96. Assume the interval is $[-1, 1]$. Let $x \in [-1, 1]$,

$$f(1) = f(x) + (1 - x)f'(x) + \tfrac{1}{2}(1 - x)^2 f''(c),\ c \in (x, 1)$$

$$f(-1) = f(x) + (-1 - x)f'(x) + \tfrac{1}{2}(-1 - x)^2 f''(d),\ d \in (-1, x).$$

So, $f(1) - f(-1) = 2f'(x) + \tfrac{1}{2}(1 - x)^2 f''(c) - \tfrac{1}{2}(1 + x)^2 f''(d)$

$$2f'(x) = f(1) - f(-1) - \tfrac{1}{2}(1 - x)^2 f''(c) + \tfrac{1}{2}(1 + x)^2 f''(d).$$

Because $|f(x)| \le 1$ and $|f''(x)| \le 1$,

$$2|f'(x)| \le |f(1)| + |f(-1)| + \tfrac{1}{2}(1 - x)^2|f''(c)| + \tfrac{1}{2}(1 + x)^2|f''(d)| \le 1 + 1 + \tfrac{1}{2}(1 - x^2) + \tfrac{1}{2}(1 + x)^2 = 3 + x^2 \le 4.$$

So, $|f'(x)| \le 2$.

Note: Let $f(x) = \tfrac{1}{2}(x + 1)^2 - 1$. Then $|f'(x)| \le 1, |f''(x)| = 1$ and $f'(1) = 2$.

Review Exercises for Chapter 9

1. $a_n = 5^n$

$a_1 = 5^1 = 5$

$a_2 = 5^2 = 25$

$a_3 = 5^3 = 125$

$a_4 = 5^4 = 625$

$a_5 = 5^5 = 3125$

2. $a_n = \dfrac{3^n}{n!}$

$a_1 = \dfrac{3^1}{1!} = 3$

$a_2 = \dfrac{3^2}{2!} = \dfrac{9}{2}$

$a_3 = \dfrac{3^3}{3!} = \dfrac{9}{2}$

$a_4 = \dfrac{3^4}{4!} = \dfrac{27}{8}$

$a_5 = \dfrac{3^5}{5!} = \dfrac{81}{40}$

3. $a_n = \left(-\tfrac{1}{4}\right)^n$

$a_1 = \left(-\tfrac{1}{4}\right)^1 = -\tfrac{1}{4}$

$a_2 = \left(-\tfrac{1}{4}\right)^2 = \tfrac{1}{16}$

$a_3 = \left(-\tfrac{1}{4}\right)^3 = -\tfrac{1}{64}$

$a_4 = \left(-\tfrac{1}{4}\right)^4 = \tfrac{1}{256}$

$a_5 = \left(-\tfrac{1}{4}\right)^5 = -\tfrac{1}{1024}$

4. $a_n = \dfrac{2n}{n + 5}$

$a_1 = \dfrac{2(1)}{1 + 5} = \dfrac{1}{3}$

$a_2 = \dfrac{2(2)}{2 + 5} = \dfrac{4}{7}$

$a_3 = \dfrac{2(3)}{3 + 5} = \dfrac{3}{4}$

$a_4 = \dfrac{2(4)}{4 + 5} = \dfrac{8}{9}$

$a_5 = \dfrac{2(5)}{5 + 5} = 1$

5. $a_n = 4 + \dfrac{2}{n}$: 6, 5, 4.67, …

Matches (a).

6. $a_n = 4 - \dfrac{n}{2}$: 3.5, 3, …

Matches (c).

7. $a_n = 10(0.3)^{n-1}$: 10, 3, …

Matches (d).

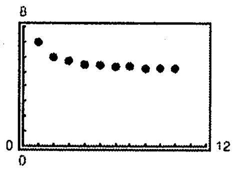
8. $a_n = 6\left(-\tfrac{2}{3}\right)^{n-1}$: 6, −4, …

Matches (b).

9. $a_n = \dfrac{5n + 2}{n}$

The sequence seems to converge to 5.

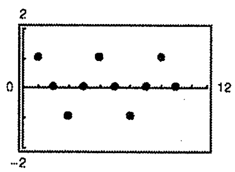
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{5n + 2}{n} = \lim_{n\to\infty}\left(5 + \frac{2}{n}\right) = 5$$



10. $a_n = \sin\dfrac{n\pi}{2}$

The sequence seems to diverge (oscillates).

$\sin\dfrac{n\pi}{2}$: 1, 0, −1, 0, 1, 0, …



11. $\lim_{n\to\infty}\left[\left(\dfrac{2}{5}\right)^n + 5\right] = 0 + 5 = 5$

Converges

12. $\lim_{n\to\infty}\left[3 - \dfrac{2}{n^2 - 1}\right] = 3 - 0 = 3$

Converges

13. $\lim_{n\to\infty}\dfrac{n^3 + 1}{n^2} = \infty$

Diverges

14. $\lim_{n\to\infty}\dfrac{1}{\sqrt{n}} = 0$

Converges

15. $\lim_{n\to\infty}\dfrac{n}{n^2 + 1} = 0$

Converges

16. $\lim_{n\to\infty}\dfrac{n}{\ln(n)} = \lim_{n\to\infty}\dfrac{1}{1/n} = \infty$

Diverges

17. $$\lim_{n\to\infty}\left(\sqrt{n+1} - \sqrt{n}\right) = \lim_{n\to\infty}\left(\sqrt{n+1} - \sqrt{n}\right)\frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} = \lim_{n\to\infty}\frac{1}{\sqrt{n+1} + \sqrt{n}} = 0$$

Converges

18. $\lim_{n\to\infty}\dfrac{\sin\sqrt{n}}{\sqrt{n}} = 0$

Converges

19. $a_n = 5n - 2$

20. $a_n = n^2 - 6$

21. $a_n = \dfrac{1}{n! + 1}$

22. $a_n = \dfrac{n}{n^2 + 1}$

23. (a) $A_n = 8000\left(1 + \dfrac{0.05}{4}\right)^n, \quad n = 1, 2, 3, \ldots$

$A_1 = 8000\left(1 + \dfrac{0.05}{4}\right)^1 = \$8100.00$

$A_2 = \$8201.25$

$A_3 = \$8303.77$

$A_4 = \$8407.56$

$A_5 = \$8512.66$

$A_6 = \$8619.07$

$A_7 = \$8726.80$

$A_8 = \$8835.89$

(b) $A_{40} = \$13{,}148.96$

24. (a) $V_n = 175{,}000(0.70)^n, \quad n = 1, 2, \ldots$

(b) $V_5 = 175{,}000(0.70)^5 \approx \$29{,}412.25$

25. $S_1 = 3$

$S_2 = 3 + \dfrac{3}{2} = \dfrac{9}{2} = 4.5$

$S_3 = 3 + \dfrac{3}{2} + 1 = \dfrac{11}{2} = 5.5$

$S_4 = 3 + \dfrac{3}{2} + 1 + \dfrac{3}{4} = \dfrac{25}{4} = 6.25$

$S_5 = 3 + \dfrac{3}{2} + 1 + \dfrac{3}{4} + \dfrac{3}{5} = \dfrac{137}{20} = 6.85$

26. $S_1 = -\frac{1}{2} = -0.5$

$S_2 = -\frac{1}{2} + \frac{1}{4} = -\frac{1}{4} = -0.25$

$S_3 = -\frac{1}{2} + \frac{1}{4} - \frac{1}{8} = -\frac{3}{8} = -0.375$

$S_4 = -\frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} = -\frac{5}{16} = -0.3125$

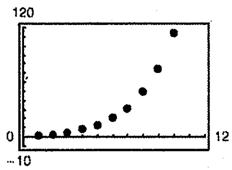
$S_5 = -\frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} = -\frac{11}{32} = -0.34375$

27. (a)

n	5	10	15	20	25
S_n	13.2	113.3	873.8	6648.5	50,500.3

The series diverges $\left(\text{geometric } r = \frac{3}{2} > 1\right)$.

(b)

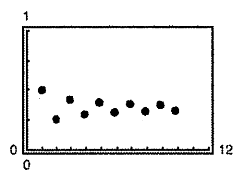


28. (a)

n	5	10	15	20	25
S_n	0.3917	0.3228	0.3627	0.3344	0.3564

The series converges by the Alternating Series Test.

(b)

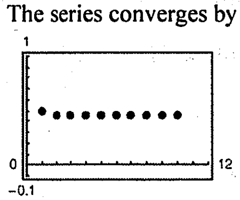


29. (a)

n	5	10	15	20	25
S_n	0.4597	0.4597	0.4597	0.4597	0.4597

The series converges by the Alternating Series Test.

(b)

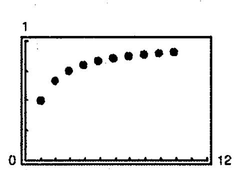


30. (a)

n	5	10	15	20	25
S_n	0.8333	0.9091	0.9375	0.9524	0.9615

The series converges, by the Limit Comparison Test with $\sum \frac{1}{n^2}$.

(b)



31. $\sum_{n=0}^{\infty}\left(\frac{2}{5}\right)^n = \frac{1}{1-(2/5)} = \frac{5}{3}$ (Geometric series)

32. $\sum_{n=0}^{\infty}\frac{3^{n+2}}{7^n} = 9\sum_{n=0}^{\infty}\left(\frac{3}{7}\right)^n = 9\left(\frac{1}{1-(3/7)}\right)$

$= 9\cdot\frac{7}{4} = \frac{63}{4}$ (Geometric series)

33. $\sum_{n=1}^{\infty}\left[(0.6)^n + (0.8)^n\right] = \sum_{n=0}^{\infty}0.6(0.6)^n + \sum_{n=0}^{\infty}0.8(0.8)^n = (0.6)\frac{1}{1-0.6} + (0.8)\frac{1}{1-0.8} = \frac{6}{10}\cdot\frac{10}{4} + \frac{8}{10}\cdot\frac{10}{2} = \frac{11}{2} = 5.5$

34. $\sum_{n=0}^{\infty}\left[\left(\frac{2}{3}\right)^n - \frac{1}{(n+1)(n+2)}\right] = \sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n - \sum_{n=0}^{\infty}\left(\frac{1}{n+1} - \frac{1}{n+2}\right)$

$= \frac{1}{1-(2/3)} - \left[\left(1-\frac{1}{2}\right) + \left(\frac{1}{2}-\frac{1}{3}\right) + \left(\frac{1}{3}-\frac{1}{4}\right) + \cdots\right] = 3 - 1 = 2$

35. (a) $0.\overline{09} = 0.09 + 0.0009 + 0.000009 + \cdots = 0.09(1 + 0.01 + 0.0001 + \cdots) = \sum_{n=0}^{\infty}(0.09)(0.01)^n$

(b) $0.\overline{09} = \frac{0.09}{1-0.01} = \frac{1}{11}$

36. (a) $0.\overline{64} = 0.64 + 0.0064 + 0.000064 + \cdots = 0.64(1 + 0.01 + 0.0001 + \cdots) = 0.64\sum_{n=0}^{\infty}(0.01)^n$

(b) $0.\overline{64} = \frac{0.64}{1-0.01} = \frac{64}{99}$

37. Diverges. Geometric series with $a = 1$ and $|r| = 1.67 > 1$.

38. Converges. Geometric series with $a = 1$ and $|r| = |0.36| < 1$.

39. Diverges. nth-Term Test. $\lim_{n\to\infty} a_n \neq 0$.

40. Diverges. nth-Term Test, $\lim_{n\to\infty} a_n = \frac{2}{3}$.

41. $D_1 = 8$

$D_2 = 0.7(8) + 0.7(8) = 16(0.7)$

$\vdots$

$D = 8 + 16(0.7) + 16(0.7)^2 + \cdots + 16(0.7)^n + \cdots$

$= -8 + \sum_{n=0}^{\infty}16(0.7)^n = -8 + \frac{16}{1-0.7} = 45\frac{1}{3}$ meters

42. (See Exercise 84 in Section 9.2)

$A = P\left(\frac{12}{r}\right)\left[\left(1+\frac{r}{12}\right)^{12t} - 1\right]$

$= 125\left(\frac{12}{0.035}\right)\left[\left(1+\frac{0.035}{12}\right)^{12(10)} - 1\right] \approx \$17{,}929.06$

43. $\sum_{n=1}^{\infty}\frac{2}{6n+1}$

Let $f(x) = \frac{2}{6x+1}$, $f'(x) = \frac{-12}{(6x+1)^2} < 0$ for $x \geq 1$

f is positive, continuous, and decreasing for $x \geq 1$.

$\int_1^{\infty}\frac{2}{6x+1}\,dx = \left[\frac{1}{3}\ln(6x+1)\right]_1^{\infty} = \infty$, diverges.

So, the series diverges by Theorem 9.10.

44. $\sum_{n=1}^{\infty}\frac{1}{\sqrt[4]{n^3}} = \sum_{n=1}^{\infty}\frac{1}{n^{3/4}}$

Divergent p-series, $p = \frac{3}{4} < 1$

45. $\sum_{n=1}^{\infty}\frac{1}{n^{5/2}}$ is a p-series with $p = \frac{5}{2} > 1$.

So, the series converges.

46. $\sum_{n=1}^{\infty} \frac{1}{5^n}$

Let $f(x) = \frac{1}{5^x}$, $f'(x) = -(\ln 5)5^{-x} < 0$ for $x \geq 1$.

f is positive, continuous, and decreasing for $x \geq 1$.

$$\int_1^{\infty} \frac{1}{5^x}\,dx = \left[\frac{-1}{(\ln 5)5^x}\right]_1^{\infty} = \frac{1}{5 \ln 5}$$

So, the series converges by Theorem 9.10.

47. $\sum_{n=1}^{\infty}\left(\frac{1}{n^2} - \frac{1}{n}\right) = \sum_{n=1}^{\infty}\frac{1}{n^2} - \sum_{n=1}^{\infty}\frac{1}{n}$

Because the second series is a divergent p-series while the first series is a convergent p-series, the difference diverges.

48. $\sum_{n=1}^{\infty} \frac{\ln n}{n^4}$

Let $f(x) = \frac{\ln x}{x^4}$, $f'(x) = \frac{1}{x^5} - \frac{4 \ln x}{x^3} < 0$.

f is positive, continuous, and decreasing for $x > 1$.

$$\int_1^{\infty} x^{-4} \ln x\,dx = \lim_{b\to\infty}\left[\frac{-\ln x}{3x^3} - \frac{1}{9x^3}\right]_1^b = 0 + \frac{1}{9} = \frac{1}{9}$$

So, the series converges by Theorem 9.10.

49. $\sum_{n=2}^{\infty} \frac{1}{\sqrt[3]{n} - 1}$

$$\frac{1}{\sqrt[3]{n} - 1} > \frac{1}{\sqrt[3]{n}}$$

Therefore, the series diverges by comparison with the divergent p-series

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt[3]{n}} = \sum_{n=2}^{\infty} \frac{1}{n^{1/3}}.$$

50. $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^3 + 3n}}$

$$\lim_{n\to\infty} \frac{n/\sqrt{n^3 + 3n}}{1/\sqrt{n}} = \lim_{n\to\infty} \frac{n^{3/2}}{\sqrt{n^3 + 3n}} = 1$$

By a limit comparison test with the divergent p-series $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$, the series diverges.

51. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3 + 2n}}$

$$\lim_{n\to\infty} \frac{1/\sqrt{n^3 + 2n}}{1/\left(n^{3/2}\right)} = \lim_{n\to\infty} \frac{n^{3/2}}{\sqrt{n^3 + 2n}} = 1$$

By a limit comparison test with the convergent p-series $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$, the series converges.

52. $\sum_{n=1}^{\infty} \frac{n + 1}{n(n + 2)}$

$$\lim_{n\to\infty} \frac{(n + 1)/n(n + 2)}{1/n} = \lim_{n\to\infty} \frac{n + 1}{n + 2} = 1$$

By a limit comparison test with $\sum_{n=1}^{\infty} \frac{1}{n}$, the series diverges.

53. $\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n - 1)}{2 \cdot 4 \cdot 6 \cdots (2n)}$

$$a_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n - 1)}{2 \cdot 4 \cdot 6 \cdots (2n)} = \left(\frac{3}{2} \cdot \frac{5}{4} \cdots \frac{2n - 1}{2n - 2}\right)\frac{1}{2n} > \frac{1}{2n}$$

Because $\sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (harmonic series), so does the original series.

54. Because $\sum_{n=1}^{\infty} \frac{1}{3^n}$ converges, $\sum_{n=1}^{\infty} \frac{1}{3^n - 5}$ converges by the Limit Comparison Test.

55. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^5}$ converges by the Alternating Series Test.

$$\lim_{n\to\infty} \frac{1}{n^5} = 0 \text{ and } a_{n+1} = \frac{1}{(n + 1)^5} < \frac{1}{n^5} = a_n.$$

56. $\sum_{n=1}^{\infty} \frac{(-1)^n(n + 1)}{n^2 + 1}$ converges by the Alternating Series Test. $\lim_{n\to\infty} \frac{n + 1}{n^2 + 1} = 0$ and if

$$f(x) = \frac{x + 1}{x^2 + 1},\ f'(x) = \frac{-\left(x^2 + 2x - 1\right)}{\left(x^2 + 1\right)^2} < 0 \Rightarrow \text{terms}$$

are decreasing. So, $a_{n+1} < a_n$.

57. $\sum_{n=2}^{\infty} \frac{(-1)^n - n}{n^2 - 3}$ converges by the Alternating Series Test.

$\lim_{n\to\infty} \frac{n}{n^2 - 3} = 0$ and if

$f(x) = \frac{n}{n^2 - 3}, f'(x) = \frac{-(n^2 + 3)}{(n^2 - 3)^2} < 0 \Rightarrow$ terms are decreasing. So, $a_{n+1} < a_n$.

58. $\sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{n}}{n + 1}$

$$a_{n+1} = \frac{\sqrt{n+1}}{n+2} \le \frac{\sqrt{n}}{n+1} = a_n$$

$$\lim_{n\to\infty} \frac{\sqrt{n}}{n+1} = 0$$

By the Alternating Series Test, the series converges.

59. Diverges by the nth-Term Test.

$$\lim_{n\to\infty} \frac{n}{n-3} = 1 \ne 0$$

60. Converges by the Alternating Series Test.

$$a_{n+1} = \frac{3\ln(n+1)}{n+1} < \frac{3\ln n}{n} = a_n, \lim_{n\to\infty} \frac{3\ln n}{n} = 0$$

61. $\lim_{n\to\infty} \sqrt[n]{\left(\frac{3n-1}{2n+5}\right)^n} = \lim_{n\to\infty} \left(\frac{3n-1}{2n+5}\right) = \frac{3}{2} > 1$

Diverges by Root Test.

62. $\lim_{n\to\infty} \sqrt[n]{\left(\frac{4n}{7n-1}\right)^n} = \lim_{n\to\infty} \left(\frac{4n}{7n-1}\right) = \frac{4}{7} < 1$

Converges by Root Test.

63. $\sum_{n=1}^{\infty} \frac{n}{e^{n^2}}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{n+1}{e^{(n+1)^2}} \cdot \frac{e^{n^2}}{n}\right|$$
$$= \lim_{n\to\infty} \left|\frac{e^{n^2}(n+1)}{e^{n^2+2n+1}n}\right|$$
$$= \lim_{n\to\infty} \left(\frac{1}{e^{2n+1}}\right)\left(\frac{n+1}{n}\right)$$
$$= (0)(1) = 0 < 1$$

By the Ratio Test, the series converges.

64. $\sum_{n=1}^{\infty} \frac{n!}{e^n}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)!}{e^{n+1}} \cdot \frac{e^n}{n!}\right|$$
$$= \lim_{n\to\infty} \frac{n+1}{e} = \infty$$

By the Ratio Test, the series diverges.

65. $\sum_{n=1}^{\infty} \frac{2^n}{n^3}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{2^{n+1}}{(n+1)^3} \cdot \frac{n^3}{2^n}\right| = \lim_{n\to\infty} \frac{2n^3}{(n+1)^3} = 2$$

By the Ratio Test, the series diverges.

66. $\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 5 \cdot 8 \cdots (3n-1)}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{1 \cdot 3 \cdots (2n-1)(2n+1)}{2 \cdot 5 \cdots (3n-1)(3n+2)} \cdot \frac{2 \cdot 5 \cdots (3n-1)}{1 \cdot 3 \cdots (2n-1)}\right| = \lim_{n\to\infty} \frac{2n+1}{3n+2} = \frac{2}{3}$$

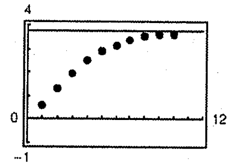
By the Ratio Test, the series converges.

67. (a) Ratio Test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)(3/5)^{n+1}}{n(3/5)^n} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)\left(\frac{3}{5}\right) = \frac{3}{5} < 1$, converges

(b)

n	5	10	15	20	25
S_n	2.8752	3.6366	3.7377	3.7488	3.7499

(c)



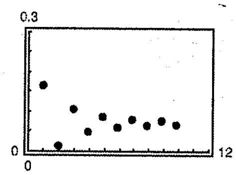
(d) The sum is approximately 3.75.

68. (a) The series converges by the Alternating Series Test.

(b)

n	5	10	15	20	25
S_n	0.0871	0.0669	0.0734	0.0702	0.0721

(c)



(d) The sum is approximately 0.0714.

69. $f(x) = e^{-2x}, \quad f(0) = 1$

$f'(x) = -2e^{-2x}, \quad f'(0) = -2$

$f''(x) = 4e^{-2x}, \quad f''(0) = 4$

$f'''(x) = -8e^{-2x}, \quad f'''(0) = -8$

$$P_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$$
$$= 1 - 2x + 2x^2 - \frac{4}{3}x^3$$

70. $f(x) = \cos \pi x, \quad f(0) = 1$

$f'(x) = -\pi \sin \pi x, \quad f'(0) = 0$

$f''(x) = -\pi^2 \cos \pi x, \quad f''(0) = -\pi^2$

$f'''(x) = \pi^3 \sin \pi x, \quad f'''(0) = 0$

$f^{(4)}(x) = \pi^4 \cos \pi x, \quad f^{(4)}(0) = \pi^4$

$$P_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$
$$= 1 - \frac{\pi^2 x^2}{2} + \frac{\pi^4 x^4}{24}$$

71. $f(x) = e^{-3x} \quad f(0) = 1$

$f'(x) = -3e^{-3x} \quad f'(0) = -3$

$f''(x) = 9e^{-3x} \quad f''(0) = 9$

$f'''(x) = -27e^{-3x} \quad f'''(0) = -27$

$$P_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$$
$$= 1 - 3x + \frac{9}{2}x^2 - \frac{9}{2}x^3$$

72. $f(x) = \tan x$ $\quad f\left(-\frac{\pi}{4}\right) = -1$

$f'(x) = \sec^2 x$ $\quad f'\left(-\frac{\pi}{4}\right) = 2$

$f''(x) = 2\sec^2 x \tan x$ $\quad f''\left(-\frac{\pi}{4}\right) = -4$

$f'''(x) = 4\sec^2 x\tan^2 x + 2\sec^4 x$ $\quad f'''\left(-\frac{\pi}{4}\right) = 16$

$$P_3(x) = f\left(-\frac{\pi}{4}\right) + f'\left(-\frac{\pi}{4}\right)\left(x + \frac{\pi}{4}\right) + \frac{f''\left(-\frac{\pi}{4}\right)}{2!}\left(x + \frac{\pi}{4}\right)^2 + \frac{f'''\left(-\frac{\pi}{4}\right)}{3!}\left(x + \frac{\pi}{4}\right)^3$$

$$= -1 + 2\left(x + \frac{\pi}{4}\right) - 2\left(x + \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x + \frac{\pi}{4}\right)^3$$

73. $f(x) = \cos x$

$\left|f^{(n+1)}(x)\right| \le 1$ for all x and all n.

$$\left|R_n(x)\right| = \left|\frac{f^{(n+1)}(z)\,x^{n+1}}{(n+1)!}\right| \le \frac{(0.75)^{n+1}}{(n+1)!} < 0.001$$

By trial and error, $n = 5$. (3 terms)

74. $f(x) = e^x, f^{(n+1)} = e^x$

Maximum on $[-0.25, 0]$ is $e^0 = 1$.

$$|R_n| \le \frac{f^{(n+1)}(z)\,x^{n+1}}{(n+1)!} \le \frac{(-0.25)^{n+1}}{(n+1)!} < 0.001$$

By trial and error, $n = 3$.

75. $\displaystyle\sum_{n=0}^{\infty}\left(\frac{x}{10}\right)^n$

Geometric series which converges only if $|x/10| < 1$ or $-10 < x < 10$.

76. $\displaystyle\sum_{n=0}^{\infty}(5x)^n$

Geometric series which converges only if $|5x| < 1 \Rightarrow |x| < \frac{1}{5}$ or $-\frac{1}{5} < x < \frac{1}{5}$.

77. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n(x-2)^n}{(n+1)^2}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(x-2)^{n+1}}{(n+2)^2}\cdot\frac{(n+1)^2}{(-1)^n(x-2)^n}\right| = |x-2|$$

$R = 1$

Center: 2

Because the series converges when $x = 1$ and when $x = 3$, the interval of convergence is $[1, 3]$.

78. $\displaystyle\sum_{n=1}^{\infty}\frac{3^n(x-2)^n}{n}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{3^{n+1}(x-2)^{n+1}}{n+1}\cdot\frac{n}{3^n(x-2)^n}\right| = 3|x-2|$$

$R = \frac{1}{3}$

Center: 2

Because the series converges at $\frac{5}{3}$ and diverges at $\frac{7}{3}$, the interval of convergence is $\left[\frac{5}{3}, \frac{7}{3}\right)$.

79. $\displaystyle\sum_{n=0}^{\infty}n!(x-2)^n$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!(x-2)^{n+1}}{n!(x-2)^n}\right| = \infty$$

which implies that the series converges only at the center $x = 2$.

80. $\displaystyle\sum_{n=0}^{\infty} \frac{(x-2)^n}{2^n} = \sum_{n=0}^{\infty} \left(\frac{x-2}{2}\right)^n$

Geometric series which converges only if

$\left|\frac{x-2}{2}\right| < 1$ or $0 < x < 4$.

81. (a) $f(x) = \displaystyle\sum_{n=0}^{\infty} \left(\frac{x}{5}\right)^n, (-5, 5)$ (Geometric)

(b) $f'(x) = \displaystyle\sum_{n=1}^{\infty} \frac{n}{5}\left(\frac{x}{5}\right)^{n-1}, (-5, 5)$

(c) $f''(x) = \displaystyle\sum_{n=2}^{\infty} \frac{n(n-1)}{25}\left(\frac{x}{5}\right)^{n-2}, (-5, 5)$

(d) $\displaystyle\int f(x)\,dx = \sum_{n=0}^{\infty} \frac{5}{n+1}\left(\frac{x}{5}\right)^{n+1}, (-5, 5)$

$\left[\displaystyle\sum_{n=0}^{\infty} \frac{5}{n+1}\left(\frac{-5}{5}\right)^{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}5}{n+1}, \quad \text{converges}\right]$

82. (a) $f(x) = \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-4)^n}{n}, (3, 5)$

$\left[\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1}\frac{(3-4)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{2n+1}}{n}, \quad \text{diverges}\right]$

$\left[\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1}\frac{(5-4)^n}{n} = \sum \frac{(-1)^{n+1}}{n}, \quad \text{converges}\right]$

(b) $f'(x) = \displaystyle\sum_{n=1}^{\infty} (-1)^{n+1}(x-4)^{n-1}, (3, 5)$

(c) $f''(x) = \displaystyle\sum_{n=2}^{\infty} (-1)^{n+1}(n-1)(x-4)^{n-2}, (3, 5)$

(d) $\displaystyle\int f(x)\,dx = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{(x-4)^{n+1}}{n(n+1)}, [3, 5]$

$\left[\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1}\frac{(3-4)^{n+1}}{n(n+1)} \text{ and } \sum_{n=1}^{\infty} (-1)^{n+1}\frac{5-4}{n(n+1)}, \quad \text{both converge}\right]$

83.
$$y = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{4^n (n!)^2}$$

$$y' = \sum_{n=1}^{\infty} \frac{(-1)^n (2n) x^{2n-1}}{4^n (n!)^2} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2n+2)x^{2n+1}}{4^{n+1}\left[(n+1)!\right]^2}$$

$$y'' = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2n+2)(2n+1)x^{2n}}{4^{n+1}\left[(n+1)!\right]^2}$$

$$x^2y'' + xy' + x^2y = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2n+2)(2n+1)x^{2n+2}}{4^{n+1}\left[(n+1)!\right]^2} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2n+2)x^{2n+2}}{4^{n+1}\left[(n+1)!\right]^2} + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{4^n (n!)^2}$$

$$= \sum_{n=0}^{\infty} \left[(-1)^{n+1} \frac{(2n+2)(2n+1)}{4^{n+1}\left[(n+1)!\right]^2} + \frac{(-1)^{n+1}(2n+2)}{4^{n+1}\left[(n+1)!\right]^2} + \frac{(-1)^n}{4^n (n!)^2} \right] x^{2n+2}$$

$$= \sum_{n=0}^{\infty} \left[\frac{(-1)^{n+1}(2n+2)(2n+1+1)}{4^{n+1}\left[(n+1)!\right]^2} + (-1)^n \frac{1}{4^n (n!)^2} \right] x^{2n+2}$$

$$= \sum_{n=0}^{\infty} \left[\frac{(-1)^{n+1} 4(n+1)^2}{4^{n+1}\left[(n+1)!\right]^2} + (-1)^n \frac{1}{4^n (n!)^2} \right] x^{2n+2}$$

$$= \sum_{n=0}^{\infty} \left[\frac{(-1)^{n+1} 1}{4^n (n!)^2} + (-1)^n \frac{1}{4^n (n!)^2} \right] x^{2n+2} = 0$$

84.
$$y = \sum_{n=0}^{\infty} \frac{(-3)^n x^{2n}}{2^n n!}$$

$$y' = \sum_{n=1}^{\infty} \frac{(-3)^n (2n) x^{2n-1}}{2^n n!} = \sum_{n=0}^{\infty} \frac{(-3)^{n+1}(2n+2)x^{2n+1}}{2^{n+1}(n+1)!}$$

$$y'' = \sum_{n=0}^{\infty} \frac{(-3)^{n+1}(2n+2)(2n+1)x^{2n}}{2^{n+1}(n+1)!}$$

$$y'' + 3xy' + 3y = \sum_{n=0}^{\infty} \frac{(-3)^{n+1}(2n+2)(2n+1)x^{2n}}{2^{n+1}(n+1)!} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}3^{n+2}(2n+2)x^{2n+2}}{2^{n+1}(n+1)!} + \sum_{n=0}^{\infty} \frac{(-1)^n 3^{n+1} x^{2n}}{2^n n!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}3^{n+1}(2n+2)x^{2n}}{2^n n!} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}3^{n+2}x^{2n+2}}{2^n n!} + \sum_{n=0}^{\infty} \frac{(-1)^n 3^{n+1} x^{2n}}{2^n n!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 3^{n+1} x^{2n}}{2^n n!}\left[-(2n+1)+1\right] + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}3^{n+2}x^{2n+2}}{2^n n!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 3^{n+1} x^{2n}}{2^n n!}(-2n) + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}3^{n+2}x^{2n+2}}{2^n n!}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 3^{n+1} x^{2n}}{2^n n!}(2n) + \sum_{n=1}^{\infty} \frac{(-1)^n 3^{n+1} x^{2n}}{2^{n-1}(n-1)!} \cdot \frac{2n}{2n}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n 3^{n+1} x^{2n}}{2^n n!}\left[-2n + 2n\right] = 0$$

85. $\dfrac{2}{3-x} = \dfrac{2/3}{1-(x/3)} = \dfrac{a}{1-r}$

$$\sum_{n=0}^{\infty} \frac{2}{3}\left(\frac{x}{3}\right)^n = \sum_{n=0}^{\infty} \frac{2x^n}{3^{n+1}}$$

86. $\dfrac{3}{2+x} = \dfrac{3/2}{1+(x/2)} = \dfrac{3/2}{1-(-x/2)} = \dfrac{a}{1-r}$

$$\sum_{n=0}^{\infty} \frac{3}{2}\left(-\frac{x}{2}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n 3x^n}{2^{n+1}}$$

87. $\dfrac{6}{4-x} = \dfrac{6}{3-(x-1)} = \dfrac{2}{1-\left(\dfrac{x-1}{3}\right)} = \dfrac{a}{1-r}$

$$= \sum_{n=0}^{\infty} 2\left(\frac{x-1}{3}\right)^n = 2\sum_{n=0}^{\infty} \frac{(x-1)^n}{3^n}$$

Interval of convergence:

$\left|\dfrac{x-1}{3}\right| < 1 \Rightarrow |x-1| < 3 \Rightarrow (-2, 4)$

88. $\dfrac{1}{3-2x} = \dfrac{1/3}{1-\left(\dfrac{2}{3}x\right)} = \dfrac{a}{1-r}$

$$= \sum_{n=0}^{\infty} \frac{1}{3}\left(\frac{2x}{3}\right)^n = \frac{1}{3}\sum_{n=0}^{\infty} \left(\frac{2x}{3}\right)^n$$

Interval of convergence:

$\left|\dfrac{2x}{3}\right| < 1 \Rightarrow |2x| < 3 \Rightarrow |x| < \dfrac{3}{2} \Rightarrow \left(-\dfrac{3}{2}, \dfrac{3}{2}\right)$

89. $\ln x = \displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x-1)^n}{n}, \quad 0 < x \le 2$

$\ln\left(\dfrac{5}{4}\right) = \displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{(5/4)-1}{n}\right)^n = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{4^n n} \approx 0.2231$

90. $\ln x = \displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x-1)^n}{n}, \quad 0 < x \le 2$

$\ln\left(\dfrac{6}{5}\right) = \displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{(6/5)-1}{n}\right)^n$

$= \displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{5^n n} \approx 0.1823$

91. $e^x = \displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!}, \quad -\infty < x < \infty$

$e^{1/2} = \displaystyle\sum_{n=0}^{\infty}\frac{(1/2)^n}{n!} = \sum_{n=0}^{\infty}\frac{1}{2^n n!} \approx 1.6487$

92. $e^x = \displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!}, \quad -\infty < x < \infty$

$e^{2/3} = \displaystyle\sum_{n=0}^{\infty}\frac{(2/3)^n}{n!} = \sum_{n=0}^{\infty}\frac{2^n}{3^n n!} \approx 1.9477$

93. $\cos x = \displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}, \quad -\infty < x < \infty$

$\cos\left(\dfrac{2}{3}\right) = \displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{2^{2n}}{3^{2n}(2n)!} = 0.7859$

94. $\sin x = \displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}, \quad -\infty < x < \infty$

$\sin\left(\dfrac{1}{3}\right) = \displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{1}{3^{2n+1}(2n+1)!} \approx 0.3272$

95. $f(x) = \sin x$

$f'(x) = \cos x$

$f''(x) = -\sin x$

$f'''(x) = -\cos x, \cdots$

$$\sin(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(x)[x-(3\pi/4)]^n}{n!}$$

$$= \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\left(x-\frac{3\pi}{4}\right) - \frac{\sqrt{2}}{2\cdot 2!}\left(x-\frac{3\pi}{4}\right)^2 + \cdots = \frac{\sqrt{2}}{2}\sum_{n=0}^{\infty}\frac{(-1)^{n(n+1)/2}[x-(3\pi/4)]^n}{n!}$$

96. $f(x) = \cos x$

$f'(x) = -\sin x$

$f''(x) = -\cos x$

$f'''(x) = \sin x$

$$\cos x = \sum_{n=0}^{\infty}\frac{f^{(n)}(-\pi/4)[x+(\pi/4)]^n}{n!} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x+\frac{\pi}{4}\right) - \frac{\sqrt{2}}{2\cdot 2!}\left(x+\frac{\pi}{4}\right)^2 - \frac{\sqrt{2}}{2\cdot 3!}\left(x+\frac{\pi}{4}\right)^3 + \frac{\sqrt{2}}{2\cdot 4!}\left(x+\frac{\pi}{4}\right)^4 + \cdots$$

$$= \frac{\sqrt{2}}{2}\left[1 + \left(x+\frac{\pi}{4}\right) + \sum_{n=1}^{\infty}\frac{(-1)^{[n(n+1)]/2}[x+(\pi/4)]^{n+1}}{(n+1)!}\right]$$

97. $3^x = \left(e^{\ln(3)}\right)^x = e^{x\ln(3)}$ and because $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, you have

$$3^x = \sum_{n=0}^{\infty} \frac{(x \ln 3)^n}{n!} = 1 + x\ln 3 + \frac{x^2[\ln 3]^2}{2!} + \frac{x^3[\ln 3]^3}{3!} + \frac{x^4[\ln 3]^4}{4!} + \cdots.$$

98.

$$\begin{aligned}
f(x) &= \csc(x) \\
f'(x) &= -\csc(x)\cot(x) \\
f''(x) &= \csc^3(x) + \csc(x)\cot^2(x) \\
f'''(x) &= -5\csc^3(x)\cot(x) - \csc(x)\cot^3(x) \\
f^{(4)}(x) &= 5\csc^5(x) + 15\csc^3(x)\cot^2(x) + \csc(x)\cot^4(x) \\
\csc(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(\pi/2)[x - (\pi/2)]^n}{n!} = 1 + \frac{1}{2!}\left(x - \frac{\pi}{2}\right)^2 + \frac{5}{4!}\left(x - \frac{\pi}{2}\right)^4 + \cdots
\end{aligned}$$

99.

$$\begin{aligned}
f(x) &= \frac{1}{x} \\
f'(x) &= -\frac{1}{x^2} \\
f''(x) &= \frac{2}{x^3} \\
f'''(x) &= -\frac{6}{x^4}, \cdots \\
\frac{1}{x} &= \sum_{n=0}^{\infty} \frac{f^{(n)}(-1)(x+1)^n}{n!} = \sum_{n=0}^{\infty} \frac{-n!(x+1)^n}{n!} = -\sum_{n=0}^{\infty}(x+1)^n, -2 < x < 0
\end{aligned}$$

100.

$$\begin{aligned}
f(x) &= \sqrt{x} = x^{1/2} \\
f'(x) &= \frac{1}{2}x^{-1/2} \\
f''(x) &= -\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)x^{-3/2} \\
f'''(x) &= \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)x^{-5/2} \\
f^{(4)}(x) &= -\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)x^{-7/2}, \cdots \\
\sqrt{x} &= \sum_{n=0}^{\infty} \frac{f^{(n)}(4)(x-4)^n}{n!} = 2 + \frac{(x-4)}{2^2} - \frac{(x-4)^2}{2^5 2!} + \frac{1\cdot 3(x-4)^3}{2^8 3!} - \frac{1\cdot 3\cdot 5(x-4)^4}{2^{11}4!} + \cdots \\
&= 2 + \frac{(x-4)}{2^2} + \sum_{n=2}^{\infty} \frac{(-1)^{n+1} 1\cdot 3\cdot 5\cdots(2n-3)(x-4)^n}{2^{3n-1}n!}
\end{aligned}$$

101.

$$\begin{aligned}
(1+x)^k &= 1 + kx + \frac{k(k-1)x^2}{2!} + \frac{k(k-1)(k-2)x^3}{3!} + \cdots \\
(1+x)^{1/5} &= 1 + \frac{x}{5} + \frac{(1/5)(-4/5)x^2}{2!} + \frac{1/5(-4/5)(-9/5)x^3}{3!} + \cdots \\
&= 1 + \frac{1}{5}x - \frac{1\cdot 4x^2}{5^2 2!} + \frac{1\cdot 4\cdot 9x^3}{5^3 3!} - \cdots = 1 + \frac{x}{5} + \sum_{n=2}^{\infty} \frac{(-1)^{n+1} 4\cdot 9\cdot 14\cdots(5n-6)x^n}{5^n n!} = 1 + \frac{x}{5} - \frac{2}{25}x^2 + \frac{6}{125}x^3 - \cdots
\end{aligned}$$

102.

$$h(x) = (1+x)^{-3}$$
$$h'(x) = -3(1+x)^{-4}$$
$$h''(x) = 12(1+x)^{-5}$$
$$h'''(x) = -60(1+x)^{-6}$$
$$h^{(4)}(x) = 360(1+x)^{-7}$$
$$h^{(5)}(x) = -2520(1+x)^{-8}$$

$$\frac{1}{(1+x)^3} = 1 - 3x + \frac{12x^2}{2!} - \frac{60x^3}{3!} + \frac{360x^4}{4!} - \frac{2520x^5}{5!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n (n+2)!x^n}{2n!} = \sum_{n=0}^{\infty} \frac{(-1)^n (n+2)(n+1)x^n}{2}$$

103. (a)

$$f(x) = e^{2x} \qquad f(0) = 1$$
$$f'(x) = 2e^{2x} \qquad f'(0) = 2$$
$$f''(x) = 4e^{2x} \qquad f''(0) = 4$$
$$f'''(x) = 8e^{2x} \qquad f'''(0) = 8$$

$$P(x) = 1 + 2x + \frac{4x^2}{2!} + \frac{8x^3}{3!} = 1 + 2x + 2x^2 + \frac{4}{3}x^3$$

(b) $$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!},\ e^{2x} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!}$$

$$P(x) = 1 + 2x + 2x^2 + \frac{4}{3}x^3$$

(c) $$e^x \cdot e^x = \left(1 + x + \frac{x^2}{2!} + \cdots\right)\left(1 + x + \frac{x^2}{2!} + \cdots\right)$$

$$P(x) = 1 + 2x + 2x^2 + \frac{4}{3}x^3$$

104. (a)

$$f(x) = \sin 2x \qquad f(0) = 0$$
$$f'(x) = 2\cos 2x \qquad f'(0) = 2$$
$$f''(x) = -4\sin 2x \qquad f''(0) = 0$$
$$f'''(x) = -8\cos 2x \qquad f'''(0) = -8$$
$$f^{(4)}(x) = 16\sin 2x \qquad f^{(4)}(0) = 0$$
$$f^{(5)}(x) = 32\cos 2x \qquad f^{(5)}(0) = 32$$
$$f^{(6)}(x) = -64\sin 2x \qquad f^{(6)}(0) = 0$$
$$f^{(7)}(x) = -128\cos 2x \qquad f^{(7)}(0) = -128$$

$$\sin 2x = 0 + 2x + \frac{0x^2}{2!} - \frac{8x^3}{3!} + \frac{0x^4}{4!} + \frac{32x^5}{5!} + \frac{0x^6}{6!} - \frac{128x^7}{7!} + \cdots = 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{315}x^7 + \cdots$$

(b) $$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\sin 2x = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{(2n+1)!} = 2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \frac{(2x)^7}{7!} + \cdots$$

$$= 2x - \frac{8x^3}{6} + \frac{32x^5}{120} - \frac{128x^7}{5040} + \cdots = 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{315}x^7 + \cdots$$

(c) $\sin 2x = 2\sin x\cos x$

$$= 2\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots\right)\left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots\right)$$

$$= 2\left[x + \left(-\frac{x^3}{2} - \frac{x^3}{6}\right) + \left(\frac{x^5}{24} + \frac{x^5}{12} + \frac{x^5}{120}\right) + \left(-\frac{x^7}{720} - \frac{x^7}{144} - \frac{x^7}{240} - \frac{x^7}{5040}\right) + \cdots\right]$$

$$= 2\left(x - \frac{2x^3}{3} + \frac{2x^5}{15} - \frac{4x^7}{315} + \cdots\right) = 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{315}x^7 + \cdots$$

105. $$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

$$e^{6x} = \sum_{n=0}^{\infty} \frac{(6x)^n}{n!} = 1 + 6x + \frac{(6x)^2}{2!} + \frac{(6x)^3}{3!} + \ldots$$

$$= 1 + 6x + 18x^2 + 36x^3 + \ldots$$

106. $$\ln x = \sum_{n=1}^{\infty} (-1)^{n-1}\frac{(x-1)^n}{n}, 0 < x \le 2$$

$$\ln(x-1) = \sum_{n=1}^{\infty} (-1)^{n-1}\frac{(x-1-1)^n}{n}$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1}\frac{(x-2)^n}{n}, 1 < x \le 3$$

107. $$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\sin 2x = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{(2n+1)!}$$

$$= 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 - \cdots$$

108. $$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\cos 3x = \sum_{n=0}^{\infty} (-1)^n \frac{(3x)^{2n}}{(2n)!}$$

$$= 1 - \frac{9}{2}x^2 + \frac{27x^4}{8} - \cdots$$

109. $$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots$$

$$\frac{\arctan x}{\sqrt{x}} = \sqrt{x} - \frac{x^{5/2}}{3} + \frac{x^{9/2}}{5} - \frac{x^{13/2}}{7} + \frac{x^{17/2}}{9} - \cdots$$

$$\lim_{x\to 0^+} \frac{\arctan x}{\sqrt{x}} = 0$$

By L'Hôpital's Rule,

$$\lim_{x\to 0^+} \frac{\arctan x}{\sqrt{x}} = \lim_{x\to 0^+} \frac{\left(\frac{1}{1+x^2}\right)}{\left(\frac{1}{2\sqrt{x}}\right)} = \lim_{x\to 0^+} \frac{2\sqrt{x}}{1+x^2} = 0.$$

110. $$\arcsin x = x + \frac{x^3}{2\cdot 3} + \frac{1\cdot 3x^5}{2\cdot 4\cdot 5} + \frac{1\cdot 3\cdot 5x^7}{2\cdot 4\cdot 6\cdot 7} + \cdots$$

$$\frac{\arcsin x}{x} = 1 + \frac{x^2}{2\cdot 3} + \frac{1\cdot 3x^4}{2\cdot 4\cdot 5} + \frac{1\cdot 3\cdot 5x^6}{2\cdot 4\cdot 6\cdot 7} + \cdots$$

$$\lim_{x\to 0} \frac{\arcsin x}{x} = 1$$

By L'Hôpital's Rule,

$$\lim_{x\to 0} \frac{\arcsin x}{x} = \lim_{x\to 0} \frac{\left(\frac{1}{\sqrt{1-x^2}}\right)}{1} = 1.$$

Problem Solving for Chapter 9

1. (a) $1\left(\frac{1}{3}\right) + 2\left(\frac{1}{9}\right) + 4\left(\frac{1}{27}\right) + \cdots = \sum_{n=0}^{\infty} \frac{1}{3}\left(\frac{2}{3}\right)^n = \frac{1/3}{1-(2/3)} = 1$

(b) $0, \frac{1}{3}, \frac{2}{3}, 1$, etc.

(c) $\lim_{n\to\infty} C_n = 1 - \sum_{n=0}^{\infty} \frac{1}{3}\left(\frac{2}{3}\right)^n = 1 - 1 = 0$