
11. $\int \csc u\,du = -\ln|\csc u + \cot u| + C$
12. $\int \sec^2 u\,du = \tan u + C$
13. $\int \csc^2 u\,du = -\cot u + C$
14. $\int \sec u \tan u\,du = \sec u + C$
15. $\int \csc u \cot u\,du = -\csc u + C$
16. $\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin\frac{u}{a} + C$
17. $\int \frac{du}{a^2 + u^2} = \frac{1}{a}\arctan\frac{u}{a} + C$
18. $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\operatorname{arcsec}\frac{|u|}{a} + C$

Ch. 8.1 Integration Methods

Here is a list of procedures that you use to make an integral fit one of the basic rules.

1) Expand a function $(1+e^x)^2 = 1+2e^x+e^{2x}$
2) Separate the numerator $\frac{1+x}{x^2+1} = \frac{1}{x^2+1} + \frac{x}{x^2+1}$
3) Complete the square $\frac{1}{\sqrt{2x-x^2}} = \frac{1}{\sqrt{1-(x-1)^2}}$
4) Long division $\frac{x^2}{x^2+1} = 1 - \frac{1}{x^2+1}$
5) Add and subtract terms in numerator $\frac{2x}{x^2+2x+1} = \frac{2x+2-2}{x^2+2x+1} = \frac{2x+2}{x^2+2x+1} - \frac{2}{x^2+2x+1}$
6) Use trigonometric identities $\cot^2 x = \csc^2 x - 1$
7) Multiply and divide by Pythagorean conjugate

$$\frac{1}{1+\sin x} = \left(\frac{1}{1+\sin x}\right)\left(\frac{1-\sin x}{1-\sin x}\right) = \frac{1-\sin x}{1-\sin^2 x} = \frac{1-\sin x}{\cos^2 x} = \sec^2 x - \frac{\sin x}{\cos^2 x}$$

Ch. 8.2 IBP and Tab

Integration by Parts (I.B.P.) -a method of integration useful for problems involving the product of two different types of functions. (example: logs and polynomial)

IBP Formula: $\int u\,dv = uv - \int v\,du$ *This theorem is derived from the product rule for derivatives

IBP Steps:

1. Determine the u-value by using the acronym L.I.P.E.T.
   a. LIPET shows the priority order for determining u-value
   b. Logs Inverse Trig Polynomial Exponential function Trigonometric function
2. Let dv be other function
3. Find u, dv, v, and dv
4. Plug into formula and integrate

Tab Steps:

Tab method is used whenever the u-value can be assigned to the polynomial

*Use LIPET to determine if polynomial is the appropriate u-value.

*Tab method is especially useful for polynomial of degree higher than 1

Steps: 1) Create 2 columns u | dv
2) u-value must be the polynomial
3) Let dv be the other portion
4) Find the derivative of polynomial (u) until reaching zero
5) Find integral of dv the same number of times
6) Assign alternating signs to each column (+/ –)
7) Add the product of diagonal terms

Ch. 8.3 Trig Integration

In this section we will evaluate integrals of the form $\int \sin^m x \cos^n x dx$ and $\int \sec^m x \tan^n x dx$ where either m or n is a positive integer. In order to find these integrals, we have to write the integrand as a combination of trig functions that we can use the Power Rule on. For example, we can integrate $\int \sin^5 x \cos x dx$ by letting u = sinx and du = cosxdx.

To break up the integral into manageable parts, use the following identities:

$\sin^2 x + \cos^2 x = 1$ Pythagorean identity

$\sin^2 x = \dfrac{1-\cos 2x}{2}$ Half-angle identity for $\sin^2 x$

$\cos^2 x = \dfrac{1+\cos 2x}{2}$ Half-angle identity for $\cos^2 x$

Guidelines for Evaluating Integrals Involving Sine and Cosine

1) If the power of sine is odd and positive, save one sine and convert the rest to cosines.
2) If the power of cosine is odd and positive, save one cosine and convert the rest to sines.
3) If the powers of both the sine and cosine are even and nonnegative, use the half-angle identities to convert the integrand to odd powers of the cosine.

8.3 (continued)

Guidelines for Evaluating Integrals Involving Secant and Tangent (Note: $1+\tan^2 x = \sec^2 x$)

1) If the power of secant is even and positive, save a secant-squared factor and convert the rest to tangents.
2) If the power of the tangent is odd and positive, save a secant-tangent and convert the rest to secants.
3) If there are no secants and the power of tangent is even and positive, convert a tangent-squared to a (secant-squared − 1). Expand and repeat as necessary.
4) If the integral is only secant with an odd positive power, use integration by parts.
5) If none of the first four guidelines apply, try to convert to sines and cosines.

8.4 Trig Substitution

Recall the Arc Trig Integral Rules:

$$\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\frac{u}{a}+C \qquad \int \frac{du}{a^2+u^2} = \frac{1}{a}\arctan\frac{u}{a}+C \qquad \int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\operatorname{arcsec}\frac{|u|}{a}+C$$

Consider the forms of these Pythagorean identities

a) $\cos^2\theta = 1-\sin^2\theta$ b) $\sec^2\theta = 1+\tan^2\theta$ c) $\tan^2\theta = \sec^2\theta - 1$

Trig Substitution (a > 0)

1. For $\sqrt{a^2-u^2}$, use $\sin\theta = \frac{u}{a}$ → Therefore, $\sqrt{a^2-u^2} = a\cos\theta$

2. For $\sqrt{a^2+u^2}$, use $\tan\theta = \frac{u}{a}$ → Therefore, $\sqrt{a^2+u^2} = a\sec\theta$

3. For $\sqrt{u^2-a^2}$, use $\sec\theta = \frac{u}{a}$ → Therefore, $\sqrt{u^2-a^2} = \pm a\tan\theta$

8.5 Partial Fraction

1) Linear Factors - Cover up method

2) Repeated Linear Factors	$\int \frac{5x^2+20x+6}{x^3+2x^2+x}dx$ $\frac{x(x^2+2x+1)}{x(x+1)(x+1)}$ $\int \frac{A}{x}+\frac{B}{x+1}+\frac{C}{(x+1)^2}dx$
3) Linear and Quadratic Factors	Find $\int \frac{2x^3-4x-8}{(x^2-x)(x^2+4)}dx$ $\frac{A}{x}+\frac{B}{x-1}+\frac{Cx+D}{x^2+4}$ $x(x-1)(x^2+4)$
4) Repeated Quadratic Factors	$\int \frac{8x^3+13x}{(x^2+2)^2}dx$ $\frac{Ax+B}{x^2+2}+\frac{Cx+D}{(x^2+2)^2}$

8.7 L'Hopital's and Indeterminate Form

L'Hôpital's Rule

Let *f* and *g* be functions that are differentiable on an open interval (a, b) containing *c*, except possibly at *c* itself. Assume that g'(x) ≠ 0 for all x in (a, b), except possibly at c itself. If the limit of f(x)/g(x) as x approaches c produces an indeterminate form (0/0 or ∞/∞), then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

a) Indeterminate Form of $0 * \infty$ (Rewrite as Numerator /denominator)

b) Indeterminate Form of 1^{∞} (First Take Ln of both sides of equation to bring down exponent)

c) Indeterminate Form of 0^{∞} (First Take Ln of both sides of equation to bring down exponent)

d) Indeterminate Form $\infty - \infty$ (First Rewrite as one fraction)

8.8 Improper Integrals

The trick to solving an improper integral is to consider the integration in terms of a limit. For instance,

$$\int_1^b \frac{dx}{x^2} = \left[\frac{-1}{x}\right]_1^b = \frac{-1}{b} - (-1) = 1 - \frac{1}{b}$$

This integral is interpreted as the area of the shaded region under the graph on the interval from [a, b]. Taking the limit as $b \to \infty$ produces

$$\int_1^{\infty} \frac{dx}{x^2} = \lim_{b \to \infty}\left(\int_1^b \frac{dx}{x^2}\right) = \lim_{b \to \infty}\left(1 - \frac{1}{b}\right) = 1$$

a) Improper Integrals can converge to a value or diverges

b) Improper Integrals may have infinite discontinuity for one of the bounds

c) Improper Integrals with Interior Discontinuity (Need to split into the sum of 2 definite integrals)

d) Doubly Improper Integrals (Need to split into the sum of 2 definite integrals)