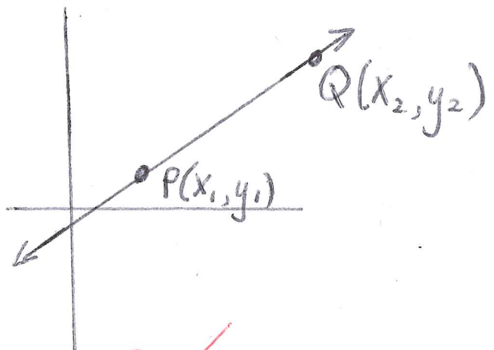
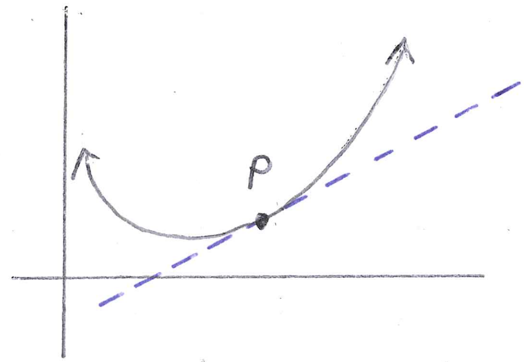
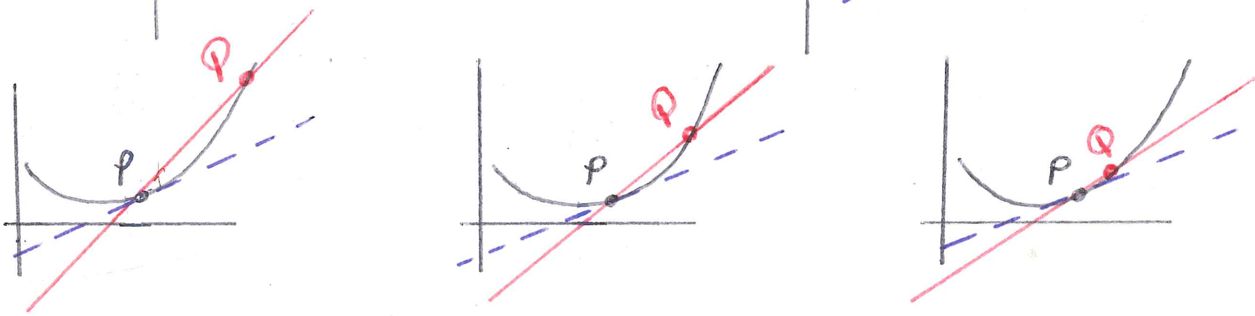
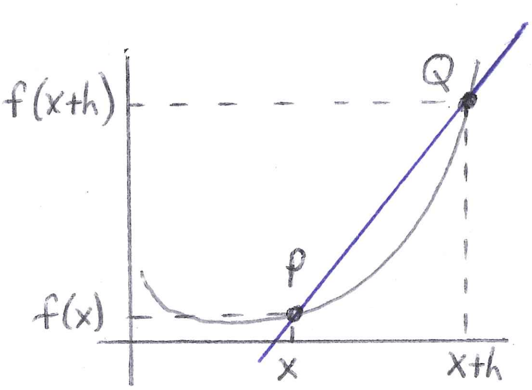
Ch. 2.1 Notes: The Derivative and Tangent Line Problem

Goal: To find a formula to calculate the slope (steepness) of all tangent lines to a curve.









slope: $\dfrac{y_2-y_1}{x_2-x_1} = \dfrac{f(x+h)-f(x)}{\cancel{x}+h-\cancel{x}}$

A. General (Limit) Definition of the Derivative

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

"f prime of x": This is the notation for the derivative function

Derivative: the slope or steepness of a curve at a single point.

* The Derivative is a slope-finding formula for a curved function, where the slope is ever-changing.

B. Alternative Derivative Definition

$$f'(c) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c}$$

Ex. 1 Find the general derivative of $f(x)=x^2$. Then write the equation of the line tangent to $f(x)$ at $x=1$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{(x+h)^2-x^2}{h}$$

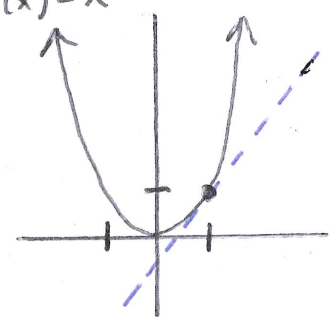
$$=\lim_{h\to 0}\frac{\cancel{x^2}+2xh+h^2-\cancel{x^2}}{h}$$

$$=\lim_{h\to 0}\frac{\cancel{h}(2x+h)}{\cancel{h}}$$

$$=2x+0$$

$$\boxed{f'(x)=2x}$$

$f(x)=x^2$



$f(x)=x^2$

- $f(x)$ is the height-finding formula
- Since $f(1)=1^2=1$, this tells us that when $x=1$, the height of graph has a y-value of 1

*Therefore, the derivative (slope-finding formula) for $f(x)=x^2$

$f'(x)=2x$

- $f'(x)$ is the slope-finding formula for the $f(x)$ graph
- Since $f'(1)=2(1)=2$ this tells us that when $x=1$, the slope of tangent line to $f(x)$ has slope of 2 (steepness)

Tangent-line equation:

* $y-y_1=m(x-x_1)$

point: $(1,1)$

slope: $m=2$

$$\boxed{y-1=2(x-1) \quad \text{or} \quad y=2x-1}$$

Ex. 2 Find equation of tangent line to $f(x)=x^2$ at $x=-5$

$f(x)=x^2$

$f'(x)=2x$

$f(-5)=(-5)^2=25$

$f'(-5)=2(-5)=-10$

point: $(-5,25)$

slope: $m=-10$

$y-y_1=m(x-x_1)$

$y-25=-10(x--5)$

$$\boxed{y-25=-10(x+5)}$$

Ex. 3 ⓐ Find derivative of $f(x)=\sqrt{x}$.

ⓑ Find the slope of function at $x=2$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}\cdot\frac{(\sqrt{x+h}+\sqrt{x})}{(\sqrt{x+h}+\sqrt{x})}$$

$$=\lim_{h\to 0}\frac{\cancel{x}+h-\cancel{x}}{h(\sqrt{x+h}+\sqrt{x})}=\lim_{h\to 0}\frac{\cancel{h}}{\cancel{h}(\sqrt{x+h}+\sqrt{x})}=\lim_{h\to 0}\frac{1}{\sqrt{x+0}+\sqrt{x}}=\frac{1}{2\sqrt{x}}\quad \boxed{f'(x)=\frac{1}{2\sqrt{x}}}$$

b) $f'(x)=\frac{1}{2\sqrt{x}}$

$f'(2)=\frac{1}{2\sqrt{2}}$, $\boxed{m=\frac{1}{2\sqrt{2}}}$

Ex. 4 Use the alternative derivative definition to find slope of $f(x)=\sqrt{x}$ at $x=2$

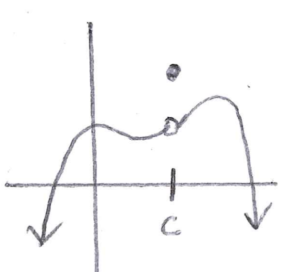
$$f'(c)=\lim_{x\to c}\frac{f(x)-f(c)}{x-c}$$

$$f'(2)=\lim_{x\to 2}\frac{\sqrt{x}-f(2)}{x-2}$$

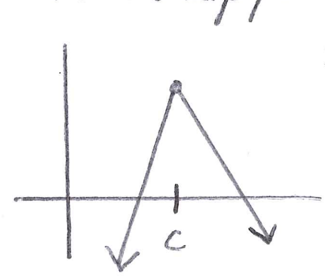
$$=\lim_{x\to 2}\frac{\sqrt{x}-\sqrt{2}}{x-2}\cdot\frac{(\sqrt{x}+\sqrt{2})}{(\sqrt{x}+\sqrt{2})}=\lim_{x\to 2}\frac{\cancel{x-2}}{\cancel{(x-2)}(\sqrt{x}+\sqrt{2})}$$

$$=\lim_{x\to 2}\frac{1}{(\sqrt{x}+\sqrt{2})}=\frac{1}{\sqrt{2}+\sqrt{2}}=\frac{1}{2\sqrt{2}}\quad \boxed{f'(2)=\frac{1}{2\sqrt{2}}}$$

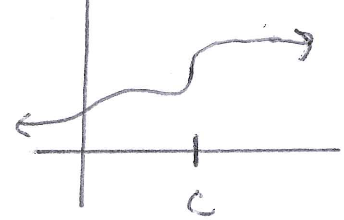
Differentiability: In order for a function to be differentiable (smooth curve) at a point, c, it must be continuous at that point, cannot contain a sharp point, cannot have vertical tangent



Graph not continuous
$f'(c)=DNE$



Sharp point at $f(c)$
$f'(c)=DNE$



vertical tangent at $f(c)$
$f'(c)=DNE$

Ch. 2.1 Homework p. 103-106 #1, 13, 17, 21, 23, 25, 27, 37-45 odd, 49-55

Find derivative by general (limit) definition

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}}$$

21) $f(x) = \frac{1}{x-1}$

$$f'(x) = \lim_{h\to 0} \frac{\frac{1}{x+h-1} - \frac{1}{x-1}}{h} = \lim_{h\to 0} \frac{\frac{x-1-(x+h-1)}{(x+h-1)(x-1)}}{h}$$

$$= \lim_{h\to 0} \frac{x-1-x-h+1}{h(x+h-1)(x-1)} = \lim_{h\to 0} \frac{-\cancel{h}}{\cancel{h}(x+h-1)(x-1)} = \lim_{h\to 0} \frac{-1}{(x+0-1)(x-1)} = \frac{-1}{(x-1)^2}$$

$$\boxed{f'(x) = \frac{-1}{(x-1)^2}}$$

23) $f(x) = \sqrt{x+1}$

$$f'(x) = \lim_{h\to 0} \frac{\sqrt{x+h+1} - \sqrt{x+1}}{h} \cdot \frac{(\sqrt{x+h+1} + \sqrt{x+1})}{(\sqrt{x+h+1} + \sqrt{x+1})} = \lim_{h\to 0} \frac{x+h+1-(x+1)}{h(\sqrt{x+h+1}+\sqrt{x+1})}$$

$$= \lim_{h\to 0} \frac{\cancel{x}+\cancel{h}+\cancel{1}-\cancel{x}-\cancel{1}}{\cancel{h}(\sqrt{x+h+1}+\sqrt{x+1})} = \lim_{h\to 0} \frac{1}{\sqrt{x+h+1}+\sqrt{x+1}} = \lim_{h\to 0} \frac{1}{\sqrt{x+0+1}+\sqrt{x+1}} = \frac{1}{2\sqrt{x+1}}$$

$$\boxed{f'(x) = \frac{1}{2\sqrt{x+1}}}$$

25) $f(x) = x^2+1$ $(2,5)$ Find equation of tangent line

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2+1-(x^2+1)}{h} = \lim_{h\to 0} \frac{\cancel{x^2}+2xh+h^2+\cancel{1}-\cancel{x^2}-\cancel{1}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}} = 2x+0 \quad \boxed{f'(x) = 2x}$$

$f'(2) = 2(2) = 4$

point: $(2,5)$

slope: $m = 4$

$y - y_1 = m(x - x_1)$

$$\boxed{y-5 = 4(x-2)}$$

Find f and c

49) $\lim\limits_{h \to 0} \dfrac{[5-3(1+h)]-2}{h}$ matches $\lim\limits_{h \to 0} \dfrac{f(x+h)-f(x)}{h}$

$$\boxed{\begin{aligned} f(x) &= 5-3x \\ c &= 1 \end{aligned}}$$

51) $\lim\limits_{x \to 6} \dfrac{-x^2+36}{x-6}$ matches $\lim\limits_{x \to c} \dfrac{f(x)-f(c)}{x-c}$

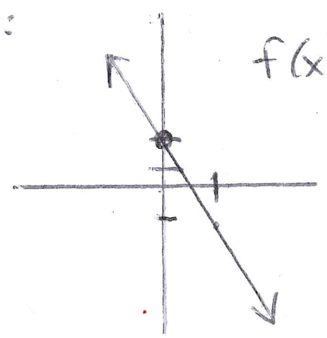
$$\boxed{\begin{aligned} f(x) &= -x^2 \\ c &= 6 \end{aligned}}$$

53) Sketch function with given characteristics:

$f(0)=2$

$f'(x)=-3, \quad -\infty < x < \infty$

* this means the slope of graph is always $m=-3$

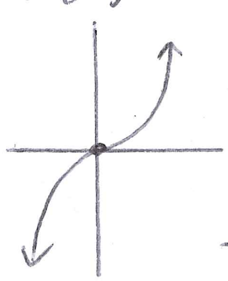


$f(x)=-3x+2$

55) $f(0)=0$

$f'(0)=0$ ← the slope of graph at $x=0$ is $m=0$

$f'(x)>0$ if $x \neq 0$ ← slope of graph is always



$f(x)=x^3$