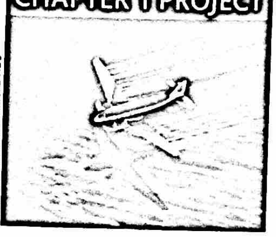
CHAPTER 1 PROJECT



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Pollution in Clear Lake

The Toxic Waste Disposal Company (TWDC) specializes in the disposal of a particularly dangerous pollutant, Agent Yellow (AY). Unfortunately, instead of safely disposing of this pollutant, the company simply dumped AY in (formerly) Clear Lake. Fortunately, they have been caught and are now defending themselves in court.

The facts below are not in dispute. As a result of TWDC's activity, the current concentration of AY in Clear Lake is now 10 ppm (parts per million). Clear Lake is part of a chain of rivers and lakes. Fresh water flows into Clear Lake, and the contaminated water flows downstream from it. The Department of Environmental Protection estimates that the level of contamination in Clear Lake will fall by 20% each year. These facts can be modeled as

$$p(0)=10 \qquad p(t+1)=0.80p(t)$$

where $p=p(t)$, measured in ppm, is the concentration of pollutants in the lake at time t, in years.

1. Explain how the above equations model the facts.
2. Create a table showing the values of t for $t=0, 1, 2, \ldots, 20$.
3. Show that $p(t)=10(0.8)^t$.
4. Use technology to graph $p=p(t)$.
5. What is $\lim_{t\to\infty} p(t)$?

Lawyers for TWDC looked at the results in 1–5 above and argued that their client has not done any real damage. They concluded that Clear Lake would eventually return to its former clear and unpolluted state. They even called in a mathematician, who wrote the following on a blackboard:

$$\lim_{t\to\infty} p(t)=0$$

and explained that this bit of mathematics means, descriptively, that after many years the concentration of AY will, indeed, be close to zero.

Concerned citizens booed the mathematician's testimony. Fortunately, one of them has taken calculus and knows a little bit about limits. She noted that, although "after many years the concentration of AY will approach zero," the townspeople like to swim in Clear Lake and state regulations prohibit swimming unless the concentration of AY is below 2 ppm. She proposed a fine of \$100,000 per year for each full year that the lake is unsafe for swimming. She also questioned the mathematician, saying, "Your testimony was correct as far as it went, but I remember from studying calculus that talking about the eventual concentration of AY after many, many years is only a small part of the story. The more precise meaning of your statement $\lim_{t\to\infty} p(t)=0$ is that given some tolerance T for the concentration of AY, there is some time N (which may be far in the future) so that for all $t>N$, $p(t)<T$."

6. Using a table or a graph for $p=p(t)$, find N so that if $t>N$, then $p(t)<2$.
7. How much is the fine?

Her words were greeted by applause. The town manager sprang to his feet and noted that although a tolerance of 2 ppm was fine for swimming, the town used Clear Lake for its drinking water and until the concentration of AY dropped below 0.5 ppm, the water would be unsafe for drinking. He proposed a fine of \$200,000 per year for each full year the water was unfit for drinking.

8. Using a table or a graph for $p=p(t)$, find N so that if $t>N$, then $p(t)<0.5$.
9. How much is the fine?
10. How would you find if you were on the jury trying TWDC? If the jury found TWDC guilty, what fine would you recommend? Explain your answers.

Chapter Review

THINGS TO KNOW

1.1 Limits of Functions Using Numerical and Graphical Techniques

- Slope of a secant line: $m_{\text{sec}}=\dfrac{f(x)-f(c)}{x-c}$ (p. 78)
- Slope of a tangent line: $m_{\text{tan}}=\lim_{x\to c}\dfrac{f(x)-f(c)}{x-c}$ (p. 78)
- $\lim_{x\to c} f(x)=L$: read, "The limit as x approaches c of $f(x)$ is equal to the number L." (p. 79)
- $\lim_{x\to c} f(x)=L$: interpreted as, "The value $f(x)$ can be made as close as we please to L, for x sufficiently close to c, but not equal to c." (p. 79)
- One-sided limits (p. 80)
- The limit L of a function $y=f(x)$ as x approaches a number c does not depend on the value of f at c. (p. 82)
- The limit L of a function $y=f(x)$ as x approaches a number c is unique. A function cannot have more than one limit as x approaches c. (p. 82)
- The limit L of a function $y=f(x)$ as x approaches a number c exists if and only if both one-sided limits exist at c and both one-sided limits are equal. That is, $\lim_{x\to c} f(x)=L$ if and only if $\lim_{x\to c^-} f(x)=\lim_{x\to c^+} f(x)=L$. (p. 83)

1.2 Analytic Techniques for Finding Limits of Functions

Basic Limits

- $\lim_{x\to c} A=A$, A a constant (p. 90)
- $\lim_{x\to c} x=c$, c a real number (p. 90)

Properties of Limits If f and g are functions for which $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ both exist and if k is any real number, then:

- $\lim_{x\to c}[f(x) \pm g(x)] = \lim_{x\to c} f(x) \pm \lim_{x\to c} g(x)$ (pp. 90, 91)
- $\lim_{x\to c}[f(x) \cdot g(x)] = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x)$ (p. 91)
- $\lim_{x\to c}[kg(x)] = k \lim_{x\to c} g(x)$ (p. 92)
- $\lim_{x\to c}[f(x)]^n = \left[\lim_{x\to c} f(x)\right]^n$, $n \geq 2$ is an integer (p. 93)
- $\lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to c} f(x)}$, provided $f(x) > 0$ if n is even (p. 94)
- $\lim_{x\to c}[f(x)]^{m/n} = \left[\lim_{x\to c} f(x)\right]^{m/n}$, provided $[f(x)]^{m/n}$ is defined for positive integers m and n (p. 94)
- $\lim_{x\to c}\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$, provided $\lim_{x\to c} g(x) \neq 0$ (p. 95)
- If P is a polynomial function, then $\lim_{x\to c} P(x) = P(c)$. (p. 95)
- If R is a rational function and if c is in the domain of R, then $\lim_{x\to c} R(x) = R(c)$. (p. 96)

1.3 Continuity

Definitions

- Continuity at a number (p. 103)
- Removable discontinuity (p. 105)
- One-sided continuity at a number (p. 105)
- Continuity on an interval (p. 106)
- Continuity on a domain (p. 107)

Properties of Continuity

- A polynomial function is continuous on its domain, all real numbers. (p. 107)
- A rational function is continuous on its domain. (p. 107)
- If the functions f and g are continuous at a number c, and if k is a real number, then the functions $f+g$, $f-g$, $f \cdot g$, and kf are also continuous at c. If $g(c) \neq 0$, the function $\dfrac{f}{g}$ is continuous at c. (p. 108)
- If a function g is continuous at c and a function f is continuous at $g(c)$, then the composite function $(f \circ g)(x) = f(g(x))$ is continuous at c. (p. 109)
- If f is a one-to-one function that is continuous on its domain, then its inverse function f^{-1} is also continuous on its domain. (p. 110)

The Intermediate Value Theorem Let f be a function that is continuous on a closed interval $[a, b]$ with $f(a) \neq f(b)$. If N is any number between $f(a)$ and $f(b)$, then there is at least one number c in the open interval (a, b) for which $f(c) = N$. (p. 110)

1.4 Limits and Continuity of Trigonometric, Exponential, and Logarithmic Functions

Basic Limits

- $\lim_{\theta\to 0} \dfrac{\sin\theta}{\theta} = 1$ (p. 119)
- $\lim_{\theta\to 0} \dfrac{\cos\theta - 1}{\theta} = 0$ (p. 122)
- $\lim_{x\to c} \sin x = \sin c$ (p. 123)
- $\lim_{x\to c} \cos x = \cos c$ (p. 123)
- $\lim_{x\to c} a^x = a^c$; $a > 0$, $a \neq 1$ (p. 124)
- $\lim_{x\to c} \log_a x = \log_a c$; $a > 0$, $a \neq 1$, and $c > 0$ (p. 124)

Squeeze Theorem If the functions f, g, and h have the property that for all x in an open interval containing c, except possibly at c, $f(x) \leq g(x) \leq h(x)$, and if $\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = L$, then $\lim_{x\to c} g(x) = L$. (p. 119)

Properties of Continuity

- The six trigonometric functions are continuous on their domains. (p. 123)
- The six inverse trigonometric functions are continuous on their domains. (p. 123)
- An exponential function is continuous on its domain, all real numbers. (p. 124)
- A logarithmic function is continuous on its domain, all positive real numbers. (p. 124)

1.5 Infinite Limits; Limits at Infinity; Asymptotes

Basic Limits

- $\lim_{x\to 0^-} \dfrac{1}{x} = -\infty$ $\quad \lim_{x\to 0^+} \dfrac{1}{x} = \infty$ (p. 128)
- $\lim_{x\to 0} \dfrac{1}{x^2} = \infty$ (p. 128)
- $\lim_{x\to 0^+} \ln x = -\infty$ (p. 129)
- $\lim_{x\to \infty} \dfrac{1}{x} = 0$ $\quad \lim_{x\to -\infty} \dfrac{1}{x} = 0$ (p. 132)
- $\lim_{x\to \infty} \ln x = \infty$ (p. 136)
- $\lim_{x\to -\infty} e^x = 0$ $\quad \lim_{x\to \infty} e^x = \infty$ (p. 136)

Definitions

- Vertical asymptote (p. 131)
- Horizontal asymptote (p. 138)

Properties of Limits at Infinity (p. 132): If k is a real number, $n \geq 2$ is an integer, and the functions f and g approach real numbers as $x \to \infty$, then:

- $\lim_{x\to\infty} A = A$, where A is a constant
- $\lim_{x\to\infty} [kf(x)] = k \lim_{x\to\infty} f(x)$
- $\lim_{x\to\infty} [f(x) \pm g(x)] = \lim_{x\to\infty} f(x) \pm \lim_{x\to\infty} g(x)$
- $\lim_{x\to\infty} [f(x)g(x)] = \left[\lim_{x\to\infty} f(x)\right]\left[\lim_{x\to\infty} g(x)\right]$
- $\lim_{x\to\infty} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to\infty} f(x)}{\lim_{x\to\infty} g(x)}$ provided $\lim_{x\to\infty} g(x) \neq 0$
- $\lim_{x\to\infty} [f(x)]^n = \left[\lim_{x\to\infty} f(x)\right]^n$
- $\lim_{x\to\infty} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to\infty} f(x)}$, where $f(x) > 0$ if n is even

1.6 The ε-δ Definition of a Limit

Definitions

- Limit of a Function (p. 145)
- Limit at Infinity (p. 150)
- Infinite Limit (p. 151)
- Infinite Limit at Infinity (p. 151)

Properties of Limits

- If $\lim_{x\to c} f(x) > 0$, then there is an open interval around c, for which $f(x) > 0$ everywhere in the interval, except possibly at c. (p. 150)
- If $\lim_{x\to c} f(x) < 0$, then there is an open interval around c, for which $f(x) < 0$ everywhere in the interval, except possibly at c. (p. 150)

OBJECTIVES

Section	You should be able to ...	Example	Review Exercises	Preparing for the AP® Exam: AP® Review Problems
1.1	1 Discuss the idea of a limit (p. 79)	1	4	
	2 Investigate a limit using a table (p. 80)	2–4	1	5
	3 Investigate a limit using a graph (p. 81)	5–8	2, 3	
1.2	1 Find the limit of a sum, a difference, and a product (p. 90)	1–6	8, 10, 12, 14, 22, 26, 29, 30, 47, 48	
	2 Find the limit of a power and the limit of a root (p. 93)	7–9	11, 18, 28, 55	
	3 Find the limit of a polynomial (p. 95)	10	10, 22	
	4 Find the limit of a quotient (p. 95)	11–14	13–17, 19–21, 23–25, 27, 56	3
	5 Find the limit of an average rate of change (p. 98)	15	37	
	6 Find the limit of a difference quotient (p. 98)	16	5, 6, 49	
1.3	1 Determine whether a function is continuous at a number (p. 103)	1–4	31–36	11
	2 Determine intervals on which a function is continuous (p. 106)	5, 6	39–42	8
	3 Use properties of continuity (p. 108)	7, 8	39–42	
	4 Use the Intermediate Value Theorem (p. 110)	9, 10	38, 44–46	6
1.4	1 Use the Squeeze Theorem to find a limit (p. 117)	1	7, 69	
	2 Find limits involving trigonometric functions (p. 119)	2, 3	9, 51–55	4, 10
	3 Determine where the trigonometric functions are continuous (p. 122)	4	63–65	
	4 Determine where an exponential or a logarithmic function is continuous (p. 124)	5	43	
1.5	1 Investigate infinite limits (p. 128)	1–3	57, 58	
	2 Find the vertical asymptotes of a graph (p. 131)	4	61, 62	1
	3 Investigate limits at infinity (p. 131)	5–10	59, 60	2
	4 Find the horizontal asymptotes of a graph (p. 138)	11	61, 62	1
	5 Find the asymptotes of the graph of a rational function (p. 139)	12	67, 68	7
1.6	1 Use the ε-δ definition of a limit (p. 144)	1–7	50, 66	

REVIEW EXERCISES

1. Use a table of numbers to investigate $\lim_{x\to 0} \frac{1-\cos x}{1+\cos x}$.

In Problems 2 and 3, use a graph to investigate $\lim_{x\to c} f(x)$.

2. $f(x) = \begin{cases} 2x-5 & \text{if } x<1 \\ 6-9x & \text{if } x\ge 1 \end{cases}$ at $c=1$

3. $f(x) = \begin{cases} x^2+2 & \text{if } x<2 \\ 2x+1 & \text{if } x\ge 2 \end{cases}$ at $c=2$

4. For $f(x) = x^2-3$:
   (a) Find the slope of the secant line joining $(1, -2)$ and $(2, 1)$.
   (b) Find the slope of the tangent line to the graph of f at $(1, -2)$.

In Problems 5 and 6, for each function find the limit of the difference quotient $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$.

5. $f(x) = \frac{3}{x}$

6. $f(x) = 3x^2+2x$

7. Find $\lim_{x\to 0} f(x)$ if $1+\sin x \le f(x) \le |x|+1$

In Problems 8–22, find each limit.

8. $\lim_{x\to 2}\left(2x-\frac{1}{x}\right)$

9. $\lim_{x\to \pi}(x\cos x)$

10. $\lim_{x\to -1}\left(x^3+3x^2-x-1\right)$

11. $\lim_{x\to 0}\sqrt[3]{x(x+2)^3}$

12. $\lim_{x\to 0}[(2x+3)(x^5+5x)]$

13. $\lim_{x\to 3}\dfrac{x^3-27}{x-3}$

14. $\lim_{x\to 3}\left(\dfrac{x^2}{x-3}-\dfrac{3x}{x-3}\right)$

15. $\lim_{x\to 2}\dfrac{x^2-4}{x-2}$

16. $\lim_{x\to -1}\dfrac{x^2+3x+2}{x^2+4x+3}$

17. $\lim_{x\to -2}\dfrac{x^3+5x^2+6x}{x^2+x-2}$

18. $\lim_{x\to 1}\left(x^2-3x+\dfrac{1}{x}\right)^{15}$

19. $\lim_{x\to 2}\dfrac{3-\sqrt{x^2+5}}{x^2-4}$

20. $\lim_{x\to 0}\left\{\dfrac{1}{x}\left[\dfrac{1}{(2+x)^2}-\dfrac{1}{4}\right]\right\}$

21. $\lim_{x\to 0}\dfrac{(x+3)^2-9}{x}$

22. $\lim_{x\to 1}[(x^3-3x^2+3x-1)(x+1)^2]$

In Problems 23–28, find each one-sided limit, if it exists.

23. $\lim_{x\to -2^+}\dfrac{x^2+5x+6}{x+2}$

24. $\lim_{x\to 5^+}\dfrac{|x-5|}{x-5}$

25. $\lim_{x\to 1^-}\dfrac{|x-1|}{x-1}$

26. $\lim_{x\to 3/2^+}\lfloor 2x\rfloor$

27. $\lim_{x\to 4^-}\dfrac{x^2-16}{x-4}$

28. $\lim_{x\to 1^+}\sqrt{x-1}$

In Problems 29 and 30, find $\lim_{x\to c^-}f(x)$ *and* $\lim_{x\to c^+}f(x)$ *for the given c. Determine whether* $\lim_{x\to c}f(x)$ *exists.*

29. $f(x)=\begin{cases}2x+3 & \text{if } x<2\\ 9-x & \text{if } x\ge 2\end{cases}$ at $c=2$

30. $f(x)=\begin{cases}3x+1 & \text{if } x<3\\ 10 & \text{if } x=3\\ 4x-2 & \text{if } x>3\end{cases}$ at $c=3$

In Problems 31–36, determine whether f is continuous at c.

31. $f(x)=\begin{cases}5x-2 & \text{if } x<1\\ 5 & \text{if } x=1\\ 2x+1 & \text{if } x>1\end{cases}$ at $c=1$

32. $f(x)=\begin{cases}x^2 & \text{if } x<-1\\ 2 & \text{if } x=-1\\ -3x-2 & \text{if } x>-1\end{cases}$ at $c=-1$

33. $f(x)=\begin{cases}4-3x^2 & \text{if } x<0\\ 4 & \text{if } x=0\\ \sqrt{16-x^2} & \text{if } 0<x\le 4\end{cases}$ at $c=0$

34. $f(x)=\begin{cases}\sqrt{4+x} & \text{if } -4\le x\le 4\\ \sqrt{\dfrac{x^2-16}{x-4}} & \text{if } x>4\end{cases}$ at $c=4$

35. $f(x)=\lfloor 2x\rfloor$ at $c=\dfrac{1}{2}$

36. $f(x)=|x-5|$ at $c=5$

37. **(a)** Find the average rate of change of $f(x)=2x^2-5x$ from 1 to x.
(b) Find the limit as x approaches 1 of the average rate of change found in (a).

38. A function f is defined on the interval $[-1, 1]$ with the following properties: f is continuous on $[-1, 1]$ except at 0, negative at -1, positive at 1, but with no zeros. Does this contradict the Intermediate Value Theorem?

In Problems 39–43, find all numbers x for which f is continuous.

39. $f(x)=\dfrac{x}{x^3-27}$

40. $f(x)=\dfrac{x^2-3}{x^2+5x+6}$

41. $f(x)=\dfrac{2x+1}{x^3+4x^2+4x}$

42. $f(x)=\sqrt{x-1}$

43. $f(x)=2^{-x}$

44. Use the Intermediate Value Theorem to determine whether $2x^3+3x^2-23x-42=0$ has a zero in the interval [3, 4].

In Problems 45 and 46, use the Intermediate Value Theorem to approximate the zero correct to three decimal places.

45. $f(x)=8x^4-2x^2+5x-1$ on the interval [0, 1].

46. $f(x)=3x^3-10x+9$; zero between -3 and -2.

47. Find $\lim_{x\to 0^+}\dfrac{|x|}{x}(1-x)$ and $\lim_{x\to 0^-}\dfrac{|x|}{x}(1-x)$.
What can you say about $\lim_{x\to 0}\dfrac{|x|}{x}(1-x)$?

48. Find $\lim_{x\to 2}\left(\dfrac{x^2}{x-2}-\dfrac{2x}{x-2}\right)$. Then comment on the statement that this limit is given by $\lim_{x\to 2}\dfrac{x^2}{x-2}-\lim_{x\to 2}\dfrac{2x}{x-2}$.

49. Find $\lim_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$ for $f(x)=\sqrt{x}$.

50. For $\lim_{x\to 3}(2x+1)=7$, find the largest possible δ that "works" for $\varepsilon=0.01$.

In Problems 51–60, find each limit.

51. $\lim_{x\to 0}\cos(\tan x)$

52. $\lim_{x\to 0}\dfrac{\sin\frac{x}{4}}{x}$

53. $\lim_{x\to 0}\dfrac{\tan(3x)}{\tan(4x)}$

54. $\lim_{x\to 0}\dfrac{\cos\frac{x}{3}-1}{x}$

55. $\lim_{x\to 0}\left(\dfrac{\cos x-1}{x}\right)^{10}$

56. $\lim_{x\to 0}\dfrac{e^{4x}-1}{e^x-1}$

57. $\lim_{x\to \pi/2^+}\tan x$

58. $\lim_{x\to -3}\dfrac{2+x}{(x+3)^2}$

59. $\lim_{x\to \infty}\dfrac{3x^3-2x+1}{x^3-8}$

60. $\lim_{x\to \infty}\dfrac{3x^4+x}{2x^2}$

In Problems 61 and 62, find any vertical and horizontal asymptotes of f.

61. $f(x)=\dfrac{4x-2}{x+3}$

62. $f(x)=\dfrac{2x}{x^2-4}$

63. Let $f(x)=\begin{cases}\dfrac{\tan x}{2x} & \text{if } x\ne 0\\ \dfrac{1}{2} & \text{if } x=0\end{cases}$. Is f continuous at 0?

64. Let $f(x)=\begin{cases}\dfrac{\sin(3x)}{x} & \text{if } x\ne 0\\ 1 & \text{if } x=0\end{cases}$. Is f continuous at 0?

65. The function $f(x)=\dfrac{\cos\left(\pi x+\frac{\pi}{2}\right)}{x}$ is not defined at 0. Decide how to define $f(0)$ so that f is continuous at 0.

66. Use the ε-δ definition of a limit to prove $\lim\limits_{x\to -3}(x^2-9)\neq 18$.

67. **(a)** Sketch a graph of a function f that has the following properties:

$$f(-1)=0 \quad \lim_{x\to\infty} f(x)=2 \quad \lim_{x\to-\infty} f(x)=2$$

$$\lim_{x\to 4^-} f(x)=-\infty \quad \lim_{x\to 4^+} f(x)=\infty$$

(b) Define a function that describes your graph.

68. **(a)** Find the domain and the intercepts (if any) of

$$R(x)=\frac{2x^2-5x+2}{5x^2-x-2}.$$

(b) Discuss the behavior of the graph of R at numbers where R is not defined.

(c) Find any vertical or horizontal asymptotes of the function R.

69. If $1-x^2\le f(x)\le \cos x$ for all x in the interval $-\frac{\pi}{2}<x<\frac{\pi}{2}$, show that $\lim\limits_{x\to 0} f(x)=1$.